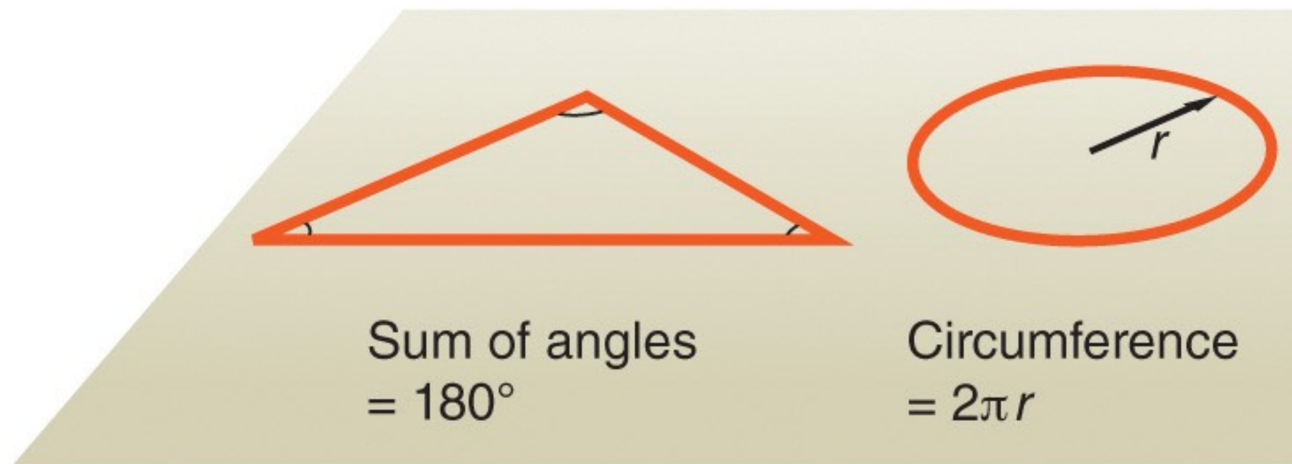
Cosmology Framework

ASTR:6782

Hai Fu

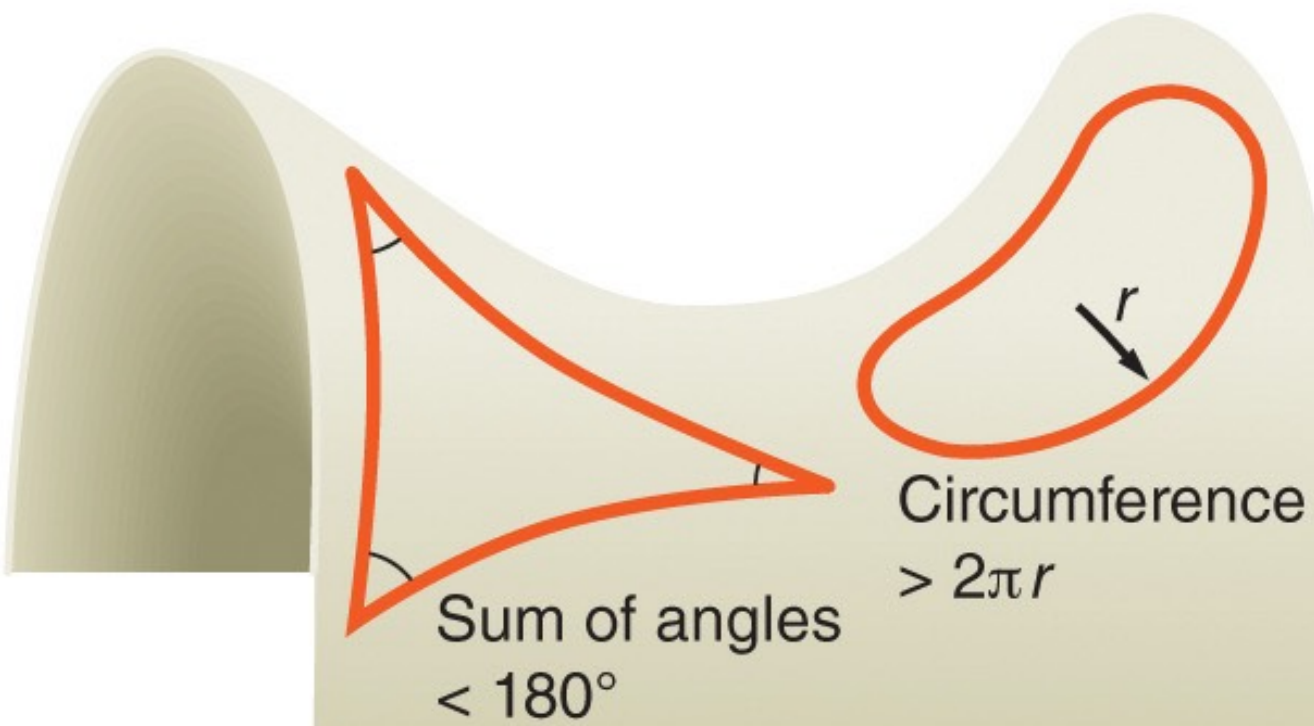
Robertson-Walker Metric: differential space-time distance

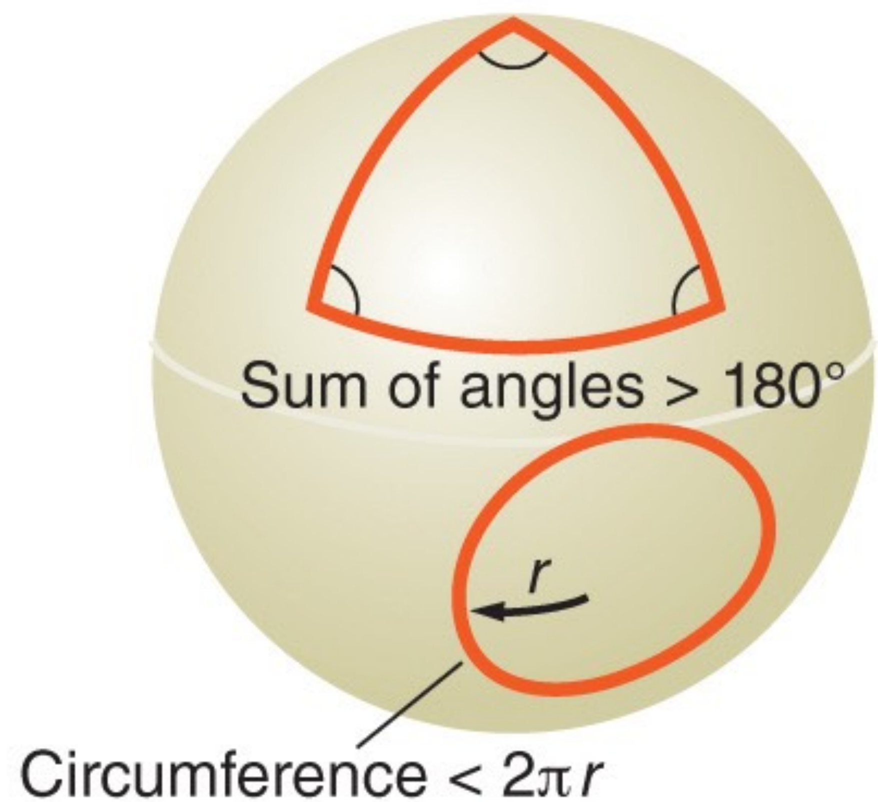
Flat vs. Curved Spaces: 2D Analogies



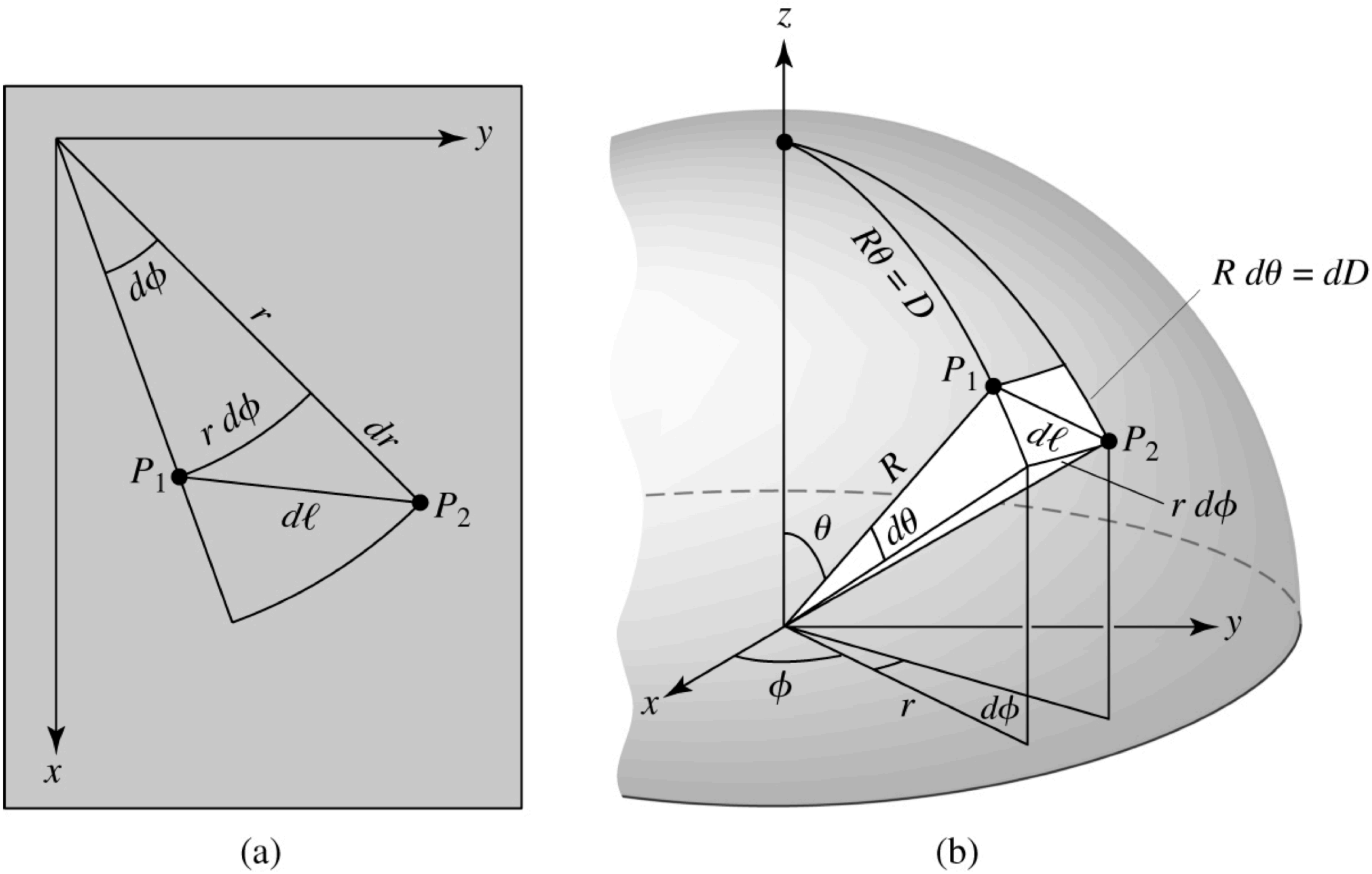
Euclid (300 BC)

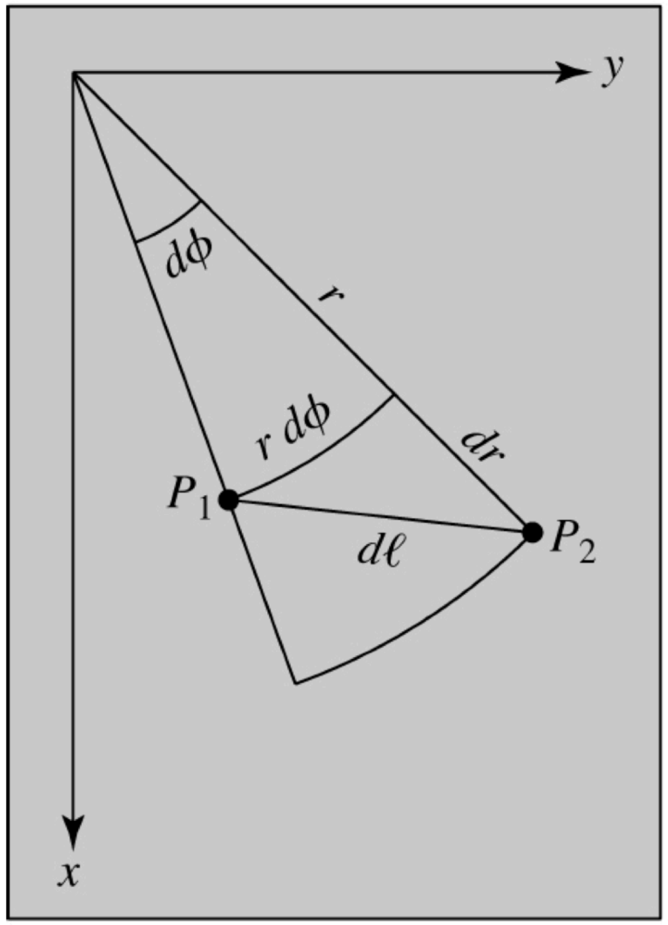
Non-Euclid

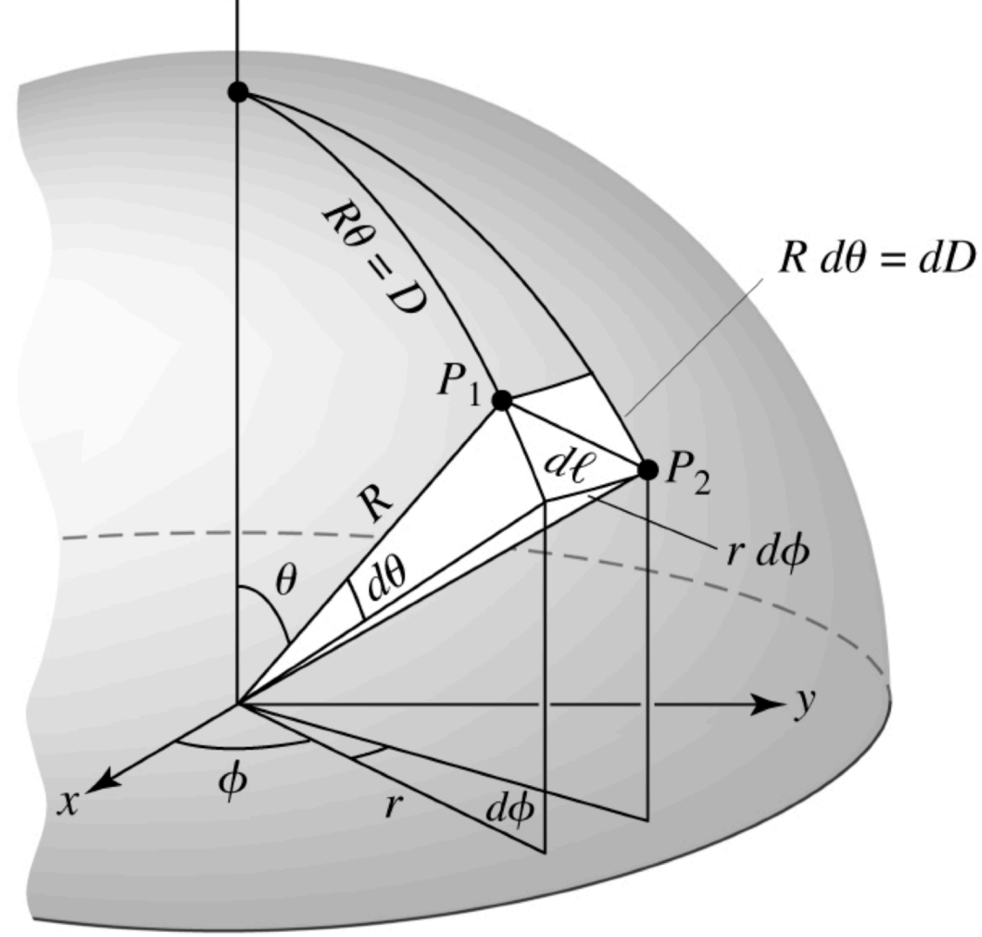




Differential Distance in a Positively Curved 2D Space







- **On a positively curved 2D surface, the natural polar coordinates are (D, ϕ). In these coordinates, the distance between P1 and P2 is:** $dl^2 = dD^2 + [R\sin(D/R)d\phi]^2$
- **To simplify the second term, recast to a new polar radial coordinate:** $r = R\sin(D/R)$ **so that**

  $D = R\arcsin(r/R)$ **and** $dD = \dfrac{1}{\sqrt{1 - r^2/R^2}}dr$
- **After the substitution:**

$$dl^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\phi^2$$

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- **After the substitution:**

$$dl^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\phi^2$$

- **On a negatively curved 2D surface, the distance between P1 and P2:** $dl^2 = dD^2 - R^2\sin^2(D/\sqrt{-R^2})d\phi^2 = dD^2 + R^2\sinh^2(D/R)d\phi^2$
- **To simplify the second term, recast to a new polar radial coordinate:** $r = R\sinh(D/R)$ **so that**

  $D = R\sinh^{-1}(r/R)$ **and** $dD = \dfrac{1}{\sqrt{1 + r^2/R^2}}dr$
- **After the substitution:**

$$dl^2 = \frac{dr^2}{1 + r^2/R^2} + r^2 d\phi^2$$

Differential Distance in a Flat & Curved 3D Spaces: Spherical Coordinates

Flat 3D Space

$$(dl)^2 = (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2$$

Flat 2D Space

$$(dl)^2 = (dr)^2 + (rd\theta)^2$$

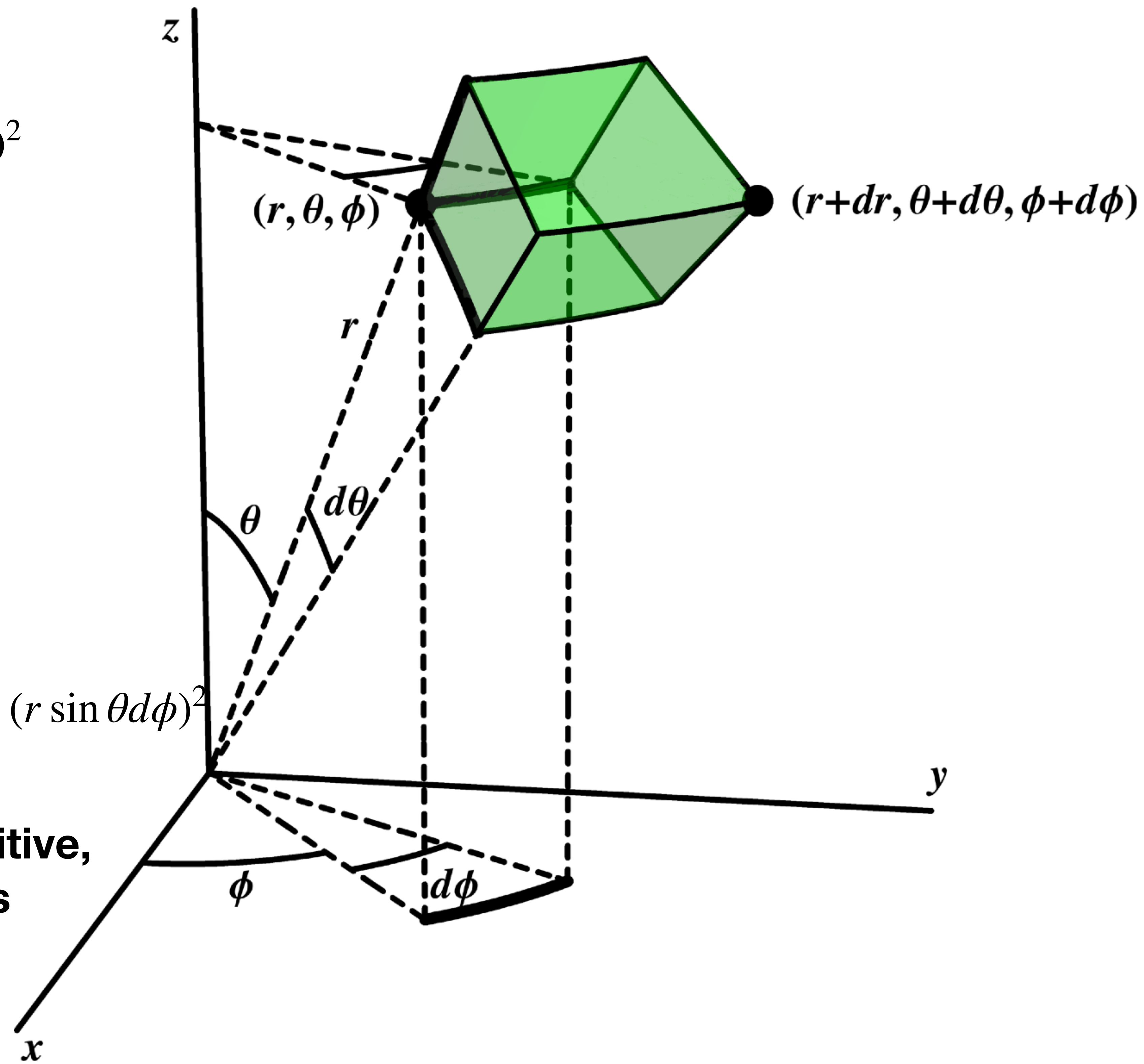
Curved 2D Space

$$(dl)^2 = \frac{dr^2}{1-\kappa r^2/R(t)^2} + (rd\theta)^2$$

Curved 3D Space

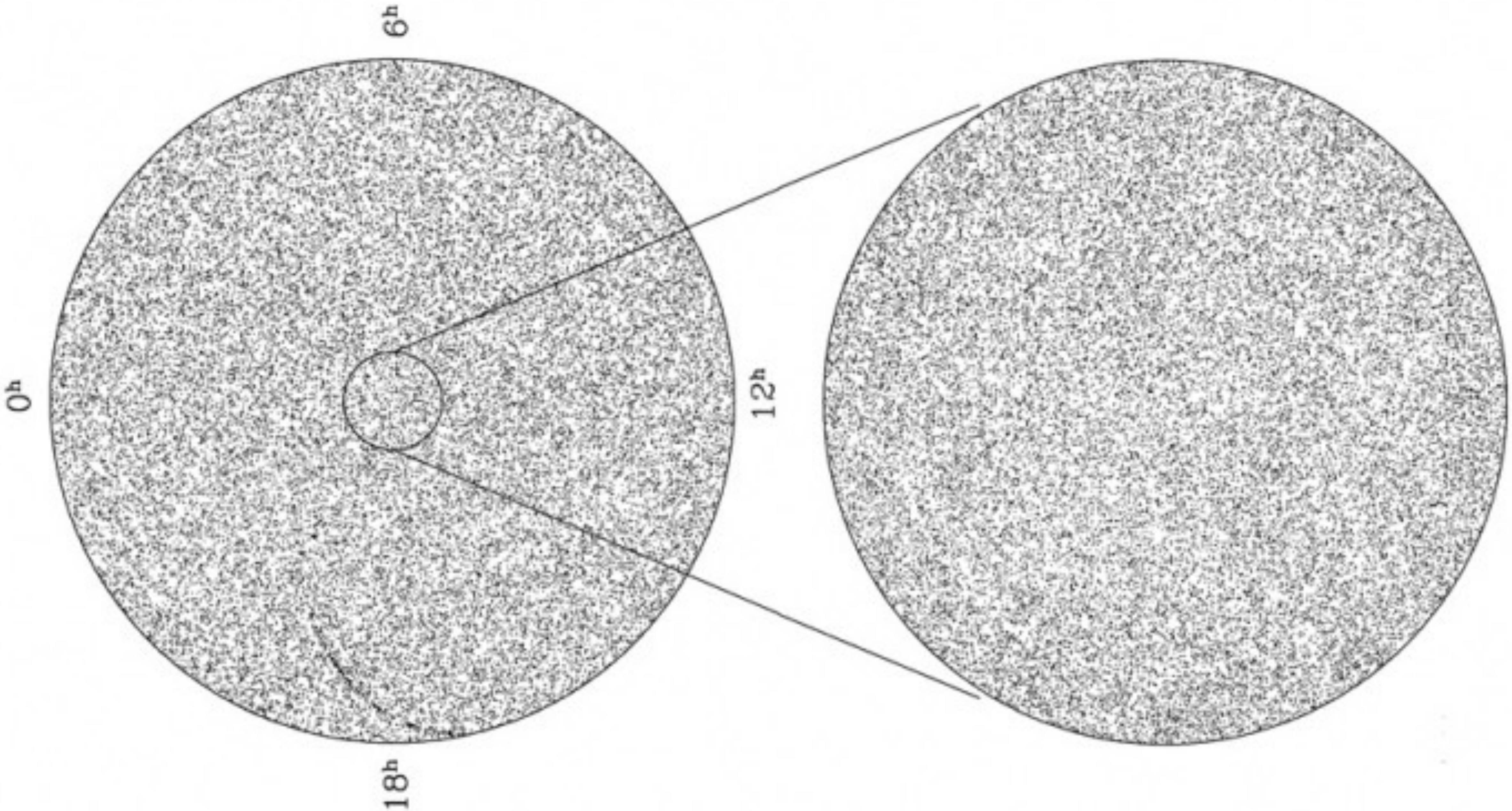
$$(dl)^2 = (\frac{dr^2}{1-\kappa r^2/R(t)^2}) + (rd\theta)^2 + (r\sin\theta d\phi)^2$$

where $\kappa = +1, 0, -1$ **for positive, zero(flat), negative curvatures**



Fundamental Assumption: The Cosmological Principle

- Although galaxies tend to clump, on the largest cosmic scales, the Universe is both **homogeneous** and **isotropic**
  - **Homogeneous:** there is no preferred **location** in the Universe
  - **Isotropic**: there is no preferred **direction** in the Universe



Differential Distance in a Uniformly Expanding, Curved, 3D Space

Distance in Curved 3D Space:

$$(dl)^2 = (\frac{dr^2}{1 - \kappa r^2/R(t)^2}) + (rd\theta)^2 + (r \sin\theta d\phi)^2$$

- For **uniform expansion**, define a time-dependent scale factor for the entire universe: ***R_U(t)*** or ***a(t)*** is the **scale factor at time t**
- Then define **comoving** coordinates that are time-independent:
  - ***x*** is the **comoving radial distance**, $x \equiv r(t)/a(t)$,
  - ***R*** is the **comoving radius of the curvature:** $R \equiv R(t)/a(t)$
  - ***k*** is the **comoving curvature**, $k \equiv \dfrac{\kappa}{R^2}$,

    where $\kappa = +1, 0, -1$ for positive, flat, and negative curvatures

- Plugging these to the distance equation at the top, we obtain the differential distance in a uniformly expanding, curved 3D space:

$$(dl)^2 = R_U(t)^2 \left(\frac{dx^2}{1 - kx^2} + (xd\theta)^2 + (x \sin\theta d\phi)^2 \right)$$

Robertson-Walker Metric: Differential Space-Time Distance

- In **General Relativity**, a **metric** is a function which measures ***differential space-time distance*** between two events**:**

$$(ds)^2 = (c \cdot dt)^2 - (dl)^2$$

- The **Robertson-Walker metric** is the metric that describes the geometry of a **homogeneous, isotropic, expanding** universe. The metric in *spherical coordinate system* is:

$$(ds)^2 = (c \cdot dt)^2 - R_U^2(t)\left[\left(\frac{dx}{\sqrt{1-kx^2}}\right)^2 + (xd\theta)^2 + (x\sin\theta d\phi)^2\right]$$

where

$\boldsymbol{R_U}$ is the **scale factor,** defined to be 1 at present day, and <1 in the past
$\boldsymbol{x}$ is the **comoving** radial distance, $x \equiv r(t)/R_U(t)$,
$\boldsymbol{k}$ is the **comoving curvature**, $k \equiv \frac{\kappa}{R^2}$, where $\kappa = +1,0,-1$ for positive, flat, and negative curvatures

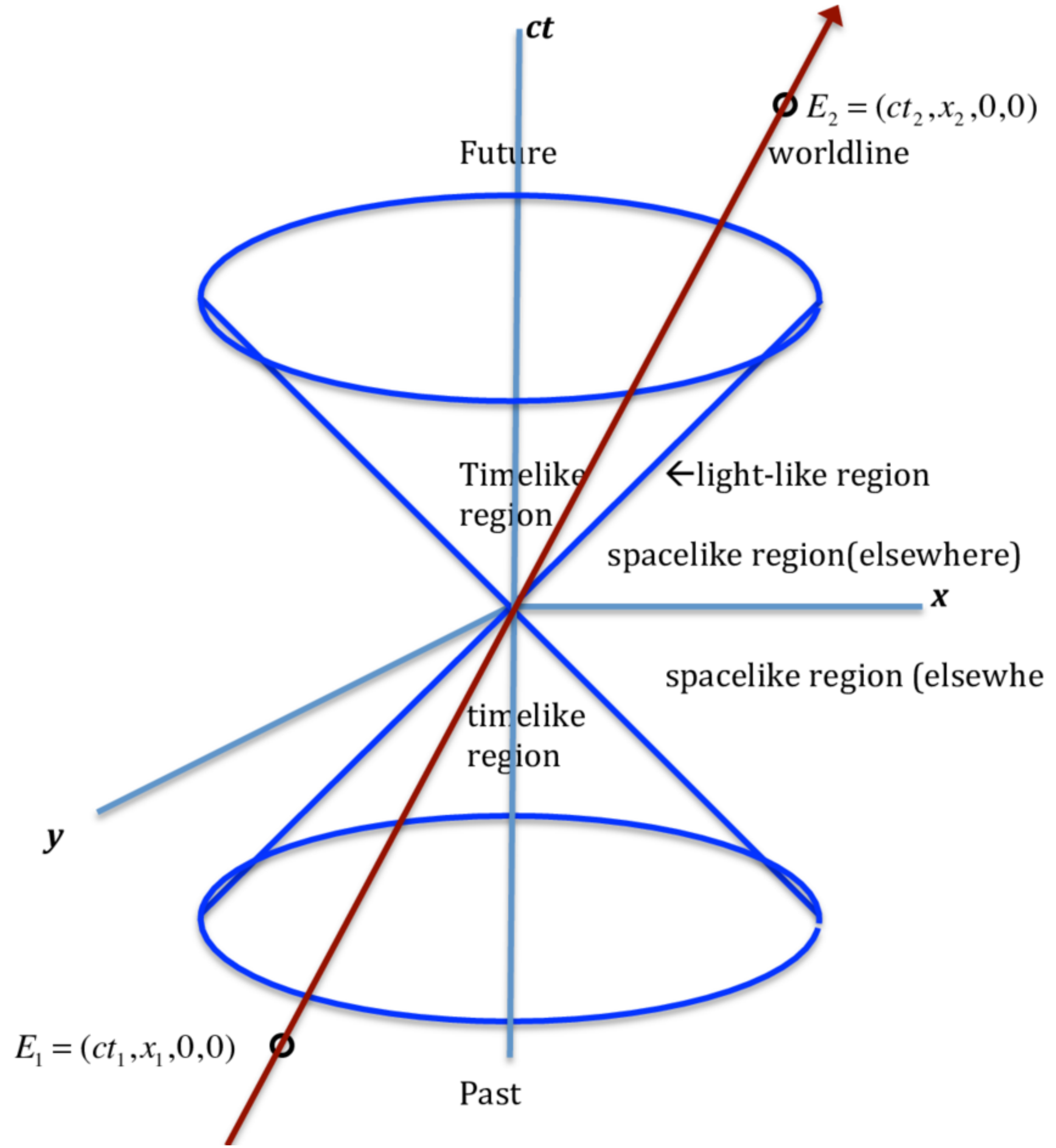
$\boldsymbol{R}$ is the **comoving** radius of the curvature.

Robertson-Walker Metric: Differential Space-Time Distance

- ***integral space-time distance***

$$\Delta s = \int_A^B \sqrt{(ds)^2}$$

- **Timelike**
  - $(\Delta s)^2 > 0$
  - events could be causally connected
  - e.g., **proper time** measured by a clock moving in a spaceship.
- **Spacelike**
  - $(\Delta s)^2 < 0$
  - events cannot be causally connected
  - e.g., **proper distance** between two locations measured simultaneously
- **Lightlike:** $(\Delta s)^2 = 0$



Robertson-Walker Metric
-> Cosmological Redshift

Redshift—Scale-factor Relation from the Robertson-Walker Metric

- Photons travel along **null geodesics** ($ds = 0$, i.e., proper time is frozen)**.** Along the **radial direction** towards the observer ($d\theta = d\phi = 0, dx < 0$), we have:

$$\frac{-dx}{\sqrt{1-kx^2}} = \frac{c\ dt}{R_U(t)}$$

- Follow the path of two adjacently emitted photons (separated by one wavelength: $\delta t_e = \lambda_e / c,\ \delta t_o = \lambda_o / c$) by integrating from the ***emitter (e)*** to the ***observer (o)***:

$$\int_{x_e}^{x_o} \frac{-dx}{\sqrt{1-kx^2}} = \int_{t_e}^{t_o} \frac{cdt}{R_u(t)}$$

$$\int_{x_e}^{x_o} \frac{-dx}{\sqrt{1-kx^2}} = \int_{t_e+\delta t_e}^{t_o+\delta t_o} \frac{cdt}{R_u(t)} = \int_{t_e}^{t_o} \frac{cdt}{R_u(t)} + \int_{t_o}^{t_o+\delta t_o} \frac{cdt}{R_u(t)} - \int_{t_e}^{t_e+\delta t_e} \frac{cdt}{R_u(t)}$$

combining the two gives us the **redshift—scale-factor relation** (true for all k)

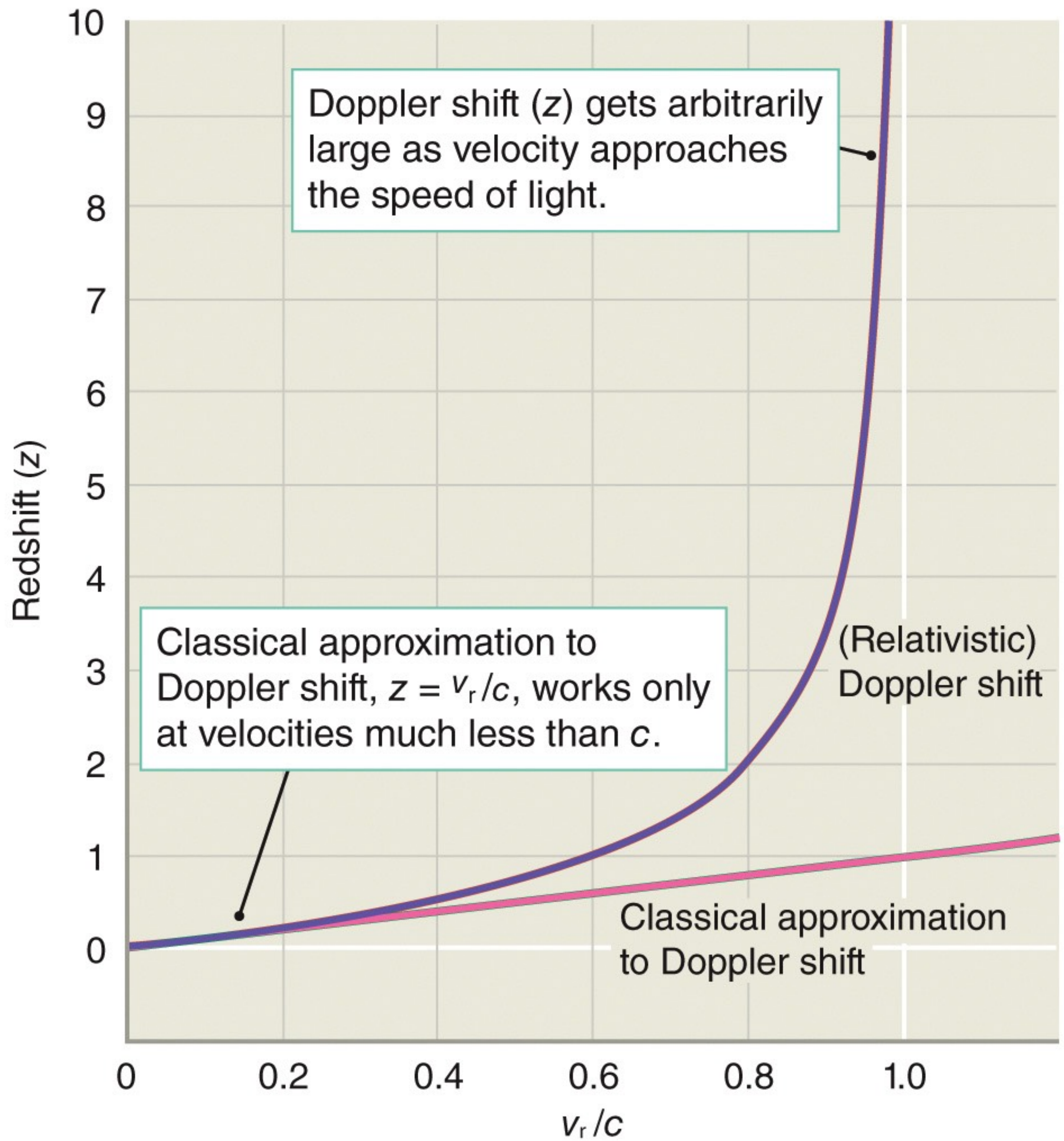
$$\Rightarrow \frac{c\delta t_o}{R_U(t_o)} = \frac{c\delta t_e}{R_U(t_e)} \Rightarrow R_U(t_e) = \frac{\delta t_e}{\delta t_o} = \frac{\lambda_e}{\lambda_o} = \frac{1}{1+z}$$

Cosmological redshift of photons emitted from a distant galaxy is caused by the increasing **scale factor** of the universe (R_u):

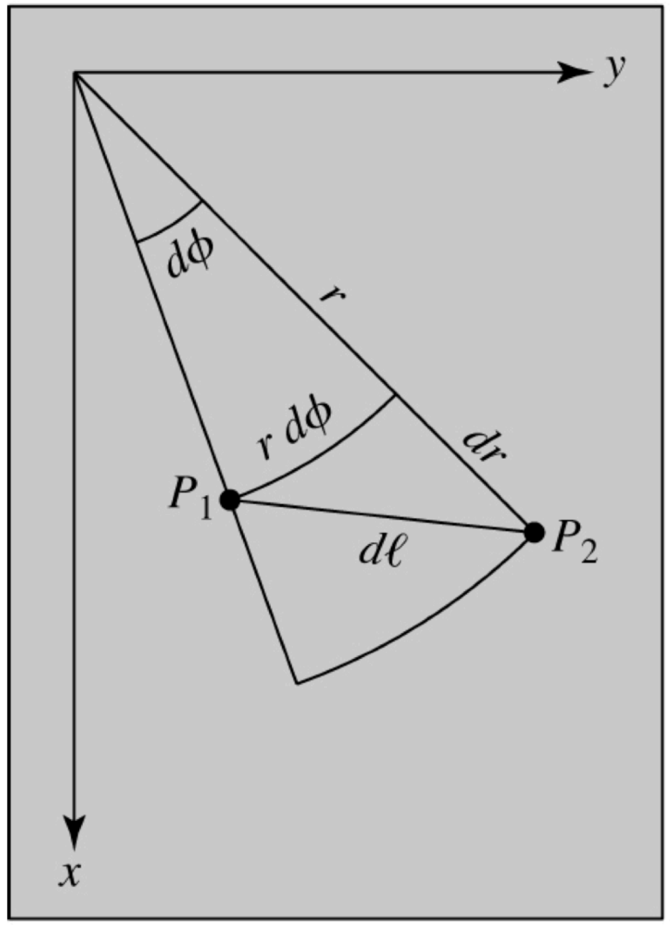
$$\frac{R_U(z)}{R_U(0)} = R_U(z) = \frac{1}{1+z} = \frac{\lambda_{\text{rest}}}{\lambda_{\text{observed}}} = \frac{\lambda(z)}{\lambda(0)}$$

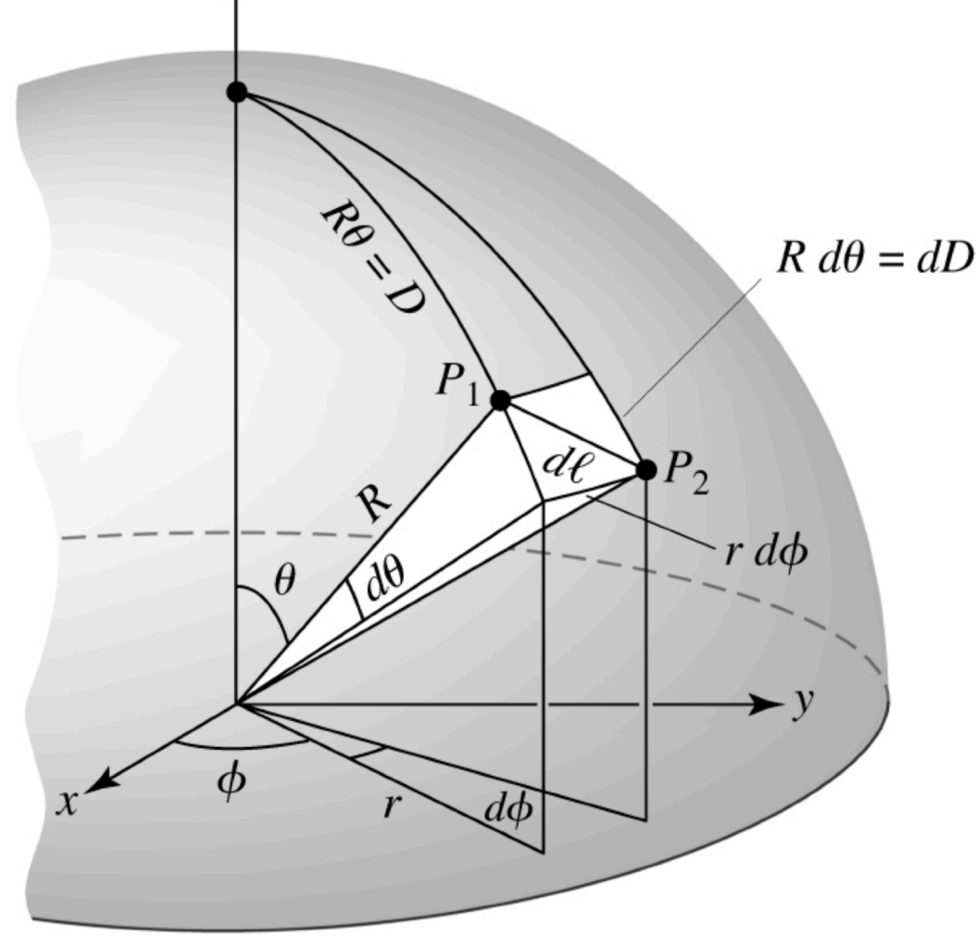
What is redshift? Classical vs. Relativistic Doppler Shift, and Scale Factors

- The **classical Doppler shift formula,**
  $$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_0} = 1 + \frac{v_r}{c}$$
  gives recession velocity:
  $$v_r = cz$$
- The **relativistic Doppler shift formula,**
  $$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_0} = \sqrt{\frac{1 + v_r/c}{1 - v_r/c}}$$
  gives recession velocity:
  $$v_r = c\frac{(1+z)^2 - 1}{(1+z)^2 + 1}$$
- But, **cosmological redshift** should be understood as a ratio of **scale factors**:
  $$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_0} = \frac{1}{R_U(z)}$$



Recap of Previous Lecture





$$(d\ell)^2 = (dD)^2 + (r\,d\phi)^2 = (R\,d\theta)^2 + (r\,d\phi)^2$$

$$r = R\,\sin\theta, \text{ so } dr = R\cos\theta\,d\theta$$

$$R\,d\theta = \frac{dr}{\cos\theta} = \frac{R\,dr}{\sqrt{R^2 - r^2}} = \frac{dr}{\sqrt{1 - r^2/R^2}}$$

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + (r\,d\phi)^2$$

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$\boldsymbol{k}$ is the **comoving** curvature, $k \equiv \frac{1}{R^2} = \frac{R_U^2}{R^2 R_U^2} = K(t)R_U^2(t)$,
$\boldsymbol{R}$ is the **comoving** radius of the curvature.

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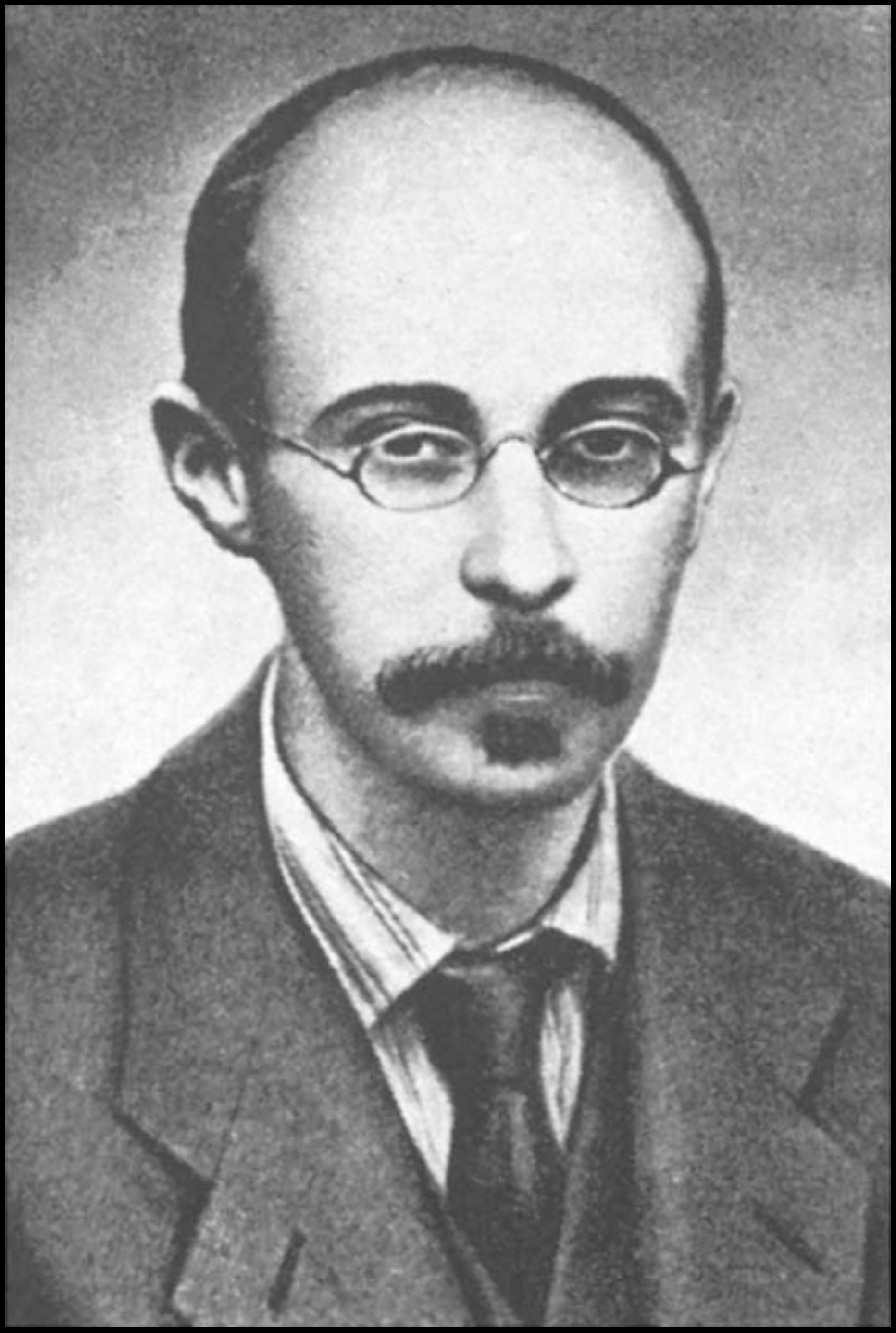
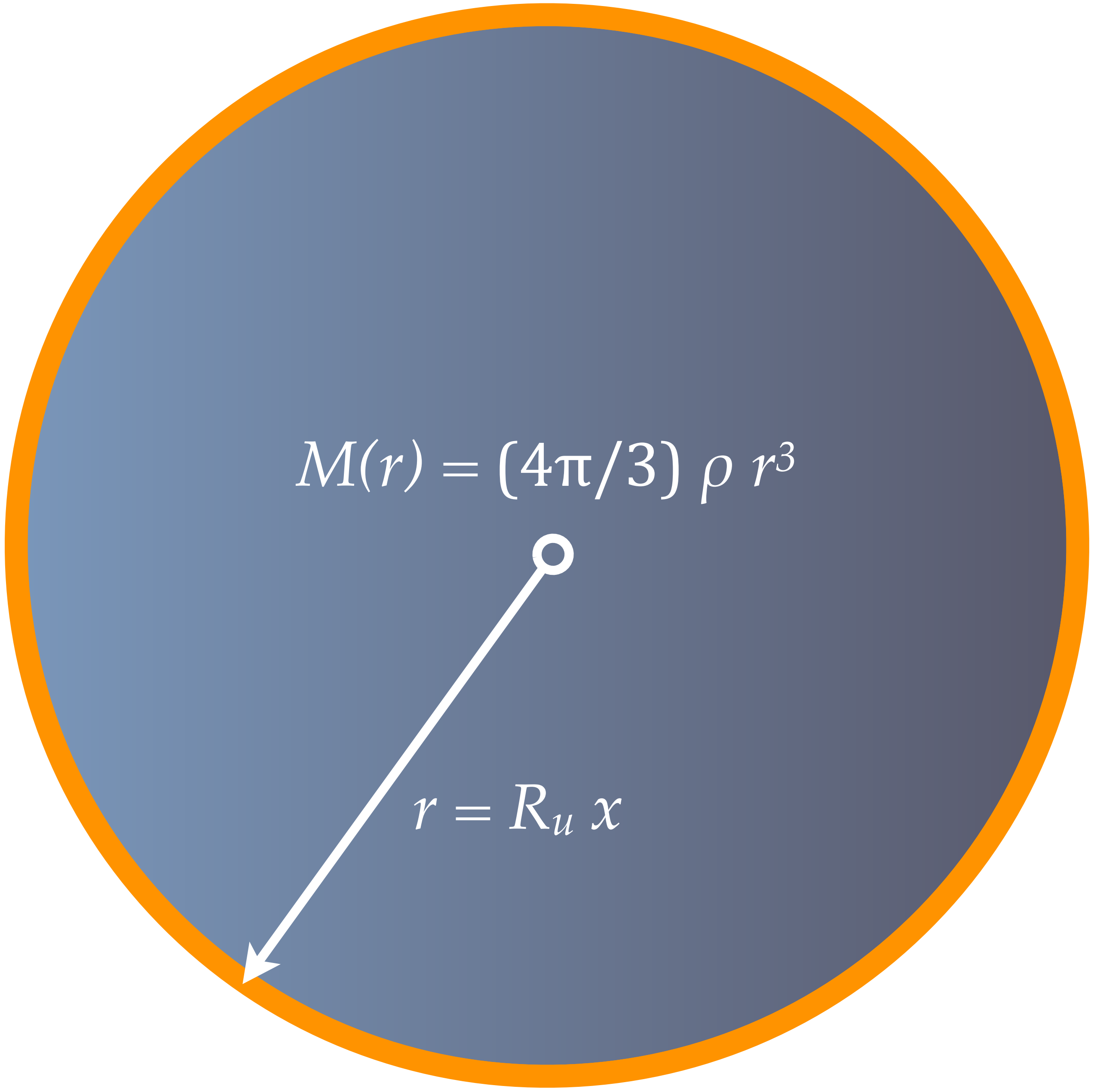
The 1st Friedmann's Equation (Newtonian Derivation)



Alexander Friedmann (1888 – 1925) was a Russian physicist and mathematician. Fought in WWI as an aviator. Died at age 37 from typhoid fever.

Newtonian Derivation based on Energy Conservation

- Imagine a **spherical shell** with **unit mass** in a **matter-only universe** with a **comoving radius** of **x**, as the universe expands:
  - its **physical radius** at time t is $r(t) = R_U(t)\, x$,
  - its **expanding velocity** is $v(t) = \dot{R}_U(t) \cdot x$,
  - the **mass enclosed** in the shell is $M(r) = \frac{4\pi}{3}[R_U(t)x]^3 \cdot \rho(t)$



Newtonian Derivation based on Energy Conservation

- Imagine a **spherical shell** with **unit mass** in a **matter-only universe** with a **comoving radius** of x, as the universe expands:
  - its **physical radius** at time *t* is $r(t) = R_U(t)\,x$,
  - its **expanding velocity** is $v(t) = \dot{R}_U(t) \cdot x$, and
  - the **mass enclosed** in the shell is $M(r) = \frac{4\pi}{3}[R_U(t)x]^3 \cdot \rho(t)$
- We can write down the **kinetic + gravitational potential energy** for the unit-mass spherical shell:

$$E = \frac{1}{2}v^2 - \frac{GM(r)}{r} = \frac{1}{2}\dot{R}_U(t)^2x^2 - \frac{4\pi}{3}G\rho(t)R_U(t)^2x^2$$

- This **energy per unit mass** must be the same for every shell with the same comoving radius **x**, so we can define *E* with a *k* parameter:

$$E \equiv -\frac{1}{2}kc^2x^2$$

- Combining the two Eqs. and cancel out x^2 on both sides, we obtain:

$$\left(\frac{\dot{R}_U^{\,2}}{R_U^2} - \frac{8}{3}\pi G\rho\right) R_U^2 = -kc^2$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi}{3}G\rho}$$

The Cosmological Constant

Friedmann's Equations derived from Einstein Field Equations

The equations connect the geometry of space-time with the local distribution of mass, energy, and momentum (pressure).

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} \mathcal{T}_{\alpha\beta}$$

where $R_{\alpha\beta}$ is Ricci tensor,

R the Ricci scaler (that describes the curvature),

$g_{\alpha\beta}$ is the metric tensor, and

$T_{\alpha\beta}$ is the stress-energy tensor

Original Friedmann's Equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi}{3} G\rho$$

$$2\,\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi}{c^2} Gp$$

Einstein's "Mistake": the Cosmological Constant (Dark Energy)

The universe must be static, right? But my field equation doesn't allow that, because when the scale factor is constant, Friedmann's equations are **nonsense**:

$$\frac{kc^2}{a^2} = \frac{8\pi}{3} G\rho_0 = -\frac{8\pi}{c^2} Gp_0$$

Let me revise my equation so that to allow a static universe:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \boxed{- \Lambda g_{\alpha\beta}} = \frac{8\pi G}{c^4} \mathcal{T}_{\alpha\beta}$$

Original Friedmann's Equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi}{3} G\rho$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi}{c^2} Gp$$

Revised Friedmann's Equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi}{3} G\rho + \frac{\Lambda c^2}{3}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi}{c^2} Gp + \Lambda c^2$$

Lambda, if positive, increases energy density, decreases pressure

Friedmann’s Equations (GR Derivation)

Einstein Field Equations

The equations connect the geometry of space-time with the local distribution of mass, energy, and momentum (pressure).

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} \mathcal{T}_{\alpha\beta}$$

where $R_{\alpha\beta}$ is Ricci tensor,

R the Ricci scaler (that describes the curvature),

$g_{\alpha\beta}$ is the metric tensor, and

$T_{\alpha\beta}$ is the stress-energy tensor

Einstein later included a constant Λ in the field Equation (explained later):

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \boxed{- \Lambda g_{\alpha\beta}} = \frac{8\pi G}{c^4} \mathcal{T}_{\alpha\beta}$$

The *t-t* and *r-r* components of the stress-energy tensor and their corresponding Ricci tensor and scaler resulted in two **Friedmann's Equations** (first derived in 1922, seven years before Hubble's discovery of an expanding universe and over 10 years before RW metric)

The Metric Tensor $g_{\alpha\beta}$: Isotropic & Homogeneous Universe

For such a universe, its geometry is described by the **Robertson-Walker Metric**,

$$(ds)^2 = (c \cdot dt)^2 - a^2(t)\left[\left(\frac{dr}{\sqrt{1-kr^2}}\right)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2\right]$$

In tensor notation, the metric equation is:

$$(ds)^2 = g_{\alpha\beta} dx_\alpha dx_\beta$$

The **metric tensor**'s diagonal components are (*c* is absorbed by *t*):

$$g_{00} = 1, \qquad g_{11} = -\frac{a^2}{1-kr^2},$$

$$g_{22} = -a^2 r^2, \qquad g_{33} = -a^2 r^2 \sin^2\theta$$

The **metric tensor**'s non-diagonal components are all zero.

Energy-Momentum Tensor $T_{\alpha\beta}$: Perfect Fluid in Thermodymamic Equilibrium

$$T_{\alpha\beta} = (\rho + \frac{p}{c^2})u_\alpha u_\beta - p g_{\alpha\beta}$$

where $\rho, p, u_\alpha, g_{\alpha\beta}$ are density, pressure, macroscopic speed, and metric tensor.

In the **comoving reference frame** of a perfect fluid, the macroscopic speed has only the temporal component:

$$(u_0, u_1, u_2, u_3) = (c,0,0,0)$$

Given the **metric tensor** from RW metric: $g_{00} = 1$

One obtains:

$$T_{00} = \rho c^2, T_{ii} = -p g_{ii}$$

For example, the **r-r** component is:

$$\mathcal{T}_{11} = \frac{pa^2}{1 - kr^2}$$

What Additional Terms are in the Einstein Field Equations?

The equations connect the geometry of space-time with the local distribution of mass, energy, and momentum (pressure).

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} \mathcal{T}_{\alpha\beta}$$

where $R_{\alpha\beta}$ is Ricci tensor,

R the Ricci scaler,

$g_{\alpha\beta}$ is the metric tensor, and

$T_{\alpha\beta}$ is the stress-energy tensor

The Ricci Tensor & Scaler: Step 1 - Christoffel Symbols

Calculate the Christoffel Symbols

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2} g^{\beta\alpha} \left\{ \frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$$

from the metric tensor of the RW metric:

$$g_{00} = 1, \qquad g_{11} = -\frac{a^2}{1 - kr^2},$$

$$g_{22} = -a^2\, r^2, \qquad g_{33} = -a^2\, r^2\, \sin^2\theta$$

The Ricci Tensor & Scaler: Step 1 - Christoffel Symbols

Calculate the Christoffel Symbols $\Gamma^{\beta}_{\mu\nu} = \frac{1}{2} g^{\beta\alpha} \left\{ \frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$

- $\Gamma^{t}_{rr} = \frac{a\dot{a}}{1-\frac{r^2}{K^2}}$
- $\Gamma^{t}_{\theta\theta} = r^2 a\dot{a}$
- $\Gamma^{t}_{\varphi\varphi} = r^2 a\dot{a} \sin^2\theta$
- $\Gamma^{r}_{tr} = \Gamma^{r}_{rt} = \Gamma^{\theta}_{t\theta} = \Gamma^{\theta}_{\theta t} = \Gamma^{\varphi}_{t\varphi} = \Gamma^{\varphi}_{\varphi t} = \frac{\dot{a}}{a}$
- $\Gamma^{r}_{rr} = \frac{r}{K^2(1-\frac{r^2}{K^2})}$
- $\Gamma^{r}_{\theta\theta} = -r(1 - \frac{r^2}{K^2})$
- $\Gamma^{r}_{\varphi\varphi} = -r(1 - \frac{r^2}{K^2}) \sin^2\theta$
- $\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \frac{1}{r}$
- $\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta$
- $\Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = \frac{1}{\tan\theta}$

Ricci Tensor & Scaler: Step 2 - Ricci Tensor from Christoffel Symbols

Calculate the components of the Ricci Tensor

$$R_{\mu\nu} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}$$

- $R_{tt} = R^m_{tmt} = R^r_{trt} + R^\theta_{t\theta t} + R^\varphi_{t\varphi t} = -3\frac{\ddot{a}}{a}$
- $R_{rr} = R^m_{rmr} = \frac{a\ddot{a}}{1-\frac{r^2}{K^2}} + \frac{2\dot{a}^2}{1-\frac{r^2}{K^2}} + \frac{2}{K^2(1-\frac{r^2}{K^2})}$
- $R_{\theta\theta} = R^m_{\theta m\theta} = r^2 a\ddot{a} + 2r^2\dot{a}^2 + 2\frac{r^2}{K^2}$
- $R_{\varphi\varphi} = R^m_{\varphi m\varphi} = r^2 a\ddot{a}\sin^2\theta + 2r^2\dot{a}^2\sin^2\theta + 2\frac{r^2}{K^2}\sin^2\theta$

Ricci Tensor & Scaler: Step 3 - Ricci Scaler from Ricci Tensor

Finally we can get the Ricci scalar:

$$R = g^{ik} R_{ik} = -6\frac{\ddot{a}}{a} - 6\left(\frac{\dot{a}}{a}\right)^2 - 6\frac{1}{K^2 a^2}$$

Ricci Tensor & Scaler: Summary

converted to our notation system by taking ***c*** out of the time derivatives, and $k = K^{-2}$:

Results that absorbed **c** in **t**:

$$R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a}$$

$$-3\frac{\ddot{a}}{a}$$

$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2kc^2}{c^2(1 - kr^2)}$$

$$\frac{a\ddot{a}}{1-\frac{r^2}{K^2}} + \frac{2\dot{a}^2}{1-\frac{r^2}{K^2}} + \frac{2}{K^2(1-\frac{r^2}{K^2})}$$

$$R = -\frac{6}{c^2}\frac{\ddot{a}}{a} - \frac{6}{c^2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{6k}{a^2}$$

$$-6\frac{\ddot{a}}{a} - 6\left(\frac{\dot{a}}{a}\right)^2 - 6\frac{1}{K^2a^2}$$

The 1st Friedmann Equation: Velocity

Einstein's Equations with the Cosmological Constant:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \boxed{- \Lambda g_{\alpha\beta}} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

Write down the time-time components of the tensors

$$R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a} \qquad g_{00} = 1 \qquad T_{00} = \rho c^2$$

$$R = -\frac{6}{c^2}\frac{\ddot{a}}{a} - \frac{6}{c^2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{6k}{a^2}$$

Plug in and simplify:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}$$

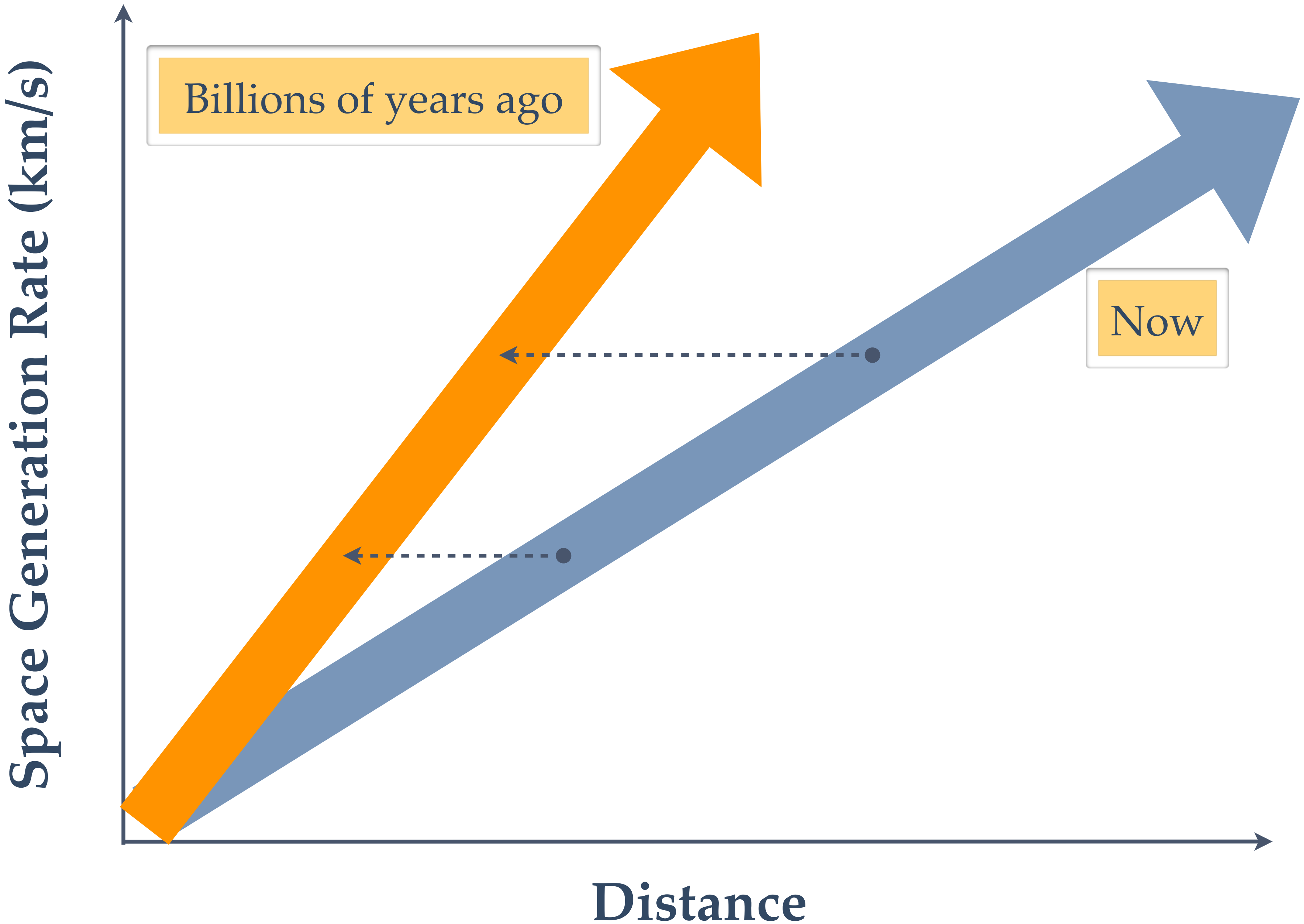
About the Hubble Parameter H(z)

- The **Hubble parameter** is defined as

$$H(t) = \dot{R}_U(t)/R_U(t)$$

- Unlike **Hubble constant**, it changes with time. For the simplest case of linearly expansion, the Hubble parameter is inversely proportional to the scale factor (as illustrated below).
- At $t = t_0$ (today), this definition gives **Hubble's Law**:

$$\dot{R}_U(t_0) = H(t_0)R_U(t_0) \Rightarrow \dot{D} = H_0 D \text{ where } D = R_U x$$



The 2nd Friedmann Equation: Acceleration

Einstein's Equations with a Cosmological Constant:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R \boxed{- \Lambda g_{\alpha\beta}} = \frac{8\pi G}{c^4}\mathcal{T}_{\alpha\beta} \qquad g_{11} = -\frac{a^2}{1-kr^2}$$

Write down the space-space components of the tensors:

$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2kc^2}{c^2(1-kr^2)} = \frac{-g_{11}}{a^2c^2}(a\ddot{a} + 2\dot{a}^2 + 2kc^2)$$

$$T_{11} = \frac{pa^2}{1-kr^2} = -pg_{11} \qquad R = -\frac{6}{c^2}\frac{\ddot{a}}{a} - \frac{6}{c^2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{6k}{a^2}$$

Plug in, cancel out g11, and simplify:

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}p + \Lambda c^2$$

The 2nd Friedmann Equation from other Spatial Dimensions

Einstein's Equations with a Cosmological Constant:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \boxed{- \Lambda g_{\alpha\beta}} = \frac{8\pi G}{c^4} \mathcal{T}_{\alpha\beta} \qquad g_{22} = -a^2 r^2$$

Write down the space-space components of the tensors:

$$R_{22} = \frac{1}{c^2}(r^2 a\ddot{a} + 2r^2\dot{a}^2 + 2kc^2r^2) = \frac{-g_{22}}{a^2c^2}(a\ddot{a} + 2\dot{a}^2 + 2kc^2)$$

$$T_{22} = -p g_{22} \qquad R = -\frac{6}{c^2}\frac{\ddot{a}}{a} - \frac{6}{c^2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{6k}{a^2}$$

Plug in, cancel out g22, and simplify, one obtains the same equation:

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}p + \Lambda c^2$$

Friedmann's Equations Combined

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}p + \Lambda c^2$$

Plug the first equation into the second one, we get the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\frac{3p}{c^2} + \rho\right) + \frac{\Lambda c^2}{3}$$

Solving the Friedmann Equations:

Prerequisite 1: Boundary Conditions

The 1st Friedmann Equation expressed in Density Parameters

- The **first** Friedmann Equation from GR is:

$$\left(\frac{\dot{R}_U}{R_U}\right)^2 + \frac{kc^2}{R_U^2} = \frac{8\pi G}{3}(\rho_m + \rho_\gamma) + \frac{\Lambda c^2}{3}$$

  which can be rearranged as:

$$R_U^2 H^2 \left[1 - \left(\frac{\rho_m}{\rho_c} + \frac{\rho_\gamma}{\rho_c} + \frac{\Lambda c^2}{8\pi G \rho_c}\right)\right] = -\,kc^2$$

  where we used the **critical density** and the **Hubble parameter**:

$$\rho_c = \frac{3H^2}{8\pi G} = 2.8 \times 10^{11} h^2 M_\odot/\mathrm{Mpc}^3,\ H \equiv \frac{\dot{R}_U}{R_U} = 100\, h\, \mathrm{km\, s^{-1}\, Mpc^{-1}}$$

- Now define the **dimensionless densities**:
  - $\Omega_m \equiv \rho_m/\rho_c$, **ordinary matter** (baryons and dark matter)
  - $\Omega_\gamma \equiv \rho_\gamma/\rho_c$, **relativistic matter** (light and neutrinos)
  - $\Omega_\Lambda \equiv \Lambda c^2/(8\pi G \rho_c)$, **dark energy** ($\Lambda$ is the cosmological constant and has the same physical unit as the curvature constant k), and $\rho_\Lambda \equiv \Lambda c^2/(8\pi G)$
  - $\Lambda = 3\Omega_{\Lambda,0} H_0^2/c^2 = 1.14 \times 10^{-7} \mathrm{Mpc}^{-2} = 1.2 \times 10^{-52} \mathrm{m}^{-2}$

- Replacing those, we obtain a new form for the **Friedmann Equation (FE1)**:

$$R_U^2 H^2\ [1 - (\Omega_m + \Omega_\gamma + \Omega_\Lambda)] = -\,kc^2$$

Critical Density, Hubble Parameter, and Cosmological Constant

- The **critical density**:

$$\rho_c = \frac{3H^2}{8\pi G} = 2.78 \times 10^{11} h^2 M_\odot/\mathrm{Mpc}^3 = 1.88 \times 10^{-26} h^2 \mathrm{kg/m}^3,$$

which is equivalent to a hydrogen density of

$$n_{H,c} = \rho_c/(4m_p/3) = 8.43 h^2 m^{-3}$$

- The **Hubble parameter:** $H \equiv \frac{\dot{R}_U}{R_U} = 100\, h\, \mathrm{km\, s^{-1}\, Mpc^{-1}}$

- The **cosmological constant**:

$$\Lambda = \frac{3\Omega_{\Lambda,0} H_0^2}{c^2} = 1.14 \times 10^{-7} \mathrm{Mpc}^{-2} = 1.2 \times 10^{-52} \mathrm{m}^{-2} \text{ for } \Omega_{\Lambda,0} = h = 0.7$$

which corresponds to a comoving curvature radius of

$$R_\Lambda = \sqrt{1/\Lambda} = 2961.7\, \mathrm{Mpc}$$

Boundary Condition: Today's Universe

- The **Friedmann Equation (FE1)**:

$$R_U^2 H^2 \; [1 - (\Omega_m + \Omega_\gamma + \Omega_\Lambda)] = -\, kc^2$$

- Boundary condition at t = t_0 gives the value of the curvature:

$$H_0^2(1 - \Omega_0) = -\, kc^2, \text{ or}$$

$$\Omega_{k,0} \equiv (1 - \Omega_0) = \frac{-kc^2}{H_0^2}$$

- The 1st Friedmann Equation with the boundary condition:

$$R_U^2 H^2 \; [1 - (\Omega_m + \Omega_\gamma + \Omega_\Lambda)] = -\, kc^2 = H_0^2(1 - \Omega_0)$$

- which can be rearranged as:

$$E^2 \equiv H^2/H_0^2 = R_U^{-2}(1 - \Omega_0)/[1 - (\Omega_m + \Omega_\gamma + \Omega_\Lambda)]$$

$$R_U = (1 + z)^{-1}$$

Recap of Previous Lecture

The 1st Friedmann Equation: Velocity

Einstein's Equations with the Cosmological Constant:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \boxed{- \Lambda g_{\alpha\beta}} = \frac{8\pi G}{c^4} \mathcal{T}_{\alpha\beta}$$

Write down the time-time components of the tensors

$$R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a} \qquad g_{00} = 1 \qquad T_{00} = \rho c^2$$

$$R = -\frac{6}{c^2}\frac{\ddot{a}}{a} - \frac{6}{c^2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{6k}{a^2}$$

Plug in and simplify:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}$$

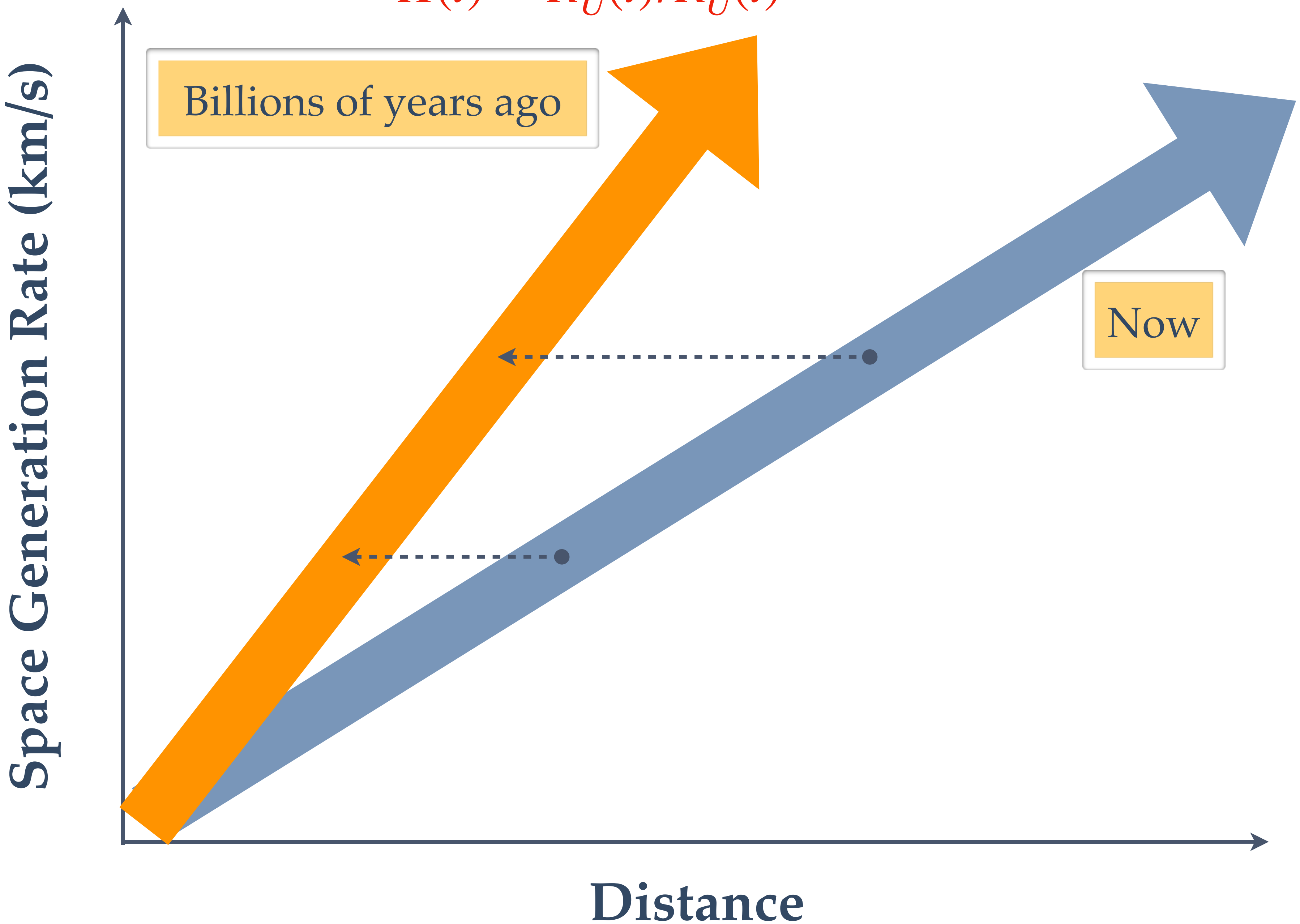
About the Hubble Parameter H(z)

- **Hubble's Law**, expressed in scale factor, gives the meaning of **Hubble constant**:

$$\dot{D} = H_0 D \text{ where } D = R_U x \Rightarrow H(t_0) = \dot{R}_U(t_0)/R_U(t_0)$$

- But unlike **a constant**, this ratio changes with time, even for the simplest case of linear expansion, as illustrated below.
- **H_0** is thus the **present-day value** of the time-dependent **Hubble parameter:**

$$H(t) = \dot{R}_U(t)/R_U(t)$$



The 2nd Friedmann Equation: Acceleration

Einstein's Equations with a Cosmological Constant:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \boxed{- \Lambda g_{\alpha\beta}} = \frac{8\pi G}{c^4} \mathcal{T}_{\alpha\beta} \qquad g_{11} = -\frac{a^2}{1 - kr^2}$$

Write down the space-space components of the tensors:

$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2kc^2}{c^2(1 - kr^2)} = \frac{-g_{11}}{a^2c^2}(a\ddot{a} + 2\dot{a}^2 + 2kc^2)$$

$$T_{11} = \frac{pa^2}{1 - kr^2} = -pg_{11} \qquad R = -\frac{6}{c^2}\frac{\ddot{a}}{a} - \frac{6}{c^2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{6k}{a^2}$$

Plug in, cancel out g11, and simplify:

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}p + \Lambda c^2$$

The 2nd Friedmann Equation from other Spatial Dimensions

Einstein's Equations with a Cosmological Constant:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \boxed{- \Lambda g_{\alpha\beta}} = \frac{8\pi G}{c^4} \mathcal{T}_{\alpha\beta} \qquad g_{22} = -a^2 r^2$$

Write down the space-space components of the tensors:

$$R_{22} = \frac{1}{c^2}(r^2 a\ddot{a} + 2r^2\dot{a}^2 + 2kc^2r^2) = \frac{-g_{22}}{a^2c^2}(a\ddot{a} + 2\dot{a}^2 + 2kc^2)$$

$$T_{22} = -p g_{22} \qquad R = -\frac{6}{c^2}\frac{\ddot{a}}{a} - \frac{6}{c^2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{6k}{a^2}$$

Plug in, cancel out g22, and simplify, one obtains the same equation:

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2} p + \Lambda c^2$$

The 1st Friedmann Equation in Various Forms

Originally derived from the *tt*-component of the field equation:

$$\left(\frac{\dot{R}_U}{R_U}\right)^2 + \frac{kc^2}{R_U^2} = \frac{8\pi G}{3}(\rho_m + \rho_\gamma) + \frac{\Lambda c^2}{3}$$

After defining Hubble parameter, critical density, density of Cosmological constant, and dimensionless density parameters:

$$H \equiv \frac{\dot{R}_U}{R_U}, \quad \rho_c \equiv \frac{3H^2}{8\pi G}, \quad \rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G}$$

It can be rearranged as:

$$R_U^2 H^2\ [1 - (\Omega_m + \Omega_\gamma + \Omega_\Lambda)] = -\,kc^2$$

After plugging in the boundary condition at present:

$$R_U^2 H^2\ [1 - (\Omega_m + \Omega_\gamma + \Omega_\Lambda)] = H_0^2[1 - (\Omega_{m,0} + \Omega_{\gamma,0} + \Omega_{\Lambda,0})]$$

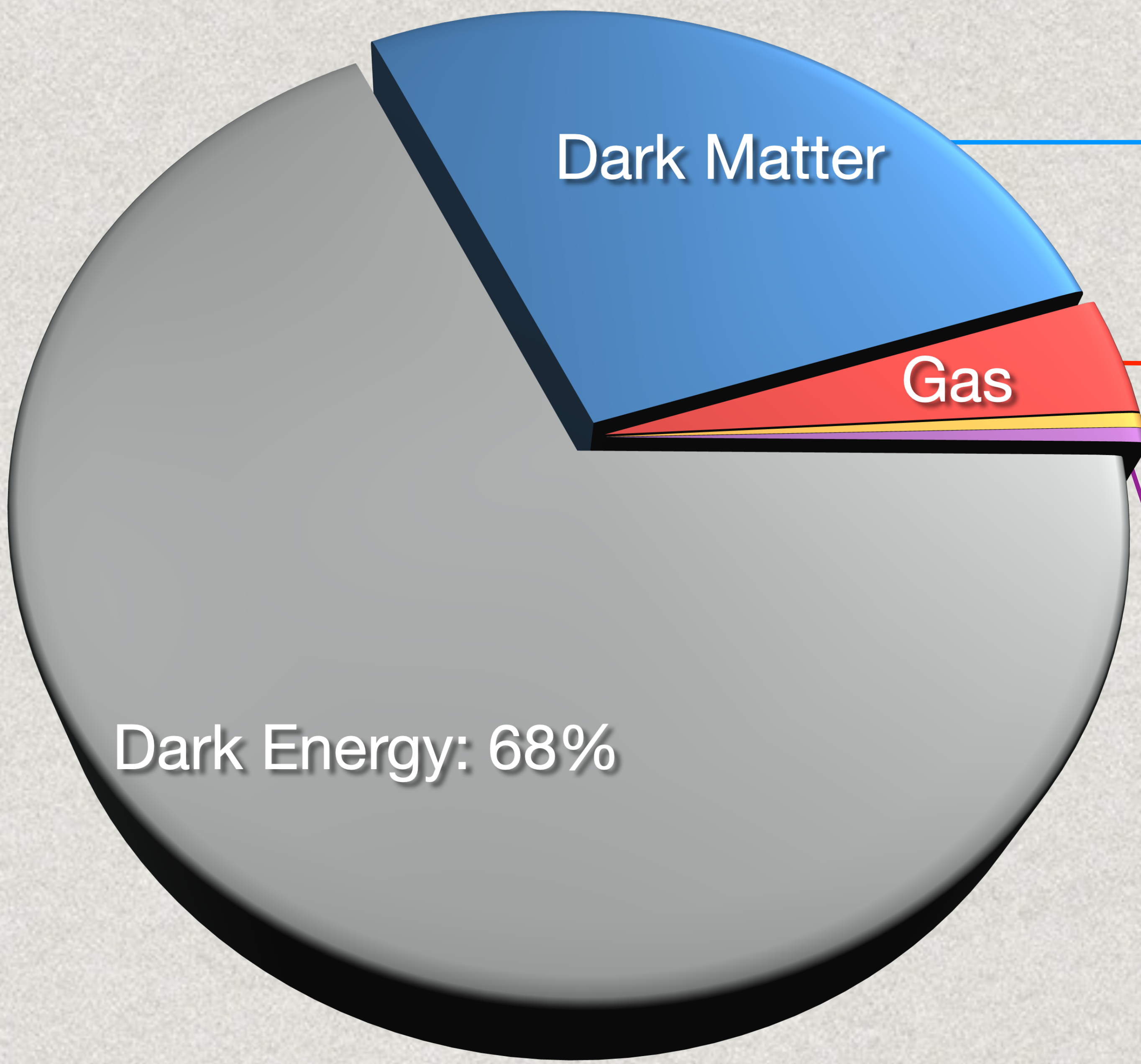
$\Omega_{m,0}, \Omega_{\gamma,0}, \Omega_{\Lambda,0}$

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}$$

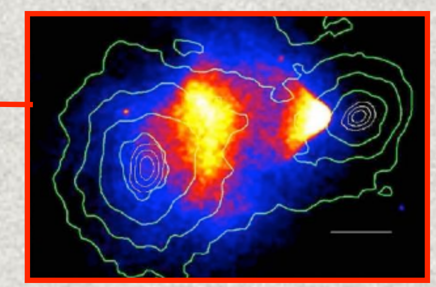
Critical density as a function of time. Value below is present value, based on present value of the Hubble parameter H

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = 9.47x10^{-27}\, kg / m^3$$

Planck Collaboration (2013)

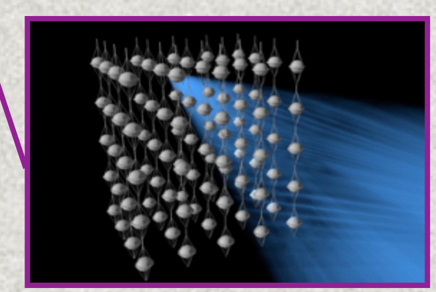


Dark matter: 27%



Gas: 4%

Galaxies: 0.5%



Neutrinos: <0.5%

Solving the Friedmann Equations:

Prerequisite 2: Pressure-Density relations & Density-Scale-factor relations

Equations of State for Matter, Radiation, and Dark Energy

The **equations of state** for different components of our universe are necessary to solve the Friedmann's equations by connecting pressure with energy density:

$$p = w\rho c^2$$

Friedmann Equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\frac{3p}{c^2} + \rho\right) + \frac{\Lambda c^2}{3}$$

Pressure from Matter is Negligible: w = 0

$$\left.\begin{array}{l} P = nkT \\ \rho c^2 = n \cdot m_H c^2 \end{array}\right\} \frac{kT}{m c^2}$$

$$\left\{\begin{array}{l} kT = 1.38 \times 10^{-22} \, J \left(T / 10 \, K \right) \\ m_H c^2 = 1.5 \times 10^{-10} \, J \end{array}\right.$$

Radiation Pressure vs. Radiation Energy Density: w = 1/3

$$P_{rad,\lambda} = \frac{1}{c}\int I_\lambda \cos^2\theta \, d\Omega = \frac{1}{c}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda \cos^2\theta \sin\theta \, d\theta \, d\phi$$

$$= \frac{2\pi}{3c}\left(I_{out} + I_{in}\right) = \frac{4\pi}{3c}\langle I_\lambda \rangle \quad \left[\int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta \, d\theta = -\frac{1}{3}\cos^3\theta \Big|_0^{\pi/2}\right]$$

Radiation Field

	general	isotropic
$\langle I_\nu \rangle$	$\frac{1}{4\pi}\int I_\nu \, d\Omega$	I_ν
F_ν	$\int I_\nu \cos\theta \, d\Omega$	~~πI_ν~~ 0
P_ν	$\frac{1}{c}\int I_\nu \cos^2\theta \, d\Omega$	$\frac{4\pi}{3c} I_\nu$
u_ν	$\frac{1}{c}\int I_\nu \, d\Omega$	$\frac{4\pi}{c} \cdot I_\nu$

The Cosmological Constant: w = -1

The Friedmann's Equations can be rearranged to reveal the equivalent density and pressure from the cosmological constant:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\left(\rho + \boxed{\frac{\Lambda c^2}{8\pi G}}\right)$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}p + \Lambda c^2 = -8\pi G\left(\frac{p}{c^2}\boxed{-\frac{\Lambda c^2}{8\pi G}}\right)$$

$$\rho_\Lambda = -\frac{p_\Lambda}{c^2} = \frac{\Lambda c^2}{8\pi G} \Rightarrow w = -1$$

Density-Scale-Factor Relations

Given the **first law of thermal dynamics,**

$$TdS = dQ = dE + pdV$$

assume **adiabatic condition** (dQ = 0) and express total energy as a product between density and volume:

$$dE = d(\rho c^2 V) = -pdV$$

Given **the equations of state** and $V \propto R_U^3(t)$, one can use the **adiabatic law** to derive **how energy density or pressure evolves with scale factor.**

$$d(\rho R_U^3) = -w\rho d(R_U^3) \Rightarrow \frac{d\rho}{\rho} = -3(1+w)\frac{dR_U}{R_U} \Rightarrow \rho \propto R_U^{-3(1+w)}$$

The resulting **density-scale-factor relations** are:

$$\rho_m(t) \propto R_U^{-3}(t),$$
$$\rho_r(t) \propto R_U^{-4}(t),$$
$$\rho_\Lambda(t) \propto R_U^{0}(t)$$

Implications: Same Matter, Less Radiation, More Dark Energy

The energy density-scale factor relations:

$$\rho_m(t) \propto R_U^{-3}(t),$$
$$\rho_r(t) \propto R_U^{-4}(t),$$
$$\rho_\Lambda(t) \propto R_U^{0}(t)$$

These relations imply that while the mass energy remain constant as universe expands, the energy contribution from radiation ***decreases***, and that from dark energy ***increases***:

$$\rho_m(t)V \propto R_U^{0}(t),$$
$$\rho_r(t)V \propto R_U^{-1}(t),$$
$$\rho_\Lambda(t)V \propto R_U^{3}(t)$$

Solution of the 1st Friedmann Equation:

Expansion History of the Universe

Solution 1: Hubble Parameter vs. redshift - E(z)

- **Boundary condition at t = t_0 gives the value of the curvature:**

$$H_0^2(1-\Omega_0) = -kc^2, \text{ or often as } \Omega_{k,0} \equiv (1-\Omega_0) = \frac{-kc^2}{H_0^2}$$

- **Relations between density parameters and scale factor**:

$$\frac{\Omega_m}{\Omega_{m,0}} = \frac{\rho_m \rho_{c,0}}{\rho_{m,0}\rho_c} = \frac{\rho_m}{\rho_{m,0}}\frac{H_0^2}{H^2} = \frac{1}{R_U^3}\frac{H_0^2}{H^2}$$

$$\frac{\Omega_\gamma}{\Omega_{\gamma,0}} = \frac{1}{R_U^4}\frac{H_0^2}{H^2} \quad \text{and} \quad \frac{\Omega_\Lambda}{\Omega_{\Lambda,0}} = \frac{H_0^2}{H^2}$$

- Given the FE1 with the boundary condition:

$$H^2 - H^2(\Omega_m + \Omega_\gamma + \Omega_\Lambda) = H_0^2(1-\Omega_0)/R_U^2$$

plug in the density-scale-factor relations and rearrange:

$$H^2 = H_0^2[(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}]$$

- Expressed scale factor as redshift and define the **dimensionless Hubble parameter** E(z):

$$E(z) = \frac{H(z)}{H_0} = \sqrt{(1-\Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{\gamma,0}(1+z)^4 + \Omega_{\Lambda,0}}$$

Solution 2: time vs. redshift - t(z) <-> $R_u(t)$

- Start from FE1 with boundary condition and density parameters:

$$H^2 = H_0^2[(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}]$$

express Hubble parameter with scale factor, $H(t) \equiv \dot{R}_U/R_U$:

$$\left(\frac{1}{R_U}\frac{dR_U}{dt}\right)^2 = \frac{H_0^2}{R_U^2}[(1-\Omega_0) + \Omega_{m,0}/R_U + \Omega_{\gamma,0}/R_U^2 + \Omega_{\Lambda,0}R_U^2]$$

- Separate time and scale factor terms, and define **Hubble time** $t_H \equiv 1/H_0$:

$$\frac{dt}{t_H} = \frac{dR_U}{\sqrt{(1-\Omega_0) + \Omega_{m,0}/R_U + \Omega_{\gamma,0}/R_U^2 + \Omega_{\Lambda,0}R_U^2}}$$

- By integrating both sides from R_U=0 (t=0, Big Bang) to R_U = 1/(1+z) (when t = t), we obtain the coordinate time *t* at redshift *z* for *any given values of the density parameters*.

$$\frac{t}{t_H} = \int_0^{1/(1+z)} \frac{dR_U}{\sqrt{(1-\Omega_0) + \Omega_{m,0}/R_U + \Omega_{\gamma,0}/R_U^2 + \Omega_{\Lambda,0}R_U^2}}$$

Alternative Derivation: time expressed in E(z)

- Start from the definition of the dimensionless Hubble parameter:

$$H(R_U) = H_0 E(R_U)$$

express Hubble parameter with scale factor, $H(t) \equiv \dot{R}_U / R_U$:

$$\frac{1}{R_U}\frac{dR_U}{dt} = H_0 E(R_U)$$

- Separate time and scale factor terms, and define **Hubble time** $t_H \equiv 1/H_0$:

$$H_0 dt = \frac{dt}{t_H} = \frac{dR_U}{E(R_U)R_U}$$

- By integrating both sides from R_U=0 (t=0, Big Bang) to R_U = 1/(1+z) (when t = t), we obtain the coordinate time *t* at redshift *z* for *any given values of the density parameters*.

$$\frac{t}{t_H} = \int_0^{1/(1+z)} \frac{dR_U}{E(R_U)R_U} = -\int_\infty^z \frac{dz'}{(1+z')E(z')} = \int_z^\infty \frac{dz'}{(1+z')E(z')}$$

A simple example: Einstein-de Sitter Universe

- The Hubble Parameter solution:

$$E(R_U) = \frac{H(R_U)}{H_0} = \sqrt{(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}}$$

- The time-scale factor solution:

$$\frac{t}{t_H} = \int_0^{1/(1+z)} \frac{dR_U}{\sqrt{(1-\Omega_0) + \Omega_{m,0}/R_U + \Omega_{\gamma,0}/R_U^2 + \Omega_{\Lambda,0}R_U^2}}$$

- A **matter-only, flat universe is known as the Einstein-de Sitter universe** have the following density parameters:

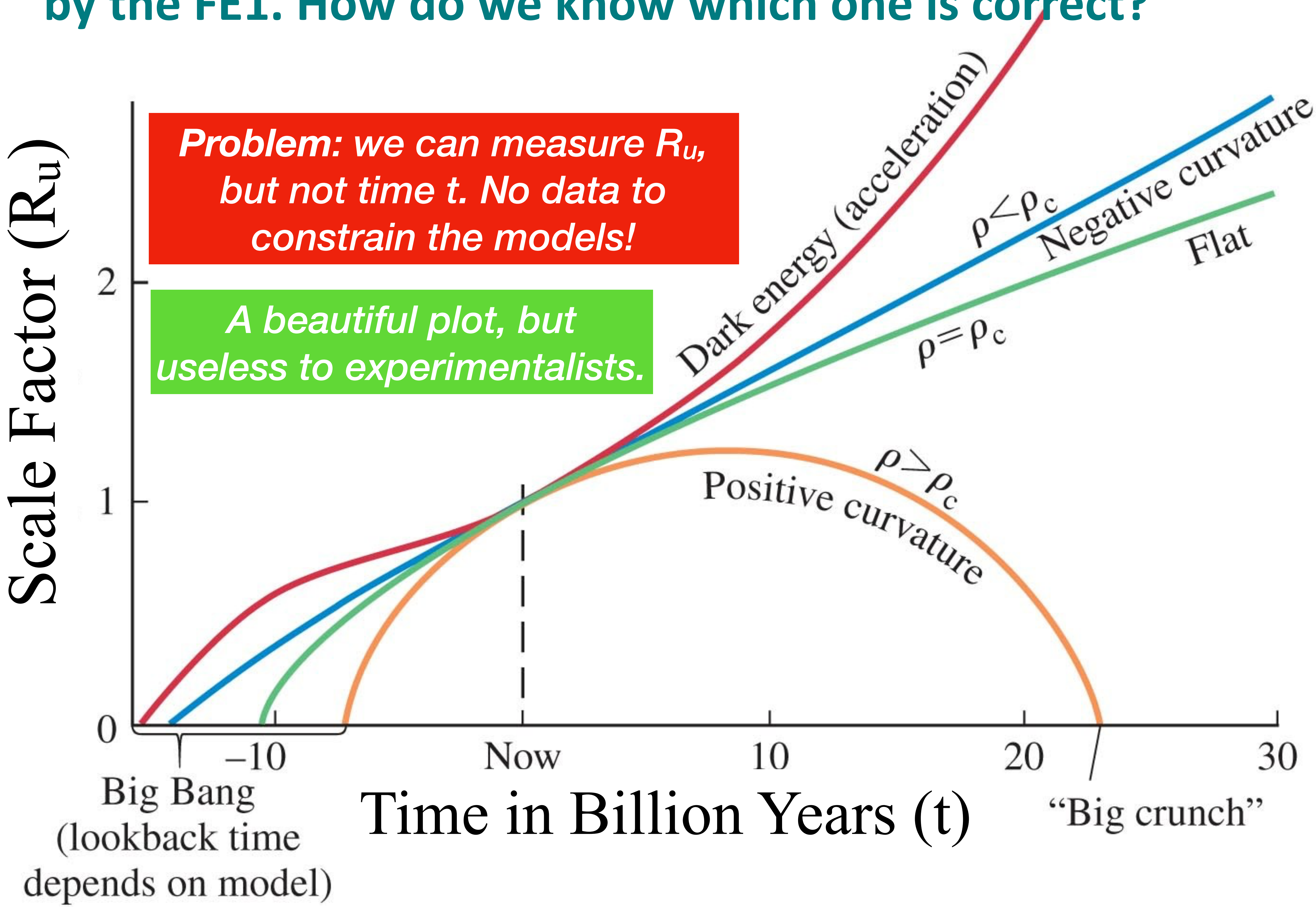
$$\Omega_0 = \Omega_{m,0} = 1,\ \Omega_{\gamma,0} = \Omega_{\Lambda,0} = 0$$

  which lead to the following analytical solution:

$$\Rightarrow H(z) = H_0/R_U^{3/2} = H_0(1+z)^{3/2}$$

$$\Rightarrow t(z) = \frac{2}{3}t_H(1+z)^{-3/2}$$

Without prior knowledge of the density parameters and Hubble constant, infinite expansion histories are allowed by the FE1. How do we know which one is correct?



Recap of Previous Lecture

Equations of State for Matter, Radiation, and Dark Energy

The **equations of state** for different components of our universe are necessary to solve the Friedmann's equations by connecting pressure with energy density:

$$p = w\rho c^2$$

w = 0, 1/3, -1 for matter, relativistic matter, and cosmological constant

First law of thermodynamics in adiabatic condition:

$$d(\rho c^2 R_U^3) = -p dR_U^3$$

Plug in the equations of state:

$$d(\rho R_U^3) = -w\rho d(R_U^3) \Rightarrow \frac{d\rho}{\rho} = -3(1+w)\frac{dR_U}{R_U} \Rightarrow \rho \propto R_U^{-3(1+w)}$$

The resulting **density-scale-factor relations** are:

$$\rho_m(t) \propto R_U^{-3}(t),$$
$$\rho_r(t) \propto R_U^{-4}(t),$$
$$\rho_\Lambda(t) \propto R_U^{0}(t)$$

FE1 in Various Forms

Originally derived from the *tt*-component of the field equation:

$$\left(\frac{\dot{R}_U}{R_U}\right)^2 + \frac{kc^2}{R_U^2} = \frac{8\pi G}{3}(\rho_m + \rho_\gamma) + \frac{\Lambda c^2}{3}$$

After defining Hubble parameter, critical density, density of Cosmological constant, and dimensionless density parameters:

$$H \equiv \frac{\dot{R}_U}{R_U}, \quad \rho_c \equiv \frac{3H^2}{8\pi G}, \quad \rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G}$$

It can be rearranged as:

$$R_U^2 H^2 \, [1 - (\Omega_m + \Omega_\gamma + \Omega_\Lambda)] = -\, kc^2$$

After plugging in the boundary condition at present:

$$R_U^2 H^2 \, [1 - (\Omega_m + \Omega_\gamma + \Omega_\Lambda)] = H_0^2[1 - (\Omega_{m,0} + \Omega_{\gamma,0} + \Omega_{\Lambda,0})]$$

Solution of FE1: Hubble Parameter vs. redshift - E(z)

- The **FE1 in density parameters and Hubble parameter**:

$$R_U^2 H^2 \; [1 - (\Omega_m + \Omega_\gamma + \Omega_\Lambda)] = -\, kc^2$$

- **Boundary condition** at t = t_0 gives the value of the curvature**:**

$$H_0^2 (1 - \Omega_0) = -\, kc^2$$

- **Equations of state** gives density-scale-factor relations:

$$\frac{\Omega_m}{\Omega_{m,0}} = \frac{\rho_m \rho_{c,0}}{\rho_{m,0} \rho_c} = \frac{\rho_m}{\rho_{m,0}} \frac{H_0^2}{H^2} = \frac{1}{R_U^3} \frac{H_0^2}{H^2}$$

$$\frac{\Omega_\gamma}{\Omega_{\gamma,0}} = \frac{1}{R_U^4} \frac{H_0^2}{H^2} \quad \text{and} \quad \frac{\Omega_\Lambda}{\Omega_{\Lambda,0}} = \frac{H_0^2}{H^2}$$

- Plug in and rearrange:

$$H^2 = H_0^2 [(1 - \Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}]$$

- Expressed scale factor as redshift and define the **dimensionless Hubble parameter** E(z):

$$E(z) = \frac{H(z)}{H_0} = \sqrt{(1 - \Omega_0)(1 + z)^2 + \Omega_{m,0}(1 + z)^3 + \Omega_{\gamma,0}(1 + z)^4 + \Omega_{\Lambda,0}}$$

Evolution of the Dimensionless Density Parameters

- Given density parameters-scale factor relations from equations of state:

$$\frac{\Omega_m}{\Omega_{m,0}} = \frac{\rho_m}{\rho_c}\frac{\rho_{c,0}}{\rho_{m,0}} = \frac{\rho_m}{\rho_{m,0}}\frac{H_0^2}{H^2} = \frac{1}{R_U^3}\frac{H_0^2}{H^2}$$

$$\frac{\Omega_\gamma}{\Omega_{\gamma,0}} = \frac{1}{R_U^4}\frac{H_0^2}{H^2} \quad \& \quad \frac{\Omega_\Lambda}{\Omega_{\Lambda,0}} = \frac{H_0^2}{H^2}$$

- And the FE1 with the boundary condition:

$$H^2 = H_0^2[(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}]$$

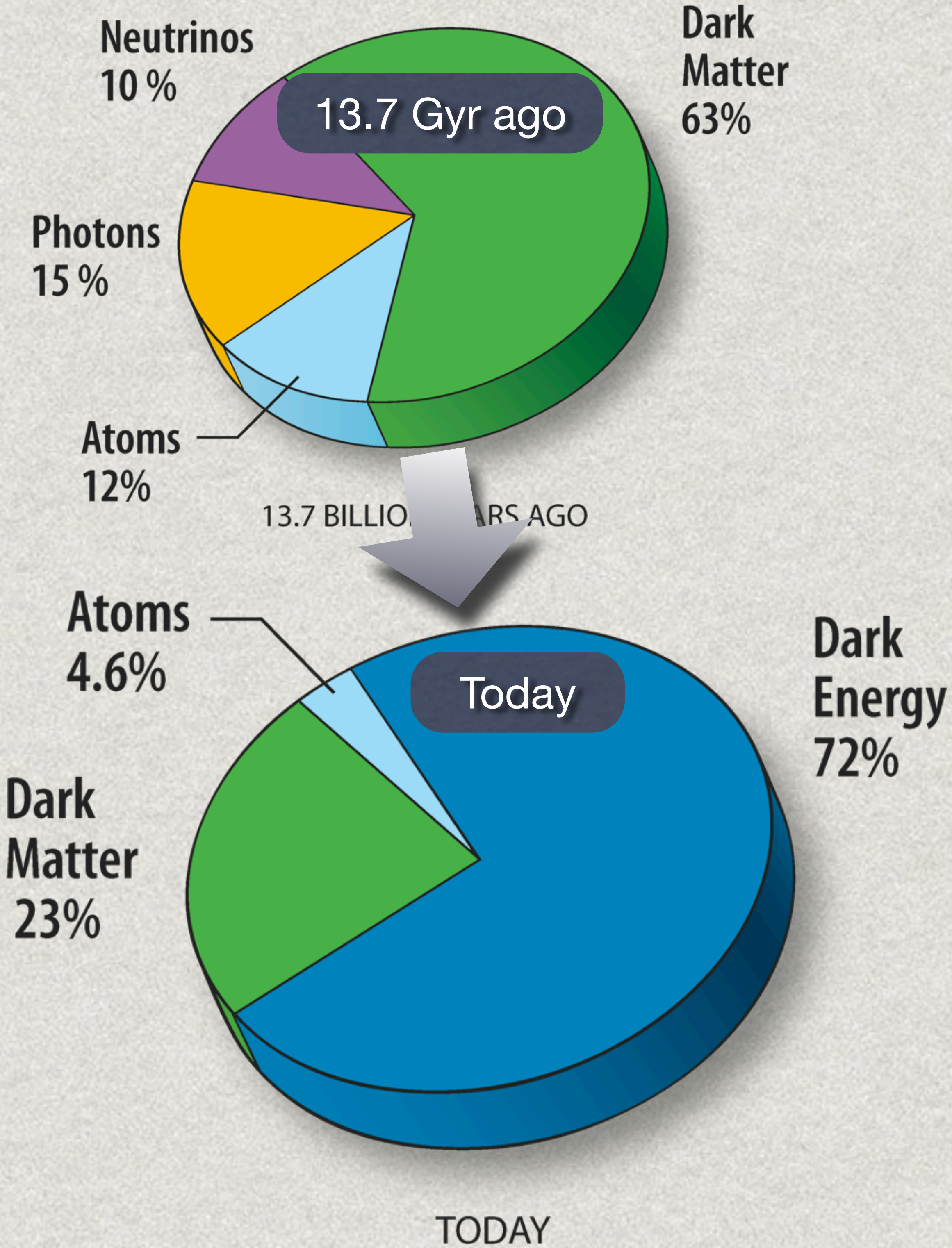
- It's easy to see that:

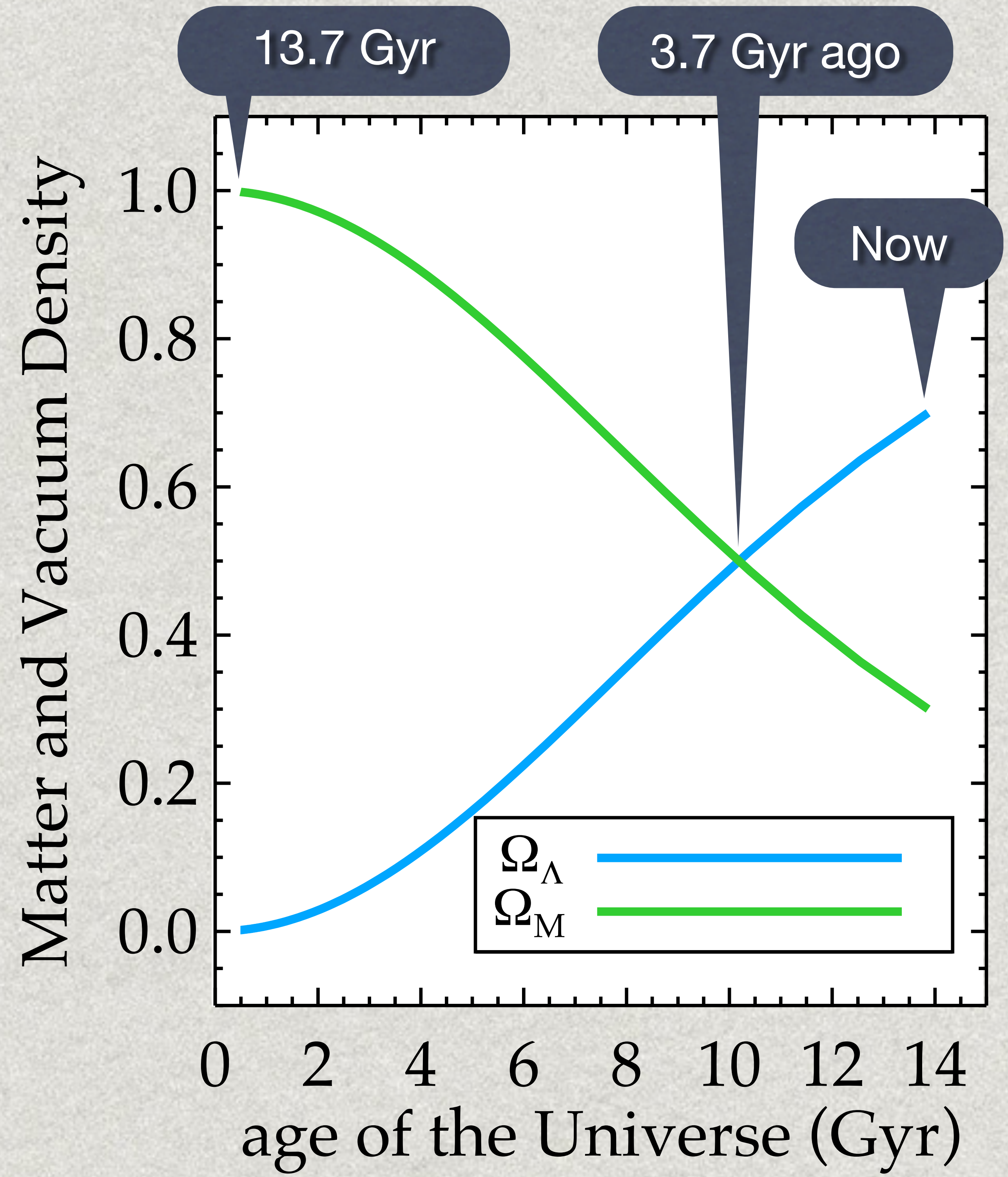
$$\frac{\Omega_m}{\Omega_{m,0}} = \frac{1}{R_U^3}\frac{H_0^2}{H^2} = \frac{1}{R_U^3[(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}]}$$

$$\frac{\Omega_\gamma}{\Omega_{\gamma,0}} = \frac{1}{R_U^4}\frac{H_0^2}{H^2} = \frac{1}{R_U^4[(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}]}$$

$$\frac{\Omega_\Lambda}{\Omega_{\Lambda,0}} = \frac{H_0^2}{H^2} = \frac{1}{(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}}$$

$$\frac{\Omega_\Lambda}{\Omega_{\Lambda,0}} = \frac{H_0^2}{H^2} = \frac{1}{(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}}$$





Expansion History: Ru-t or z-t relation

- Start from the definition of the dimensionless Hubble parameter:

$$H(R_U) = H_0 E(R_U)$$

express Hubble parameter with scale factor, $H(t) \equiv \dot{R}_U / R_U$:

$$\frac{1}{R_U}\frac{dR_U}{dt} = H_0 E(R_U)$$

- Separate time and scale factor terms, and define **Hubble time** $t_H \equiv 1/H_0$:

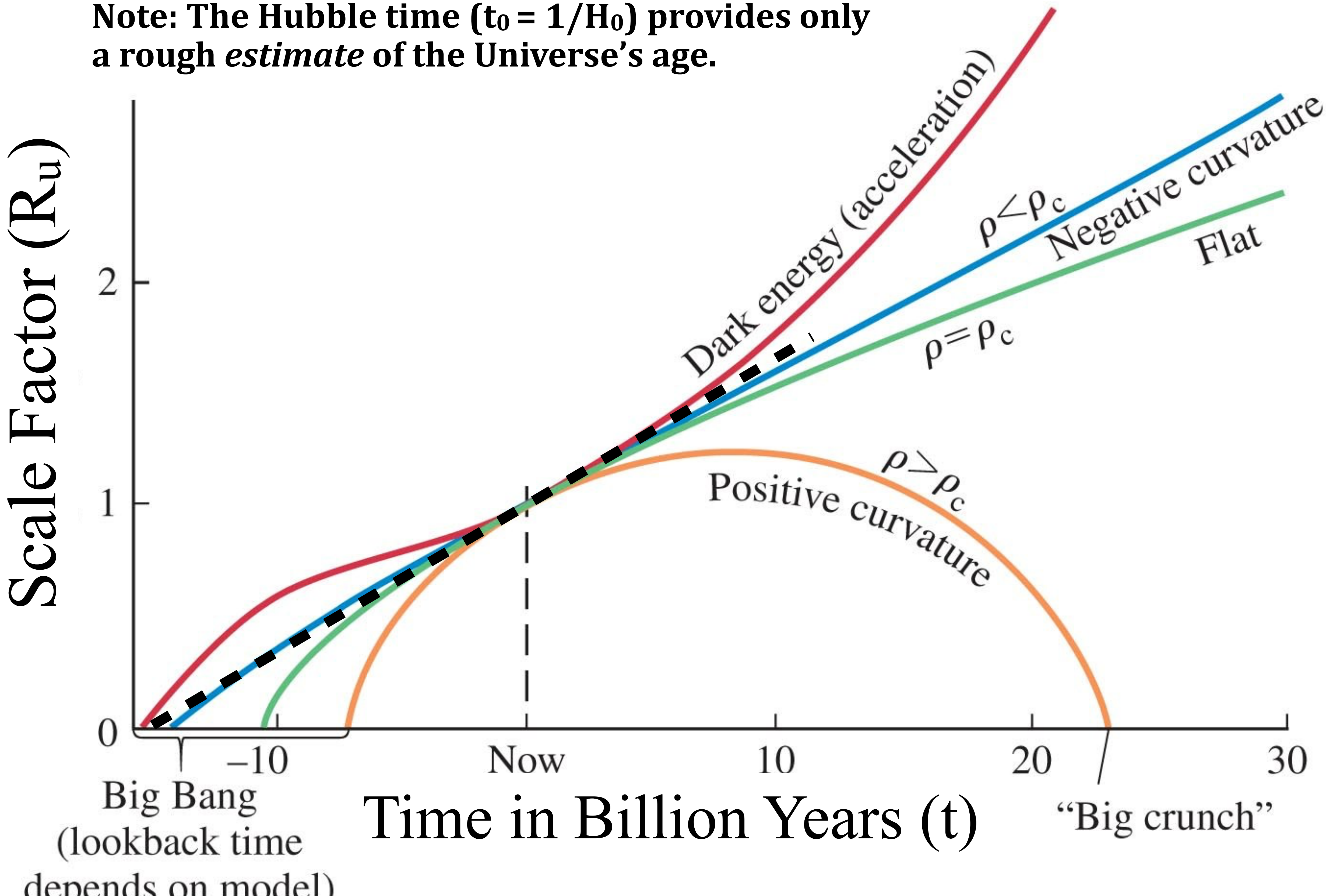
$$H_0 dt = \frac{dt}{t_H} = \frac{dR_U}{E(R_U)R_U}$$

- By integrating both sides from R_U=0 (t=0, Big Bang) to R_U = 1/(1+z) (when t = t), we obtain the coordinate time *t* at redshift *z* for *any given values of the density parameters*.

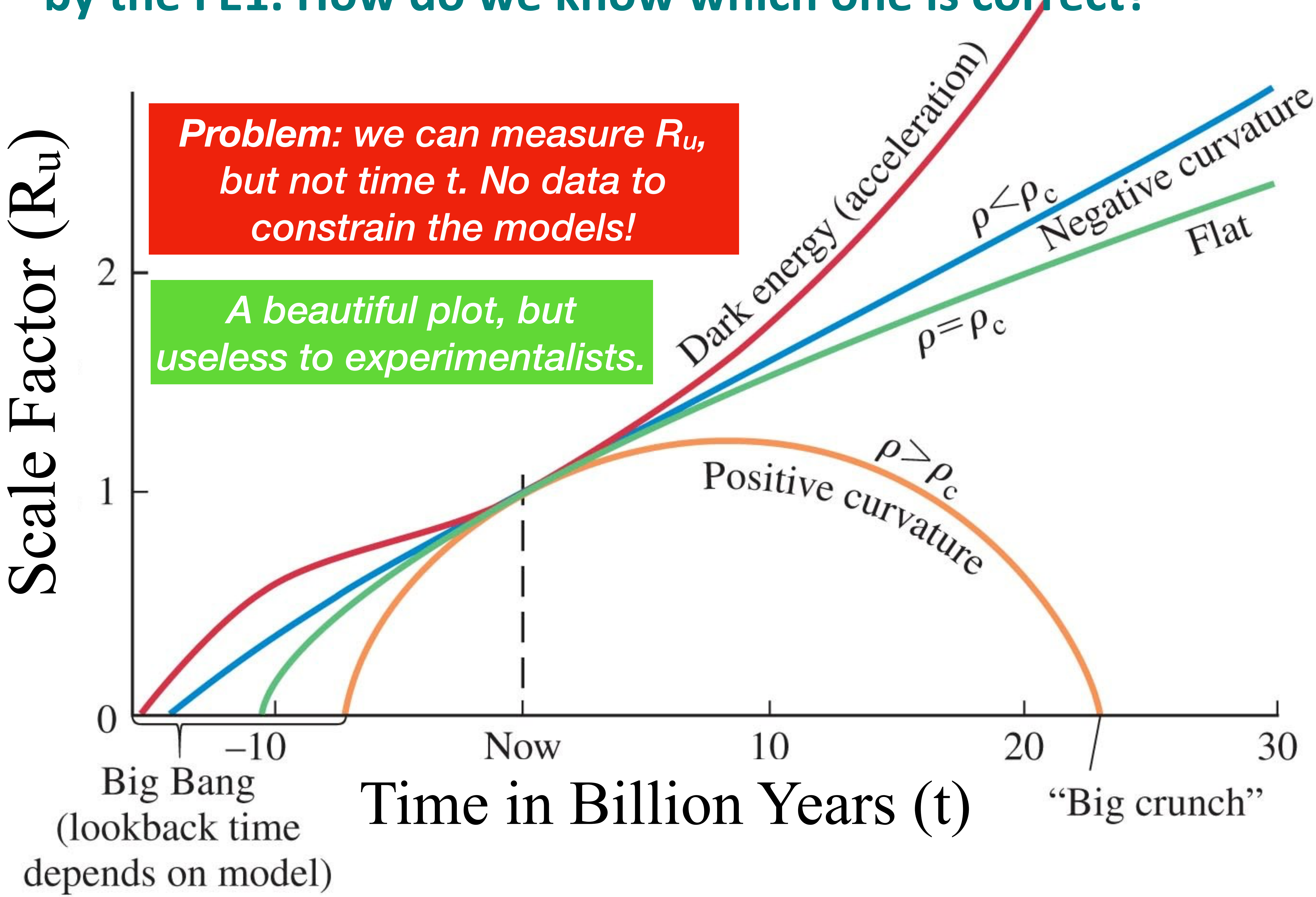
$$\frac{t}{t_H} = \int_0^{1/(1+z)} \frac{dR_U}{E(R_U)R_U} = -\int_\infty^z \frac{dz'}{(1+z')E(z')} = \int_z^\infty \frac{dz'}{(1+z')E(z')}$$

Expansion Histories for Different Compositions

Note: The Hubble time ($t_0 = 1/H_0$) provides only a rough *estimate* of the Universe's age.

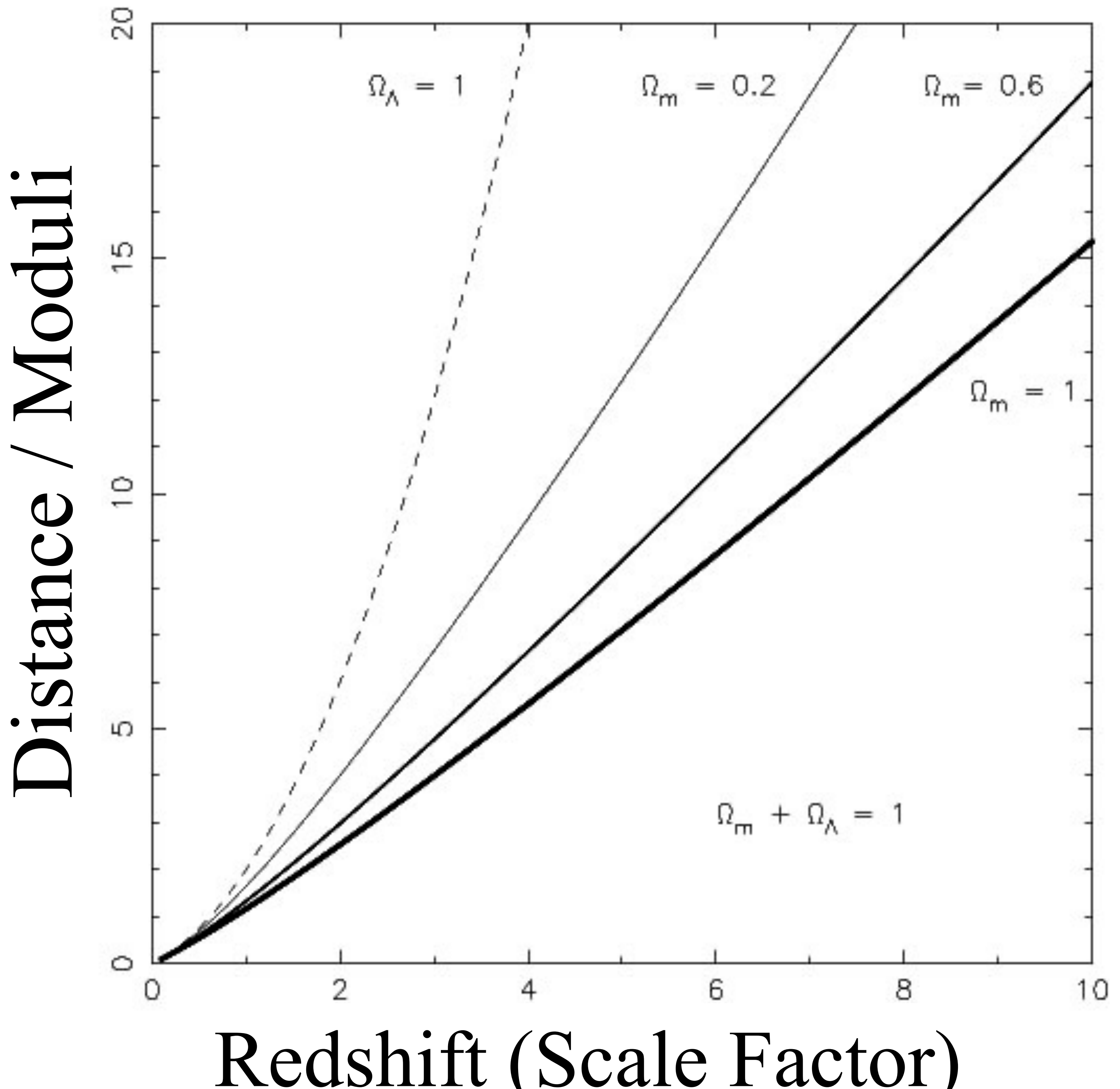


Without prior knowledge of the density parameters and Hubble constant, infinite expansion histories are allowed by the FE1. How do we know which one is correct?

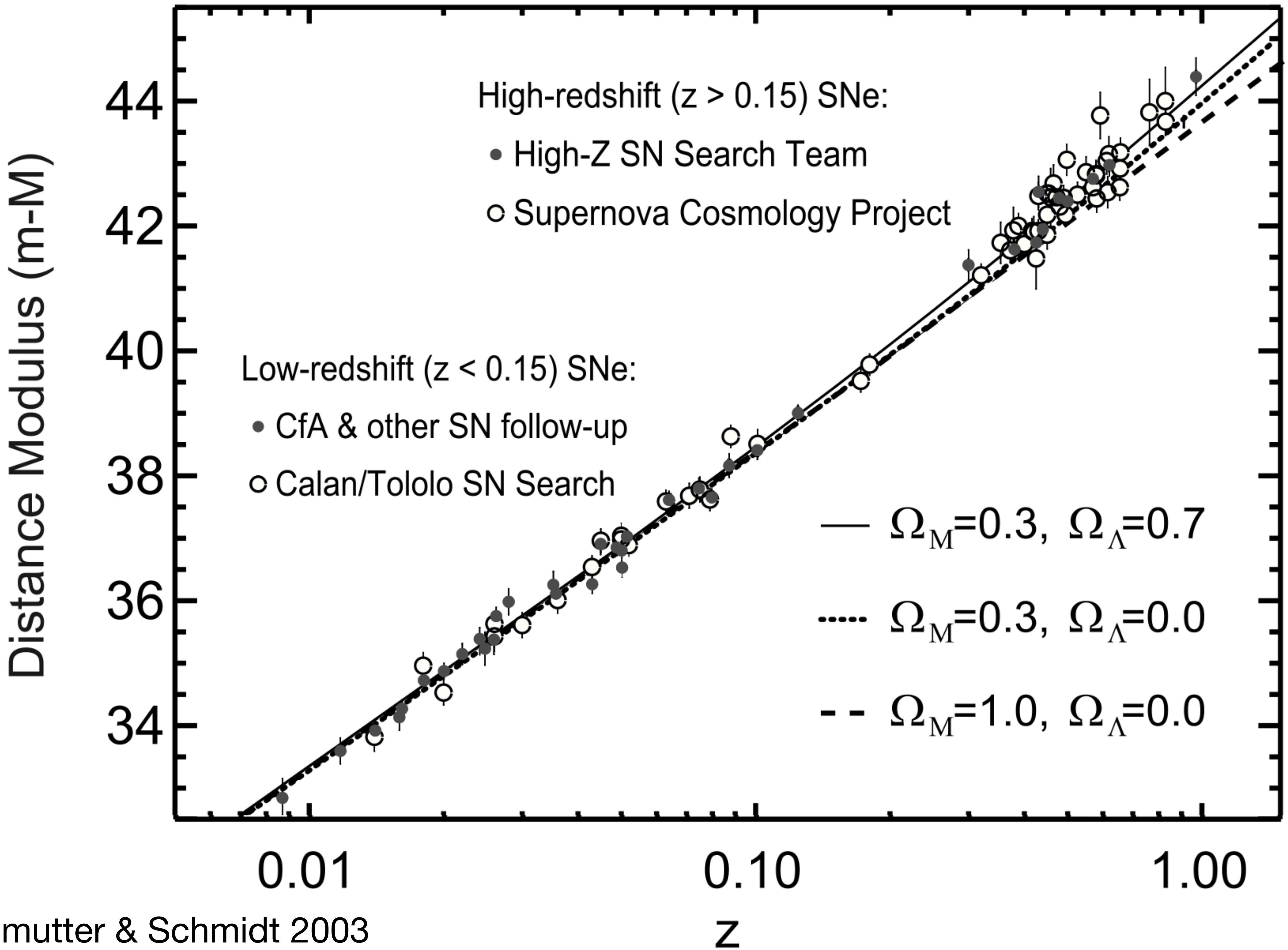


The Key is to Measure the Distance-Redshift Relation

We can measure both z (R_u) and Distances, so the models can be constrained



Hubble diagram for Cosm Parameters (z > 0.1)



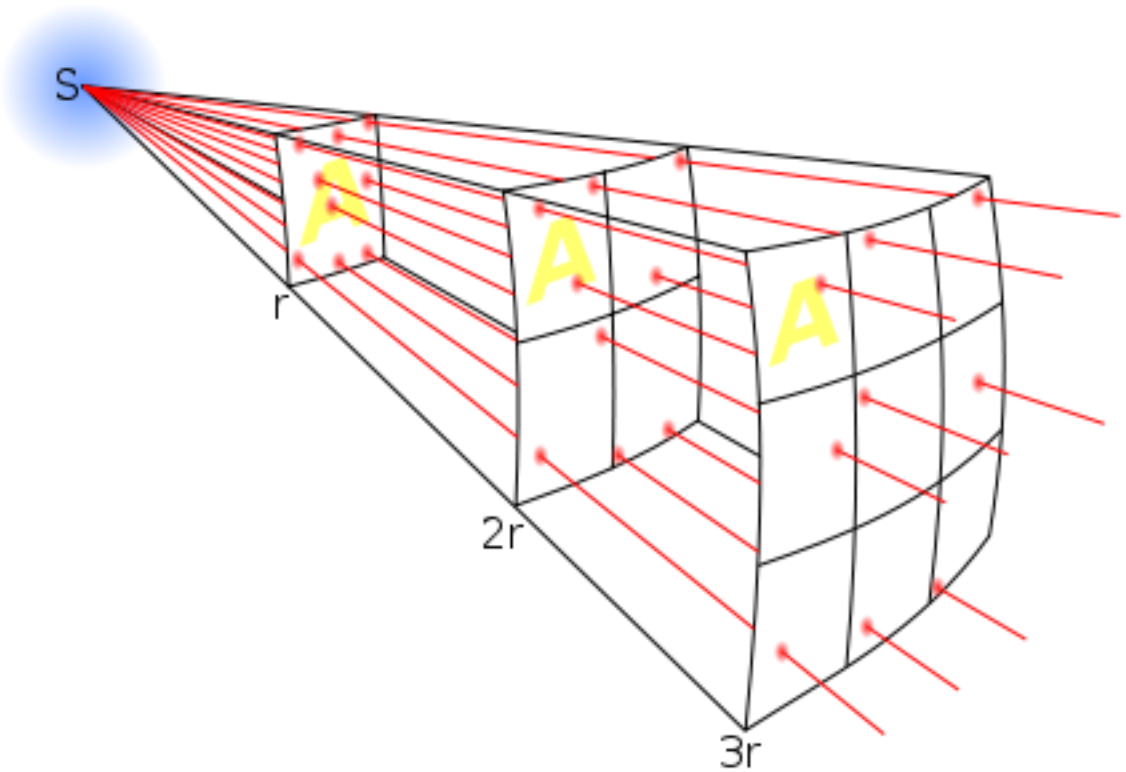
Perlmutter & Schmidt 2003

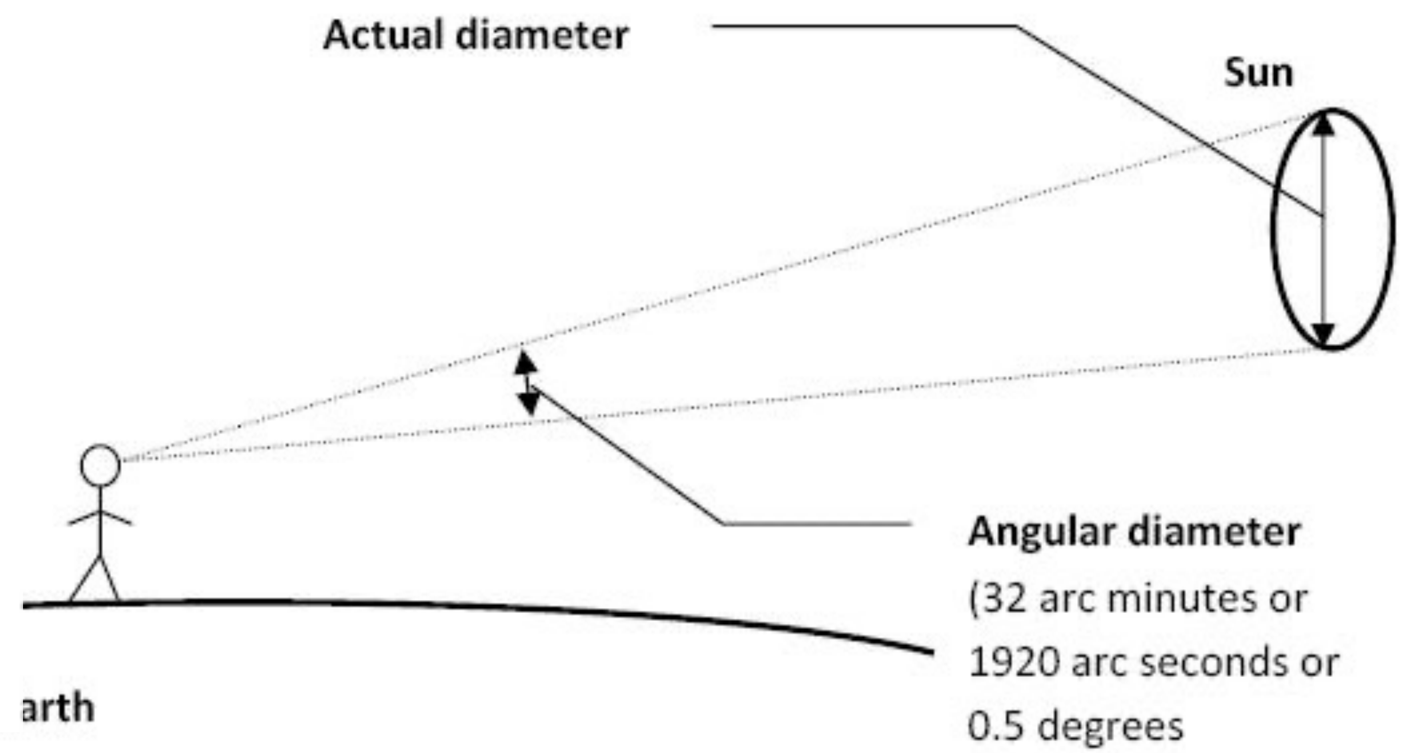
Observable Distances:

Luminosity Distance (standard candle)
Angular Diameter Distance (standard ruler)

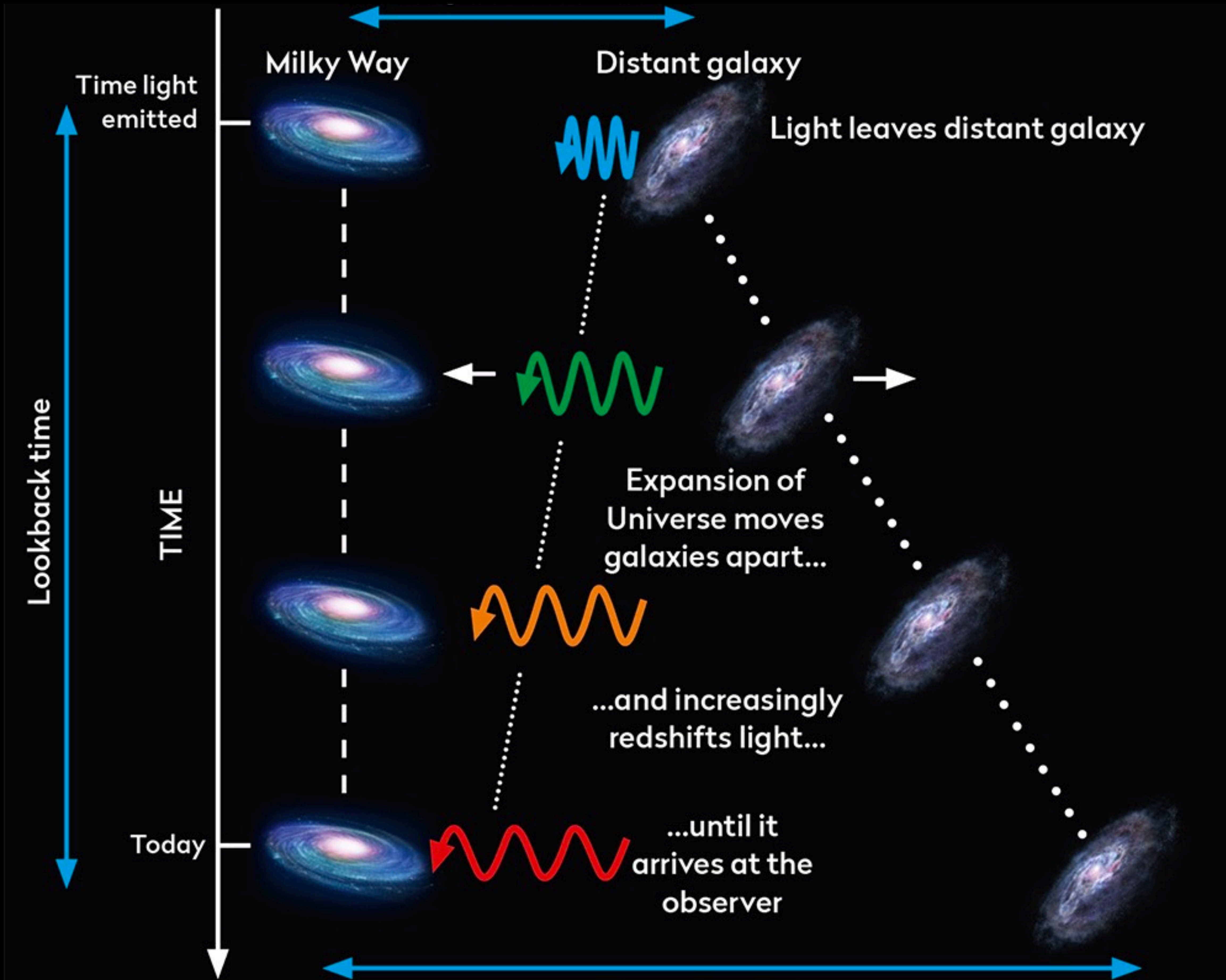
Luminosity Distance and Angular Diameter Distance

- **Luminosity distance**: $d_L = \sqrt{L/(4\pi F)}$
- **Angular diameter distance**: $d_A = l/\theta$





- In our perceived world, the two shall equal for the same object.
- **In an expanding universe, do they still equal to each other for the same object?**



@ Govert Schilling

Why Angular-Diameter Distance and Luminosity Distance are Not Equal?

- consider the luminosity of a **spherical blackbody emitter** with a radius of $\boldsymbol{R}$ with a temperature of $\boldsymbol{T_e}$ when emitted, and $\boldsymbol{T_o}$ when observed.
  - Its emitted luminosity: $L_e = 4\pi R^2 \sigma_{SB} T_e^4$
  - Its surface brightness seen by the observer: $I_o = \sigma_{SB} T_o^4 / \pi$
  - The observed flux can be derived in two ways:
    - It equals the angular size multiplied by surface brightness:
      $$F_o = \frac{\pi R^2}{d_A^2} \frac{\sigma_{SB} T_o^4}{\pi} = \sigma_{SB} T_o^4 \frac{R^2}{d_A^2}$$
    - It also equals the emitted luminosity divided by luminosity distance squared: $F_o = \frac{L_e}{4\pi d_L^2} = \sigma_{SB} T_e^4 \frac{R^2}{d_L^2}$
  - Combining the two leads to the relation between luminosity distance and angular diameter distance:
    $$\frac{d_A}{d_L} = \frac{T_o^2}{T_e^2} = a^2(t_e) = \frac{1}{(1+z)^2} \;\Rightarrow\; d_A = \frac{d_L}{(1+z)^2}$$

Luminosity Distance vs. Proper Distance

- To understand $\boldsymbol{d_L}$, consider the **luminosity (energy per unit time)** of a source when light was emitted vs. the luminosity of the source when light was observed, are they equal to each other in an expanding universe?

- because of **cosmological redshift**, each photon loses its energy by (1+z) and time is dilated by (1+z), so the **observed luminosity** is decreased by $(1+z)^2$

$$L_o = L_e a^2(t_e)$$

- So the observed flux is diluted by both **distance^2** and **$(1+z)^2$**:

$$F_o = \frac{L_o}{4\pi d_p^2} = \frac{L_e}{4\pi[d_p^2/a^2(t_e)]} \Rightarrow d_L = d_p/a(t_e) = d_p(1+z)$$

- Given the dL-dA relation, we have: $d_A = d_p/(1+z)$

Proper Distance

Robertson-Walker Metric: Differential Space-Time Distance

- In **General Relativity**, a **metric** is a function which measures ***differential space-time distance*** between two events**:**

$$(ds)^2 = (c \cdot dt)^2 - (dl)^2$$

- The **Robertson-Walker metric** is the metric that describes the geometry of a **homogeneous, isotropic, expanding** universe. The metric in *spherical coordinate system* is:

$$(ds)^2 = (c \cdot dt)^2 - R_U^2(t)\left[\left(\frac{dx}{\sqrt{1-kx^2}}\right)^2 + (xd\theta)^2 + (x\sin\theta d\phi)^2\right]$$

where

R_U is the **scale factor,** defined to be 1 at present day, and <1 in the past
x is the **comoving** radial distance, $x \equiv r(t)/R_U(t)$,
k is the **comoving** curvature, $k \equiv \frac{1}{R^2} = \frac{R_U^2}{R^2 R_U^2} = K(t)R_U^2(t)$,
R is the **comoving** radius of the curvature.

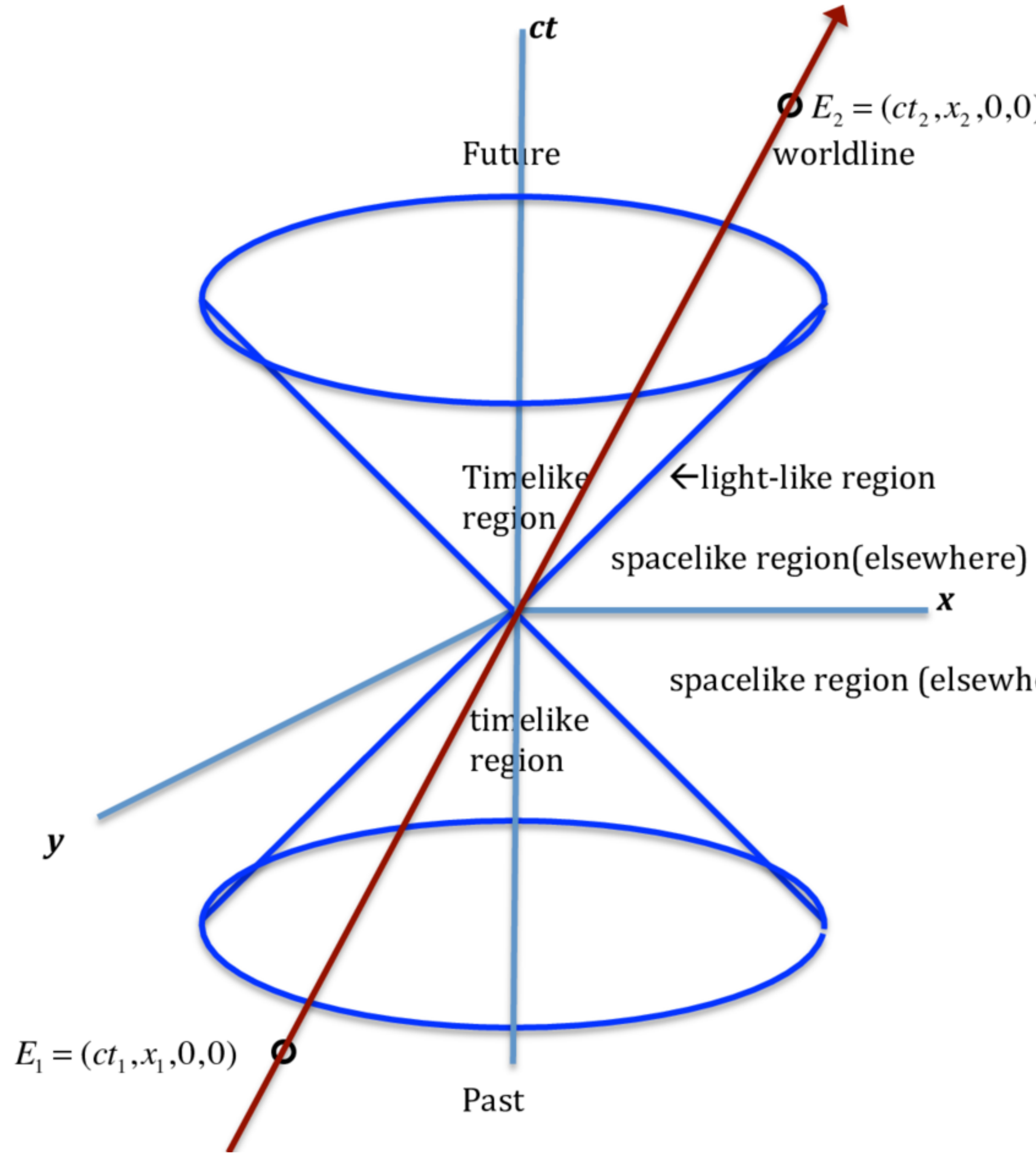
The same terms are in **Friedmann Equation**.

Robertson-Walker Metric: Differential Space-Time Distance

- ***integral space-time distance***

$$\Delta s = \int_A^B \sqrt{(ds)^2}$$

- **Timelike**
  - $(\Delta s)^2 > 0$
  - events could be causally connected
  - e.g., **proper time** measured by a clock moving in a spaceship.
- **Spacelike**
  - $(\Delta s)^2 < 0$
  - events cannot be causally connected
  - e.g., **proper distance** between two locations measured simultaneously
- **Lightlike:** $(\Delta s)^2 = 0$



Proper distance to an object at redshift z

- By definition, **proper distance** is measured between two locations **simultaneous** (dt = 0). If we also define one of the locations at the origin (x=0), and the direction to the other location is along the radial direction ($d\theta = d\phi = 0$), we can write the **proper distance** by integrating the RW metric:

$$d_p = \sqrt{-(\Delta s)^2} = R_U(t) \int_0^x \frac{dx}{\sqrt{1 - kx^2}},$$

where t is the time of the measurement, and when it is the present, $R_u = 1$.

- The integral has three analytical solutions, depending on the value of k:

$$\int_0^x \frac{dx}{\sqrt{1-kx^2}} = \begin{cases} \frac{1}{\sqrt{-k}} \int_0^{x\sqrt{-k}} \frac{dy}{\sqrt{1+y^2}} = \frac{1}{\sqrt{-k}} \sinh^{-1}\left[x\sqrt{-k}\right], & \text{open}(k<0), \\ \int_0^x dx = x, & \text{flat}(k=0), \\ \frac{1}{\sqrt{k}} \int_0^{x\sqrt{k}} \frac{dy}{\sqrt{1-y^2}} = \frac{1}{\sqrt{k}} \sin^{-1}\left[x\sqrt{k}\right], & \text{closed}(k>0)\,. \end{cases}$$

Proper Distance vs. Comoving Distance

- In the previous slide, we derived the relations between **proper distance** and **comoving distance**:

$$d_p = R_U(t)\begin{cases} \dfrac{1}{\sqrt{-k}}\sinh^{-1}\left[x\sqrt{-k}\right], & \text{open}(k<0), \\ x, & \text{flat}(k=0), \\ \dfrac{1}{\sqrt{k}}\sin^{-1}\left[x\sqrt{k}\right], & \text{closed}(k>0)\,. \end{cases}$$

- ***Proper distance*** *equals* ***comoving distance*** *only in a* ***flat universe*** *and when measured at the* ***present time***

Comoving Distance

To Measure Distances, We Shall Follow the "Money" (Light)

- In **General Relativity**, a **metric** is a function which measures ***differential space-time distance*** between two events**:**

$$(ds)^2 = (c \cdot dt)^2 - (dl)^2$$

- The **Robertson-Walker metric** is the metric that describes the geometry of a **homogeneous, isotropic, expanding** universe. The metric in *spherical coordinate system* is:

$$(ds)^2 = (c \cdot dt)^2 - R_U^2(t)\left[\left(\frac{dx}{\sqrt{1 - kx^2}}\right)^2 + (xd\theta)^2 + (x \sin\theta d\phi)^2\right]$$

where

$\boldsymbol{R_U}$ is the **scale factor,** defined to be 1 at present day, and <1 in the past
$\boldsymbol{x}$ is the **comoving** radial distance, $x \equiv r(t)/R_U(t)$,
$\boldsymbol{k}$ is the **comoving** curvature, $k \equiv \dfrac{1}{R^2} = \dfrac{R_U^2}{R^2 R_U^2} = K(t)R_U^2(t)$,
$\boldsymbol{R}$ is the **comoving** radius of the curvature.

Comoving distance expressed as a function of redshift

- To measure **comoving distance** between two locations, we **follow the light along the radial direction** ($d\theta = d\phi = 0$):

$$\frac{-dx}{\sqrt{1-kx^2}} = \frac{c\ dt}{R_U(t)}$$

- Integrating from ***emitter at x*** to ***observer at x=0,*** *on the left side:*

$$\int_0^x \frac{dx}{\sqrt{1-kx^2}} = \begin{cases} \frac{1}{\sqrt{-k}} \int_0^{x\sqrt{-k}} \frac{dy}{\sqrt{1+y^2}} = \frac{1}{\sqrt{-k}} \sinh^{-1}\left[x\sqrt{-k}\right], & \text{open}(k<0), \\ \int_0^x dx = x, & \text{flat}(k=0), \\ \frac{1}{\sqrt{k}} \int_0^{x\sqrt{k}} \frac{dy}{\sqrt{1-y^2}} = \frac{1}{\sqrt{k}} \sin^{-1}\left[x\sqrt{k}\right], & \text{closed}(k>0)\,. \end{cases}$$

- *On the right side:*

$$\int_{t_e}^{t_0} \frac{cdt}{R_U(t)} = c\int_{R_U(z)}^{1} \frac{dR_U}{\dot{R}_U R_U} = c\int_{R_U(z)}^{1} \frac{dR_U}{H(R_U)R_U^2} = c\int_z^0 \frac{-(1+z')^{-2}dz'}{H(z')(1+z')^{-2}} = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')}$$

where we used the **Hubble parameter**: $H(t) \equiv \dot{R}_U/R_U$

and the **dimensionless Hubble parameter**: $E(z) = H(z)/H_0$

Comoving distance expressed as a function of redshift

Combining the two sides, we have:

$$\int_0^{x(z)} \frac{dx}{\sqrt{1-kx^2}} = \left\{ \begin{array}{ll} \frac{1}{\sqrt{-k}} \sinh^{-1}\left[x\sqrt{-k}\right], & \text{open}(k<0), \\ x(z), & \text{flat}(k=0), \\ \frac{1}{\sqrt{k}} \sin^{-1}\left[x\sqrt{k}\right], & \text{closed}(k>0)\,. \end{array} \right\} = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')}$$

thus the **comoving distance** to the emitter is**:**

$$x(z) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{-k}} \sinh\left[\frac{\sqrt{-k}c}{H_0}\int_0^z \frac{dz'}{E(z')}\right], & \text{open}(k<0), \\ \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')}, & \text{flat}(k=0), \\ \frac{1}{\sqrt{k}} \sin\left[\frac{\sqrt{k}c}{H_0}\int_0^z \frac{dz'}{E(z')}\right], & \text{closed}(k>0)\,. \end{array} \right.$$

where the **dimensionless Hubble parameter** is from **FE1**:

$$E(z) = H(z)/H_0 = \sqrt{(1-\Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{\gamma,0}(1+z)^4 + \Omega_{\Lambda,0}}$$

the **dimensionless curvature** is set by the boundary condition: $k = -H_0^2(1-\Omega_0)/c^2$

and the **hyperbolic sine function** is defined as: $\sinh x = \frac{e^x - e^{-x}}{2}$

Proper Distance vs. Redshift

- Recall that **proper distance** expressed in **comoving distance**:

$$d_p = R_U(t)\int_0^x \frac{dx}{\sqrt{1-kx^2}} = R_U(t)\begin{cases} \frac{1}{\sqrt{-k}}\sinh^{-1}\left[x\sqrt{-k}\right], & \text{open}(k<0), \\ x, & \text{flat}(k=0), \\ \frac{1}{\sqrt{k}}\sin^{-1}\left[x\sqrt{k}\right], & \text{closed}(k>0)\,. \end{cases}$$

- Follow the path of light, **comoving distance** expressed as a fun of z**:**

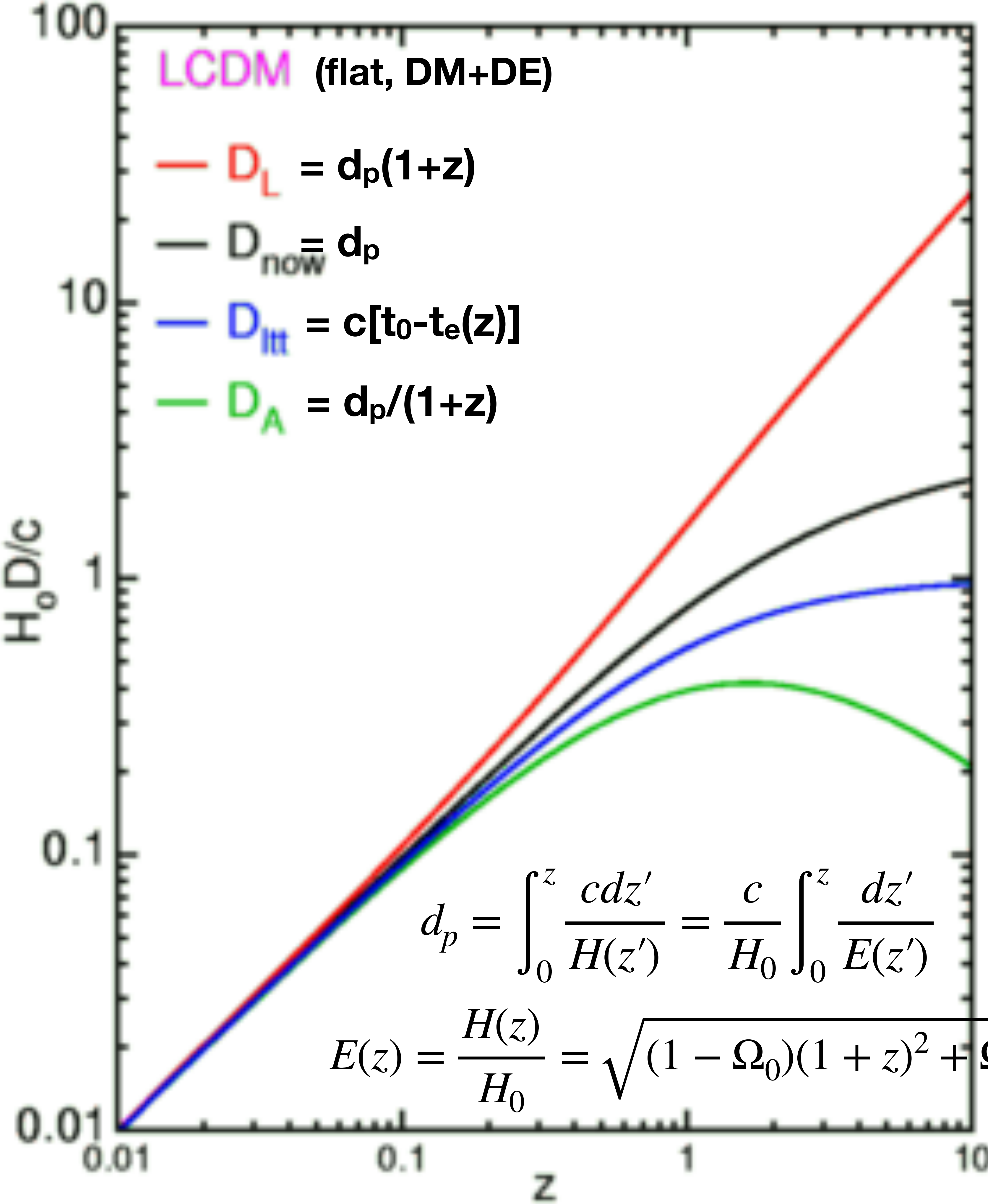
$$\int_0^{x(z)} \frac{dx}{\sqrt{1-kx^2}} = \left\{\begin{array}{ll} \frac{1}{\sqrt{-k}}\sinh^{-1}\left[x\sqrt{-k}\right], & \text{open}(k<0), \\ x(z), & \text{flat}(k=0), \\ \frac{1}{\sqrt{k}}\sin^{-1}\left[x\sqrt{k}\right], & \text{closed}(k>0)\,. \end{array}\right\} = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')}$$

- ***Combining the two,*** one obtains the **proper distance** to an emitter at redshift z**:**

$$d_p = \frac{cR_U(t)}{H_0}\int_0^z \frac{dz'}{E(z')}$$

Distance-redshift relations

D vs. z relation



Distance in unit of the Hubble scale: $D_H = c/H_0 = 3\ h^{-1}$ Gpc

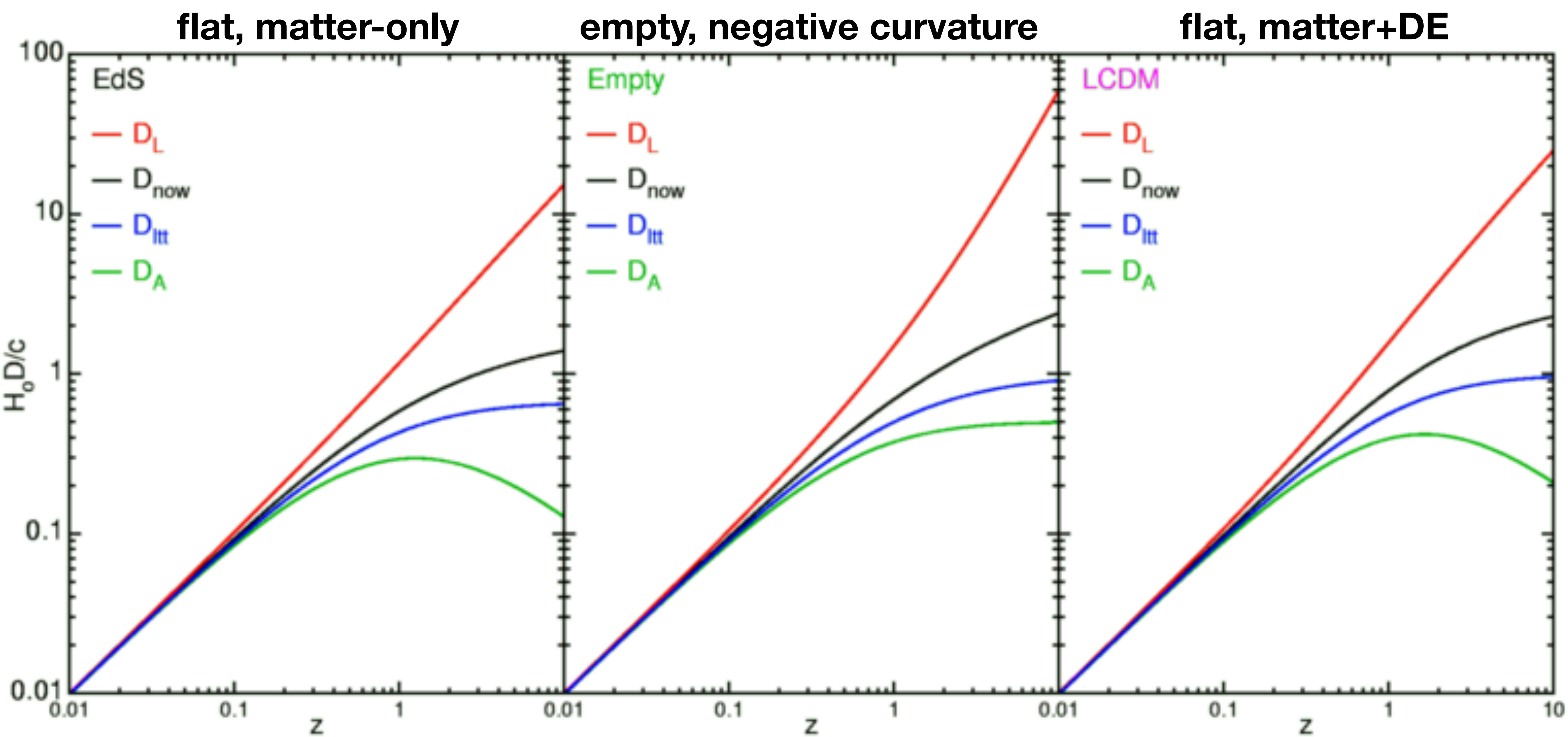
$$d_p = \int_0^z \frac{c\,dz'}{H(z')} = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')}$$

$$\frac{t}{t_H} = \int_z^\infty \frac{dz'}{(1+z')E(z')}$$

$$E(z) = \frac{H(z)}{H_0} = \sqrt{(1-\Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{\gamma,0}(1+z)^4 + \Omega_{\Lambda,0}}$$

$$d_p = \int_0^z \frac{c dz'}{H(z')} = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}$$

$$E(z) = \frac{H(z)}{H_0} = [(1 - \Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{\gamma,0}(1+z)^4 + \Omega_{\Lambda,0}]^{1/2}$$



Distance in unit of the Hubble scale: $D_H = c/H_0 = 3\ h^{-1}$ Gpc

Constrain Cosmological Parameters with the Hubble Diagram

Hubble Diagram for H_0 measurements ($z < 0.1$)

Burns+2018 w/ Cepheids

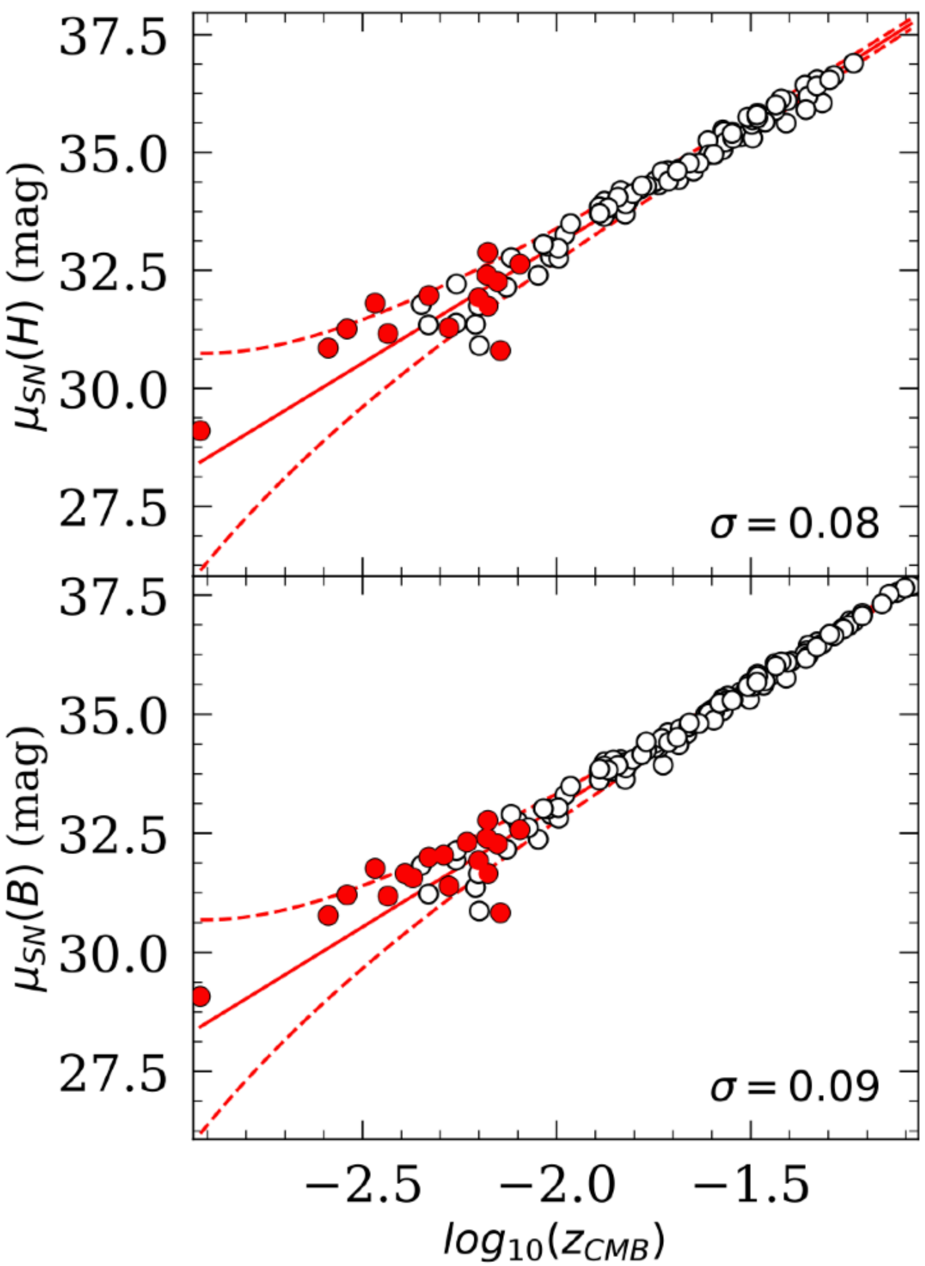
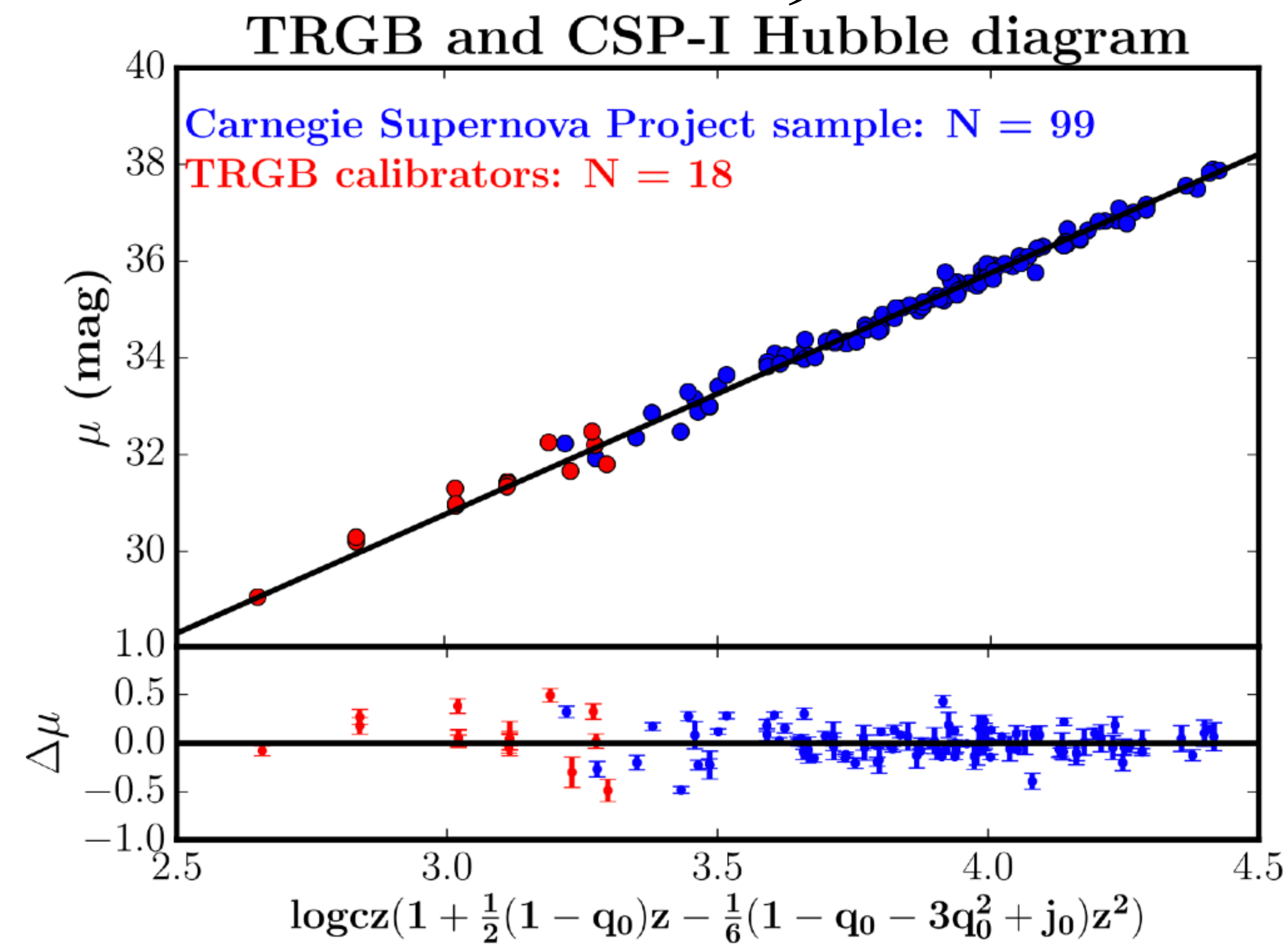
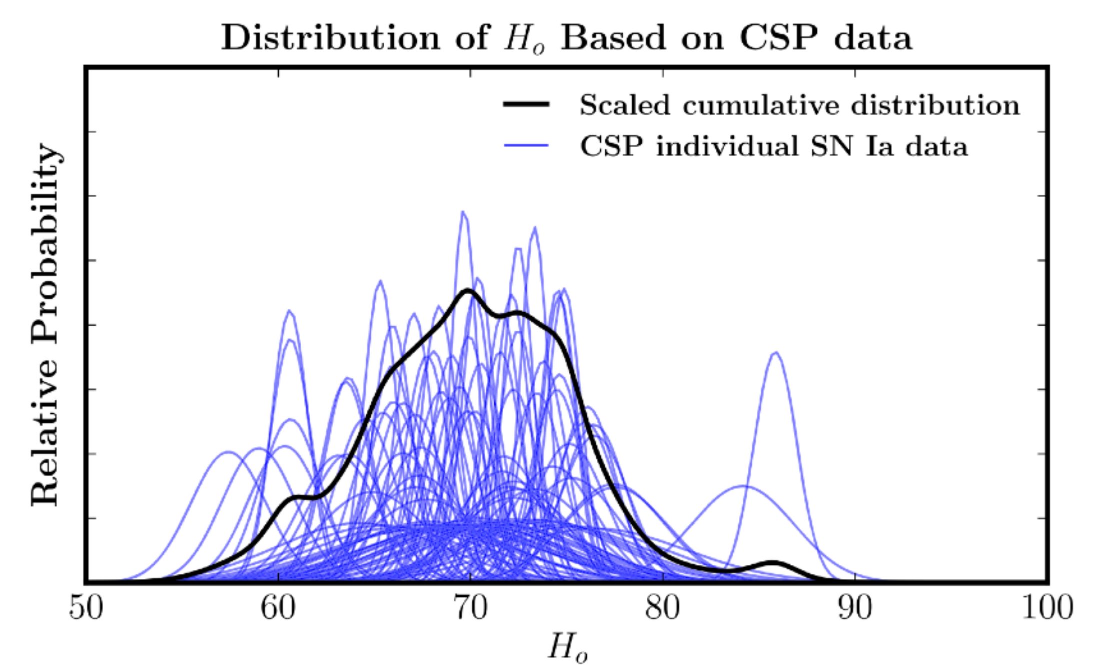


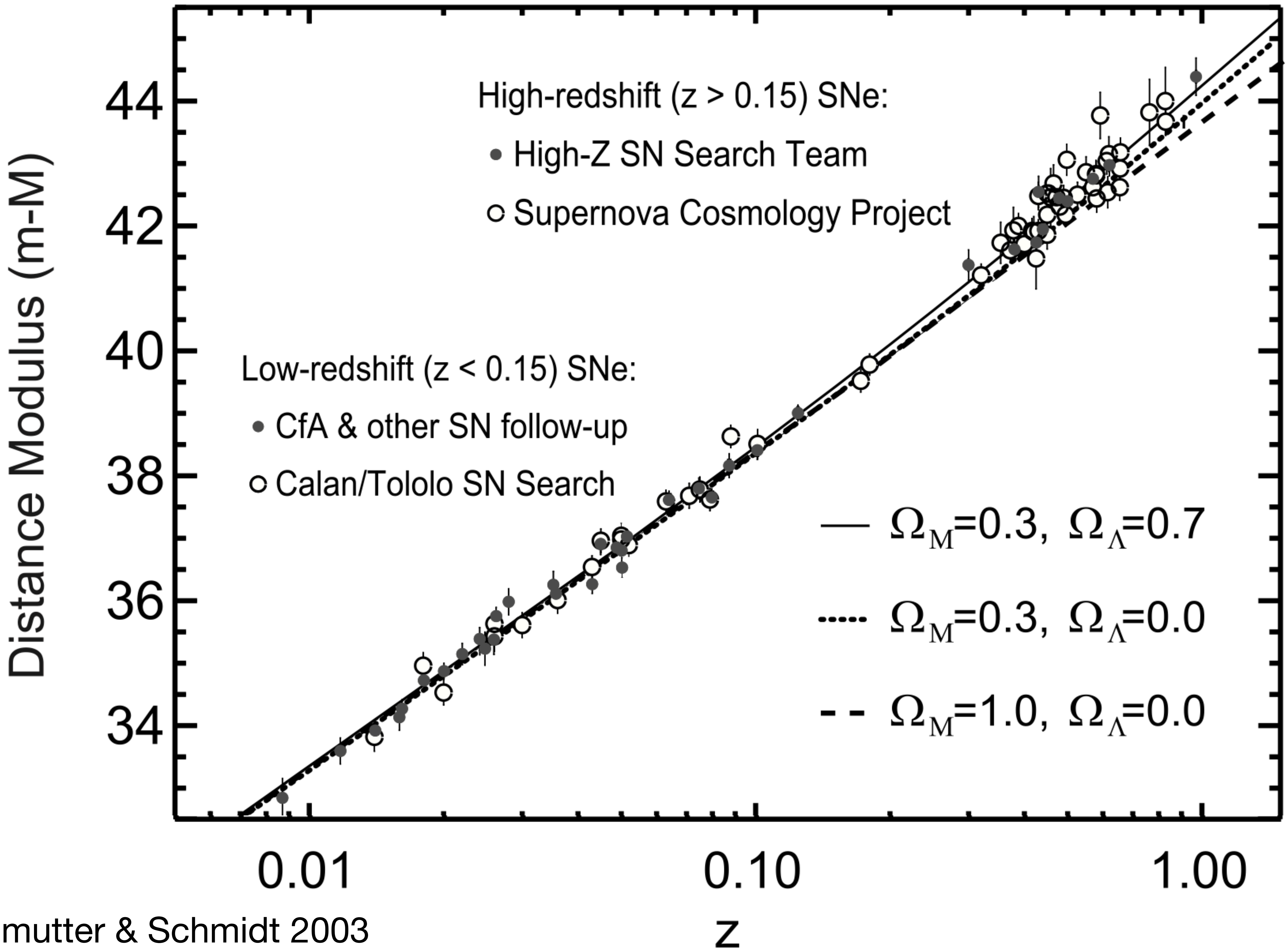
Figure 8. Hubble diagram for the *H* band (top panel) and *B* band (bottom panel) populated with SNe Ia from the CSP-I DR3 sample, as well as with those objects with Cepheid hosts (plotted as solid red circles). The best-fit $H_0 = 73.5 \pm 1.5$ km s^{-1} Mpc^{-1} from combining all filters is shown by a solid red line. The expected dispersion due to intrinsic variance and peculiar velocities is plotted as dashed red lines.

Freedman+2019 w/ TRGB



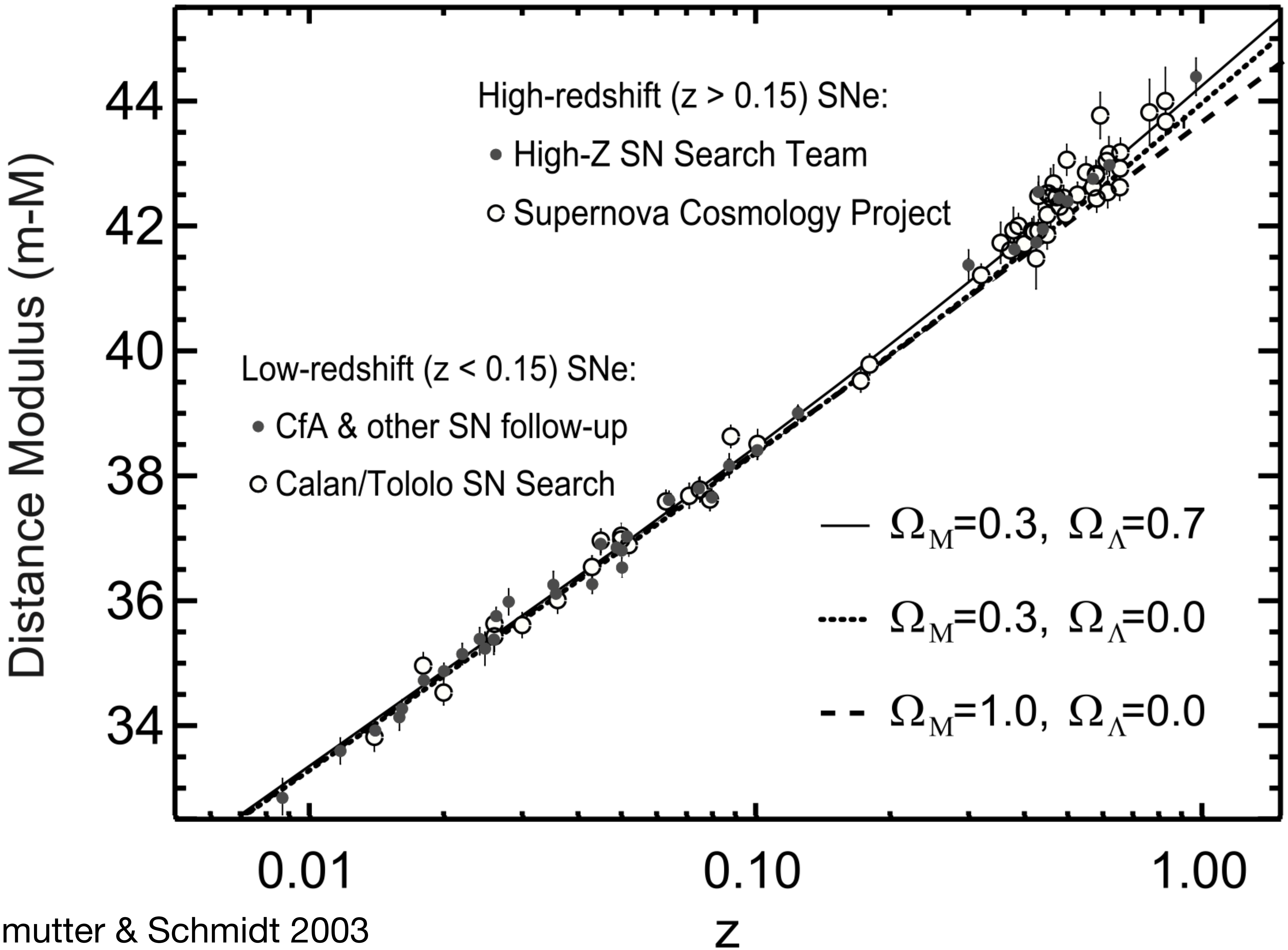


Hubble Diagram for Other Parameters (z > 0.1)

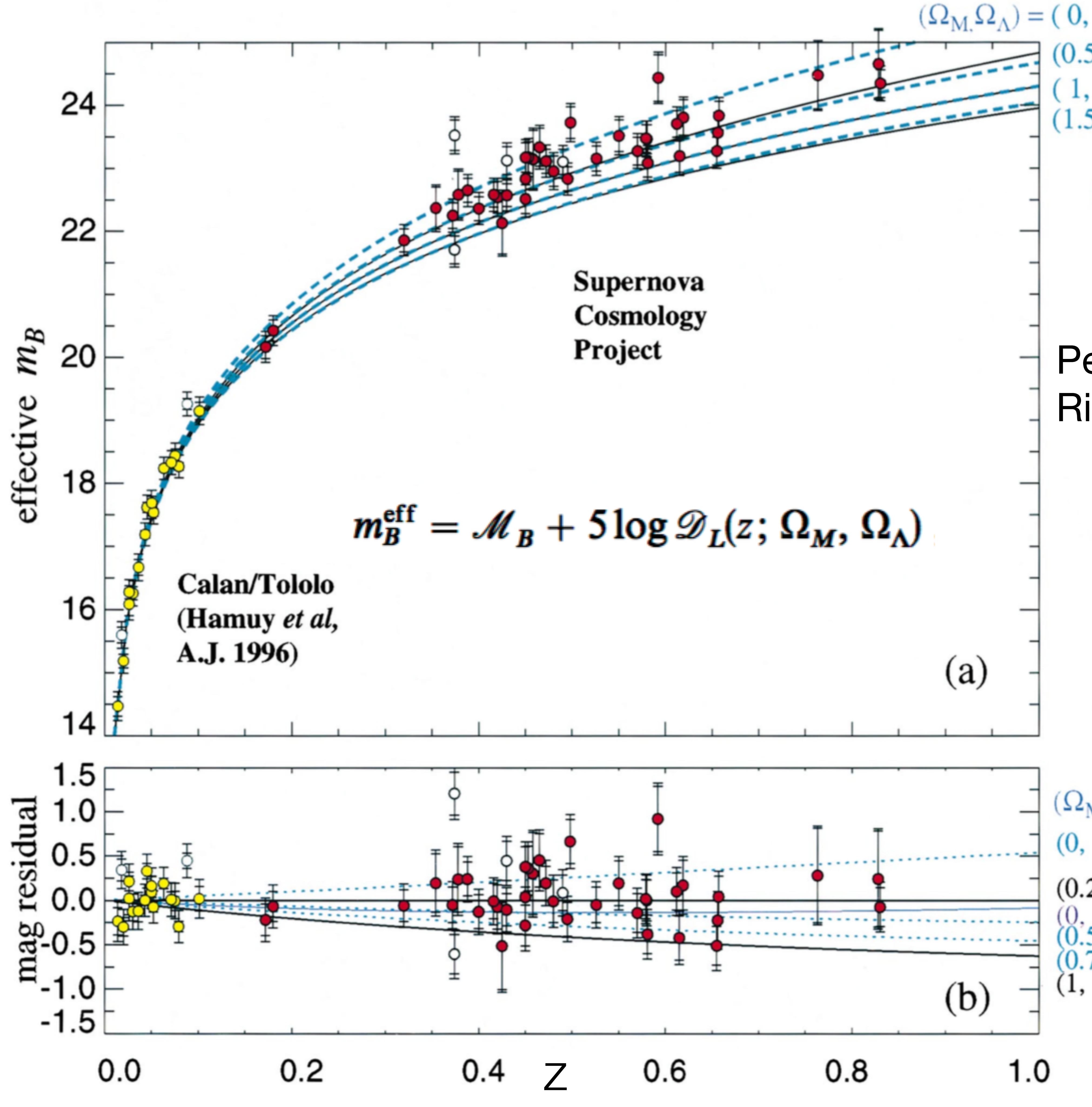


Perlmutter & Schmidt 2003

When we have distance measurement from galaxies at **z > 0.1**, **cosmological density parameters** can be constrained by **the same Hubble diagram**



Perlmutter & Schmidt 2003



Perlmutter+1999 ApJ
Riess+1998 AJ

2011

Table 1.1: Cosmic Inventory

Component	$\mathbf{\Omega}$ (ρ/ρ_c)
Dark Energy	$\mathbf{0.691 \pm 0.006}$
Matter (baryonic and non-baryonic)	$\mathbf{0.312 \pm 0.009}$
Baryons (Total)	$\mathbf{0.0488 \pm 0.0004}$
Baryons in stars and stellar remnants	$\sim \mathbf{0.003}$
Neutrinos	$\sim \mathbf{0.001}$
Photons (CMB)	$\mathbf{5 \times 10^{-5}}$

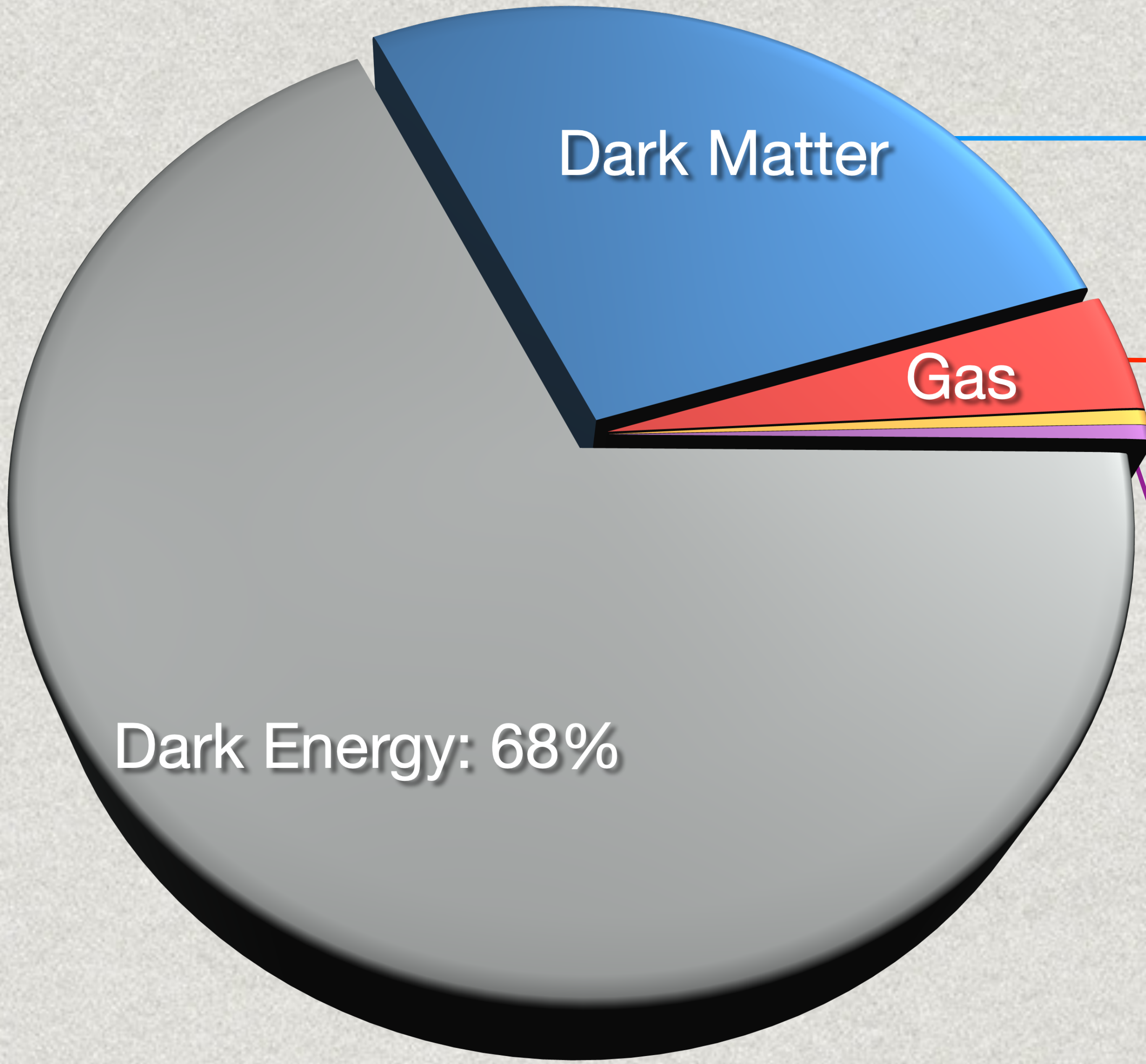
$\Omega_{m,0}, \Omega_{\gamma,0}, \Omega_{\Lambda,0}$

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}$$

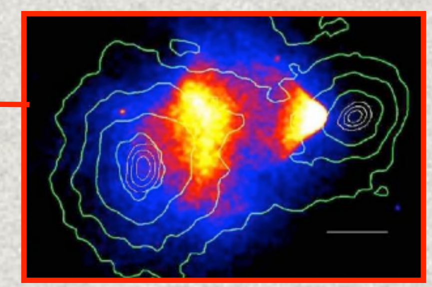
Critical density as a function of time. Value below is present value, based on present value of the Hubble parameter H

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = 9.47x10^{-27}\,kg/m^3$$

Planck Collaboration (2013)

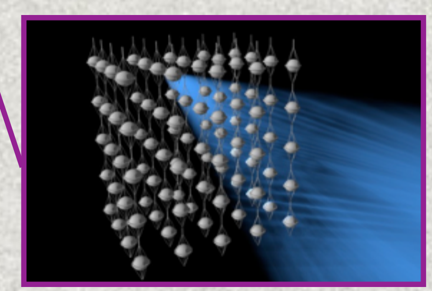


Dark matter: 27%



Gas: 4%

Galaxies: 0.5%



Neutrinos: <0.5%

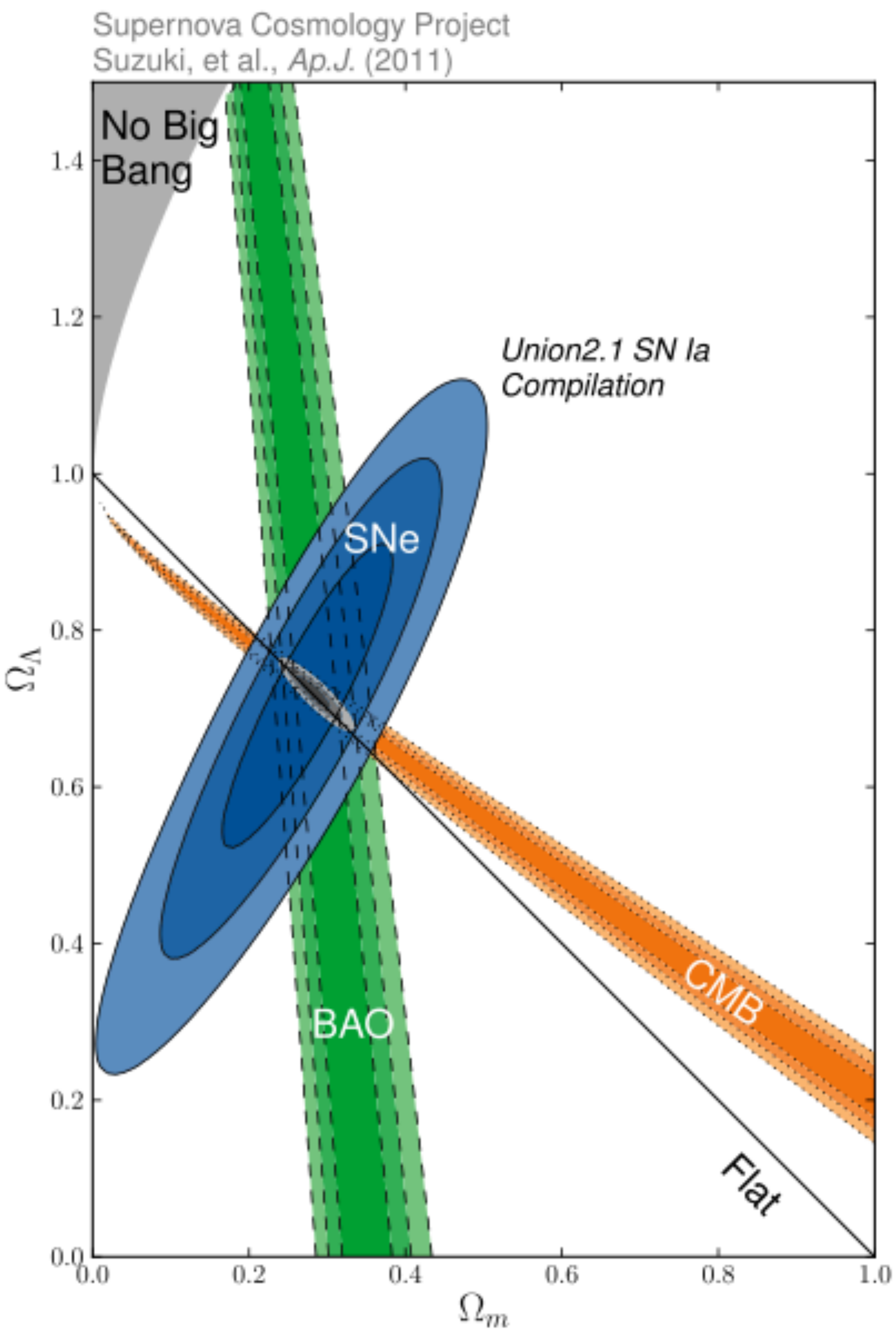


Figure 10.6: Joint likelihood contours (68%, 95%, and 99.7% confidence limits) in the $\Omega_{\rm m,0} - \Omega_{\Lambda,0}$ plane for a recent compilation of SN Ia data, together with the WMAP measure of the temperature anisotropies of the CMB, and the large-scale distribution of galaxies in the nearby Universe (BAO).

If $\Omega_{\Lambda,0} = 0.7$, what's the value of Λ?

$$\Lambda = 3\Omega_{\Lambda,0}H_0^2/c^2 = 1.14 \times 10^{-7}{\rm Mpc}^{-2} = 1.2 \times 10^{-52}{\rm m}^{-2}$$