Chap 15 Fate of Massive Stars ($M > 10 M_\odot$)

Observations:

① Luminous Blue Variables (LBVs), $M > 85 M_\odot$, strongly variable

η Carinae, $120 M_\odot$, $\dot M \sim 10^{-3} M_\odot/yr$ to $0.1 M_\odot/yr$, $L \sim 10^7 L_\odot$, $T \sim 30k$ K

$L_{Eddington} = \frac{4\pi G c}{\kappa} \cdot M < L_{actual}$, Pulsation instabilities, High rotation velocity

② Wolf-Rayet Stars (WRs) $M > 20 M_\odot$, no variability

Broad emission lines, $T \sim 25k$ to $100k$ K, $\dot M \sim 10^{-5} M_\odot/yr$

$V_{wind} = 800 - 3000$ km/s, $V_{rot} \sim 300$ km/s at equator

WN: He & N emission lines → lost H envelope

WC: He & C → lost H & N envelops

WO: O emission lines → lost H, N, C envelops

③ Of (O supergiants w/ pronounced em lines), RSG (Red Supergiant stars)

$\boxed{\text{homunculus} \sim 0.2\text{pc}}$

$\boxed{\dot M_\odot = 10^{-14} M_\odot/yr}$ Solar mass loss

$\boxed{R \propto M^{0.8}}$ M-R relation MS

$\boxed{L \propto M^{3.5}}$

$R_\odot = \frac{1}{215}$ AU

Evolution Scheme

$M > 85 M_\odot$: O → Of → LBV → WN → WC → SN Ic

$40 M_\odot < M < 85 M_\odot$: O → Of → WN → WC → SN Ic

$25 M_\odot < M < 40 M_\odot$: O → RSG → WN → WC → SN Ic

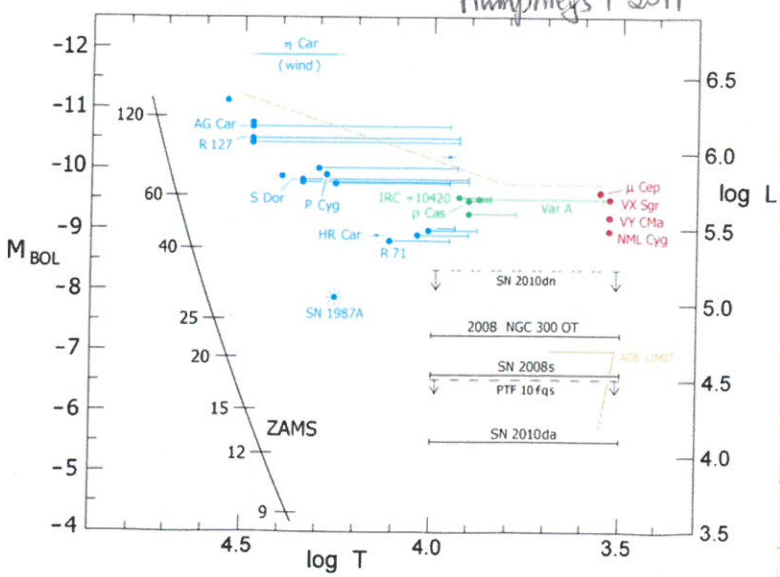
$20 M_\odot < M < 25 M_\odot$: O → RSG → WN → SN Ib

$10 M_\odot < M < 20 M_\odot$: O → RSG → BSG → SN II

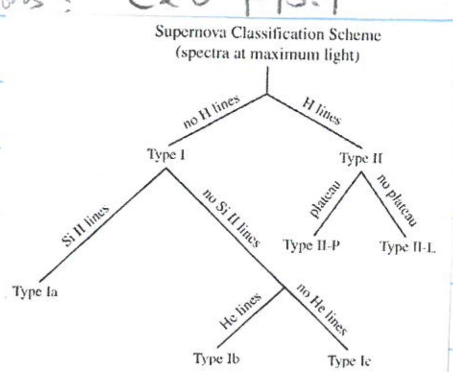
$\boxed{\text{Humphreys – Davidson Luminosity limit (1994)}}$

an empirical luminosity limit observed among LBVs.

Humphreys + 2011



SNe Classifications: C&O F15.9



Core-Collapse SNe: Type II, Ib, Ic
↗ BSG/RSG → Type II, WN → Ib, WC → Ic

Type II: SN 1054 (crab), SN 1987A (LMC)
Type I: Tycho (SN 1572), Kepler (SN 1604)

$E_{tot} \sim 10^{46}\,J \sim 10^{53}$ erg, $E_{kin} \sim 1\%\,E_{tot}$, $E_{rad} \sim 0.01\%\,E_{tot}$

$E_{tot} \sim E_{neutrino}$ $\boxed{E = mc^2 \Rightarrow m = 0.06 M_\odot}$

Energy Source
① To generate 10^{46} J, one needs to form $5.9 M_\odot$ of Fe in the explosion if SNe are driven by nuclear fusion.
② Release of Gravitational Potential Energy

$$E = K + U = -\frac{3}{10}\frac{GM^2}{R}, \quad \boxed{\text{virial theorem } K = -\tfrac{1}{2}U}$$

$$\Delta E = E_i - E_f = \frac{3}{10}\frac{GM^2}{R_f} - \frac{3}{10}\frac{GM^2}{R_i} \simeq \frac{3}{10}\frac{GM^2}{R_f} \text{ for } R_f \ll R_i$$

Given $\Delta E = 10^{46}$ J, $R_f = 50$ km $\Rightarrow M \simeq 2.5 M_\odot$

We only require $R_i \gg R_f$, and even $R_i = R_\oplus = 6000$ km works! As long as we have $2.5 M_\odot$ within $R_\oplus$ (white dwarf).

Core-collapse sequence:

① Onion-like interior with an inert ~~e⁻ degenerate~~ iron core after core Si burning

	Core H → He	He → C	C → O	O → Si	Si → Fe
$M = 20 M_\odot$:	10^7 yr	10^6 yr	300 yr	200 days	2 days

② Photodisintegration in the hot core (10^{10} K)

$${}^{56}_{26}Fe + \gamma \rightarrow 13\,{}^{4}_{2}He + 4n$$
$${}^{4}_{2}He + \gamma \rightarrow 2p^+ + 2n$$

③ electron capture

$$e^- + p^+ \rightarrow n + \nu_e$$

$${}^{56}_{26}Fe + 14\gamma + 26e^- \rightarrow 56n + 26\nu_e$$

$L_{\nu_e} \sim 10^{38}$ W for a $20 M_\odot$ star
in comparision $L \sim 10^{31}$ W for photons

Condition for e⁻ degeneracy:
$P_{degen} > P_{th}$
$\boxed{\hbar^2 \frac{n_e^{5/3}}{m_e} > n_e kT}$
or
$\boxed{\frac{\hbar c\, n_e^{4/3}}{3} > n_e kT}$
↑ relativistic case

④ lost of heat, ideal gas pressure ~~e⁻ degenerate pressure~~ → core collapse (core is not e⁻ degenerate)

⑤ decoupling of outer supersonic outer core ~~shell~~ & inner core

⑥ in ~1 second the inner core of $2.5 M_\odot$ collapse from $1 R_\oplus$ to 50 km!

$$t_{ff} = \sqrt{\frac{3\pi}{32}\frac{1}{G\rho_0}} = 3.8\times10^5\,\text{yr}\left(\rho_0/3\times10^{-17}\,\text{kg m}^{-3}\right)^{-1/2} = 1\,\text{s}$$

$\rho_0 \simeq 4\times10^9$ kg m^{-3}
$\rho_f \simeq 10^{16}$ kg m^{-3}

⑦ Core rebound due to repulsive strong nuclear force @ $R_{core} \sim 12$ km, $\rho = 8\times10^{17}$ kg/m^3
sending pressure wave outward into infalling outer core

⑧ shock wave ~~heats~~ outer iron core, drives further photodisintegration & the shock becomes stationary (ie., accretion shock)

⑨ neutrinos couple with matter behind the shock because it is so dense, the additional energy allows the shock to march outwards.
Without this boost from neutrinos, the outflowing material would fall back to the core, meaning that the explosion is unsuccessful.

⑩ when the foreground gas becomes optically thin, @ $R \sim 200$ AU, a SN is seen.
Peak luminosity $\sim 10^9 L_\odot$, comparible to a small galaxy.

⑪ Remnants:

$M_{ZAMS} < 25 M_\odot \rightarrow$ Neutron star

$M_{ZAMS} > 25 M_\odot \rightarrow$ stellar mass BH.

Light Curves: Radioactive decay from isotopes produced by the shock wave

$$\frac{dN}{dt} = -\lambda N \Rightarrow N(t) = N_0 e^{-\lambda t} = N_0 e^{-\frac{\ln 2}{\tau_{1/2}} t} = N_0 \cdot 0.5^{t/\tau_{1/2}}$$

$$L \propto \frac{dN}{dt} = -\lambda N \quad \frac{dL}{dt} \propto -\lambda \frac{dN}{dt} = +\lambda^2 N$$

$$\frac{d\log L}{dt} = \frac{1}{\ln 10} \frac{d\ln L}{dt} = \frac{1}{\ln 10} \frac{1}{L} \frac{dL}{dt} = \frac{1}{\ln 10} \frac{1}{-\lambda N} \cdot \lambda^2 N = -0.434\lambda$$

$$\frac{dM_{bol}}{dt} = -2.5 \frac{d\log L}{dt} = 1.086\lambda \quad \boxed{\lambda = \frac{\ln 2}{\tau_{1/2}} \simeq \frac{0.7}{\tau_{1/2}}}$$

Isotopes:

$^{56}_{28}Ni \rightarrow ^{56}_{27}Co + e^+ + \nu_e + \gamma$, $\tau_{1/2} = 6.1$ days (Nickel)

$^{56}_{27}Co \rightarrow ^{56}_{26}Fe + e^+ + \nu_e + \gamma$, $\tau_{1/2} = 77.7$ days (Cobalt)

$^{57}_{27}Co$ (271 days), $^{22}_{11}Na$ (2.6 yr), $^{44}_{22}Ti$ (47 yr) (Titanium)

s/r-process
important at A>60

Neutron capture	$^{A}_{Z}X + n \rightarrow ^{A+1}_{Z}X + \gamma$	$s \rightarrow t(n\ cap) \gg t(\beta\ decay)$
β decay	$^{A+1}_{Z}X \rightarrow ^{A+1}_{Z+1}X + e^- + \bar{\nu}_e + \gamma$	$r \rightarrow t(n\ cap) \ll t(\beta\ decay)$

Cosmic Rays : $E \sim 10\,\text{MeV}$ to $10^{20}\,\text{eV}$, flux $(m^2\, s\, \text{GeV})^{-1} \propto E^{-3}$

flux @ $10^{20}\,\text{eV}$ is less than 1 particle / km^2 / 100 yr

Solar wind particles not energetic enough → $E < 10\,\text{MeV}$ for proton @ $0.1c$

SNe acceleration important for $E \lesssim 10^{16}\,\text{eV}$ (note proton rest energy ~ 1 GeV)

Lorentz force: $F_B = q v B_\perp = \dfrac{\gamma m v^2}{r}$

⇒ Larmor radius / gyroradius $r = \dfrac{\gamma m v}{qB}$

when $v \sim c$, $r = \dfrac{\gamma m c^2}{qcB} = \dfrac{E}{qcB} = 1\,\text{pc}\, \dfrac{E}{10^{15}\,\text{eV}} \left(\dfrac{B}{10^{-10}\,\text{T}}\right)^{-1}$

$1\,\text{T} = 10^4$ Gauss , $10^{-10}\,\text{T} = 1\,\mu\text{G}$, Earth's surface ~ 0.5 G

this indicates that only cosmic rays with $E \lesssim 10^{15+6}\,\text{eV}$ can be confined within SNe

Convective Cores : convection occurs when $\left|\dfrac{dT}{dr}\right|_{\text{actual}} > \left|\dfrac{dT}{dr}\right|_{\text{adiabatic}}$

CNO cycle's extreme sensitivity to T cause large $\dfrac{dT}{dr}$ in the core, making the cores fully convective

SN effect on galaxies : $\dfrac{GM^2}{R} = 10^{46}\,\text{J} \left(\dfrac{M}{10^8 M_{\text{sun}}}\right)^2 \left(\dfrac{10\,\text{kpc}}{R}\right) \sim E_{SN}$

Summary : Massive stars → $10\,M_\odot < M < 150\,M_\odot$

$\boxed{\text{MS}}$: $L \propto M^{3.5}$ luminous ; $R \propto M^{0.8}$ large ; $\tau_{MS} \propto M^{-2.5}$ short-lived

↓

$\boxed{\text{RSG/BSG, LBV/WR, WN/WC/WO}}$: inert Fe core [note e^- degenerate] ; H → He → C → O → Si → Fe shells ; $dM/dt \sim 10^{-5 \to -3}\, M_\odot/\text{yr}$

↓

$\boxed{\text{SN: II, Ia, Ib}}$ — $\boxed{\text{Core collapse driven by heat loss, gravitational energy } E \sim 10^{46}\,\text{J}}$

↓

$\boxed{\text{Remnants}}$ → Pulsars (why obey Hubble flow?)

$Fe + \gamma \to He + n$; $He + \gamma \to p^+ + n$ } Heat loss when $T_{core} \sim 10^{10}\,\text{K}$; photodisintegration

$p^+ + e^- \to n + \nu_e$ } electron capture

core collapse in $\frac{1}{2}$ freefall time $= (G\rho_0)^{-\frac{1}{2}} \sim 1\,\text{s}$

$\Delta E = \dfrac{GM_{core}^2}{R_f} = 10^{46}\,\text{J}\, \dfrac{(M/2.5M_\odot)^2}{(R_f/50\,\text{km})}$

$\rho \to 10^{18}\,\text{kg/m}^3$, $R_{core,\min} \sim 12\,\text{km}$

shock wave propagates outwards & stalls due to energy loss to photodisintegration → ν_e boost

$dm/dt \sim 1/\tau_{1/2}$ light curve, radioactive ^{56}Ni : ^{56}Co

Chap 4 Special Relativity

Einstein (1905) Postulates:

① Principle of relativity: the laws of physics hold in all inertial reference frames

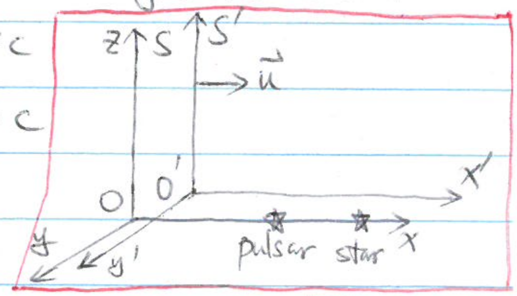
② Speed of light is constant in vacuum regardless of the motion of the light source

Lorentz transformation

$$x' = \frac{x-ut}{\sqrt{1-u^2/c^2}}, \quad t' = \frac{t-ux/c^2}{\sqrt{1-u^2/c^2}} \quad y'=y, \; z'=z$$

for a reference frame S' moving in x-direction with velocity $\vec{u}$ relative to S

Lorentz factor: $\gamma \equiv \frac{1}{\sqrt{1-u^2/c^2}} = \begin{cases} 1.0 & \text{when } u \ll c \\ \infty & \text{when } u \to c \end{cases}$



Time dilation: (clock runs slower)

$$t_2' - t_1' = \frac{(t_2-t_1)-(x_2-x_1)u/c^2}{\sqrt{1-u^2/c^2}}$$

Note the difference between this and the Doppler equation $\frac{\Delta t_{obs}}{\Delta t_{rest}} = 1+z$

for a pulsar ~~at~~ rest with S, $x_2 = x_1$, $\Delta t_{rest} = t_2 - t_1 = \boxed{\text{proper time}}$

$$\Rightarrow \Delta t_{moving} = \frac{\Delta t_{rest}}{\sqrt{1-u^2/c^2}} = \gamma \, \Delta t_{rest}$$

Can an observer sees an reversal of a time sequence? i.e. $\Delta t_{moving} < 0$?

That would require $\Delta t - \Delta x \cdot u/c^2 < 0 \Rightarrow \frac{\Delta x}{\Delta t} \cdot u > c^2 \Rightarrow \frac{\Delta x}{\Delta t} > c$

That would happen only when the two events are not related.

Length contraction: (textbook derivation incorrect)

$$\begin{cases} x_2' - x_1' = \gamma \left[(x_2 - x_1) - u(t_2 - t_1) \right] = L_{moving} \\ t_2' - t_1' = \gamma \left[(t_2 - t_1) - (x_2 - x_1) u/c^2 \right] = 0 \\ x_2 - x_1 = L_{rest} \end{cases}$$

$$\Rightarrow L_{moving} = \gamma \left(L_{rest} - L_{rest} \cdot \frac{u^2}{c^2} \right) = L_{rest} \cdot \frac{1}{\gamma}$$

Example: muon decays $N(t) = N_0 \cdot e^{-t/\tau}$ where τ is the e-folding decay time, not the half life $\tau_{1/2} = \ln 2 \cdot \tau$

$\tau = 2.2\,\mu s$ at rest,

if moving at $u = 0.9952c$, $\gamma = 10 \Rightarrow \Delta t_{moving} = \gamma \cdot \tau = 22\,\mu s$

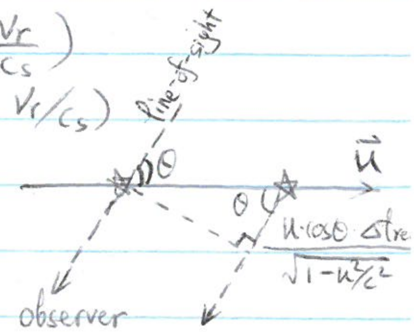
this is compatible with length contraction in the reference frame of muon

Relativistic Doppler Shift:

classic Doppler: $\dfrac{\Delta\lambda}{\lambda_{rest}} = \dfrac{V_r}{C_s}$ for sound waves or light waves due to geometrical effect

$$\frac{\lambda_{obs} - \lambda_{rest}}{\lambda_{rest}} = \frac{V_r}{C_s} \Rightarrow \lambda_{obs} = \lambda_{rest} \cdot \left(1 + \frac{V_r}{C_s}\right)$$

$$\nu_{obs} = \nu_{rest} / (1 + V_r/C_s)$$



taking into account time dilation:

$$\Delta t_{obs} = \frac{\Delta t_{rest}}{\sqrt{1-u^2/c^2}} + u \cdot \frac{\Delta t_{rest}}{\sqrt{1-u^2/c^2}} \cdot \cos\theta \cdot \frac{1}{c}$$

$$\Rightarrow \quad \nu_{obs} = \frac{\nu_{rest}\sqrt{1-u^2/c^2}}{1 + V_r/c}, \quad \text{where } V_r = u \cdot \cos\theta \text{ the radial velocity}$$

or

$$\lambda_{obs} = \lambda_{rest} \cdot \frac{1 + V_r/c}{\sqrt{1-u^2/c^2}}$$ notice when $V_r = 0$, there is still transverse Doppler

define redshift: $z = \dfrac{\lambda_{obs} - \lambda_{rest}}{\lambda_{rest}} = \dfrac{\Delta\lambda}{\lambda_{rest}}$

for classic Doppler: $z = V_r/c$

for relativistic Doppler: $z = \dfrac{1 + V_r/c}{\sqrt{1-u^2/c^2}} - 1 \Rightarrow \boxed{1 + z = \dfrac{\Delta t_{obs}}{\Delta t_{rest}} = \dfrac{1 + V_r/c}{\sqrt{1-u^2/c^2}}}$

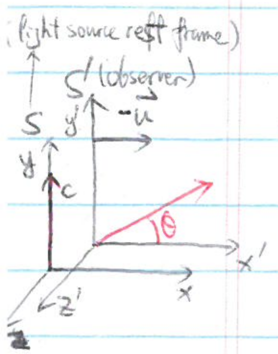
if purely radial velocity, $|u| = |V_r|$

$$z = \sqrt{\frac{1 + V_r/c}{1 - V_r/c}} - 1$$

Relativistic beaming effect:

take dt' to the Lorentz transformation, we have the velocity transformation:

$$\frac{dx'}{dt'} = V_x' = \frac{V_x - u}{1 - uV_x/c^2}, \quad V_y' = \frac{V_y\sqrt{1-u^2/c^2}}{1 - uV_x/c^2}, \quad V_z = \frac{V_z\sqrt{1-u^2/c^2}}{1 - uV_x/c^2}$$



apply to light rays traveling in y direction in rest frame of the light source

the light source is traveling towards you

$$\begin{cases} V_x = 0 \\ V_y = c \\ V_z = 0 \end{cases} \xrightarrow{\vec{u} = -u \cdot \vec{x}} \begin{cases} V_x' = u \\ V_y' = \dfrac{c \cdot \sqrt{1-u^2/c^2}}{1 + uV_x/c^2} = c \cdot \sqrt{1-u^2/c^2} \Rightarrow \sin\theta = \dfrac{V_y}{c} = \dfrac{1}{\gamma} \\ V_z' = 0 \end{cases}$$

Relativistic momentum & energy

$$E_{rest} = mc^2 \quad , \quad E_{total} = \gamma mc^2 = \sqrt{p^2c^2 + m^2c^4}$$

$$\vec{p} = \gamma m \vec{v} \leftarrow \text{a result of momentum conservation}$$

Derivation: imagine an object with mass m initially at rest, it feels an external force $\vec{F}$ for a period of time t_i to t_f, ~~the force is constant~~. Then the final kinetic energy

$$K = \int_{x_i}^{x_f} F \cdot dx = \int_{x_i}^{x_f} \frac{dp}{dt} dx = \int_{p_i}^{p_f} v \, dp \quad \left(\text{integration by parts}\right)$$

$$= vp \Big|_{p_i}^{p_f} - \int_{v_i}^{v_f} p \, dv = p_f \cdot v_f - \int_0^{v_f} \frac{mv}{\sqrt{1 - v^2/c^2}} dv$$

$$= \frac{m v_f^2}{\sqrt{1 - v_f^2/c^2}} + mc^2 \left(\sqrt{1 - v_f^2/c^2} - 1\right)$$

$$= mc^2 (\gamma - 1)$$

since $K = E_{total} - E_{rest}$, we can define $E_{rest} = mc^2$

$E_{total} = \gamma mc^2$

$$p^2c^2 + m^2c^4 = \frac{m^2v^2c^2}{1 - v^2/c^2} + m^2c^4 = \frac{m^2c^4}{1 - v^2/c^2} = (\gamma mc^2)^2 = E_{total}^2$$

therefore, $E^2 = (\gamma mc^2)^2 = p^2c^2 + m^2c^4$

Chap 17 General Relativity & Black Holes

Problem of Special Relativity:

An inertial reference frame cannot be defined in the presence of gravity, because gravity is equivalent to acceleration

$$a_g = G\frac{M_g}{r^2}\frac{m_g}{m_i} = G\frac{M_g}{r^2}$$

Principle of Equivalence: Your common sense only works in LIRFs.

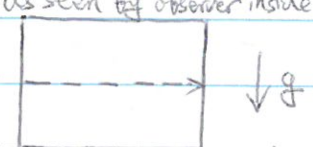
All local, freely falling, nonrotating labs are all equivalent.

These labs are $\boxed{\text{local inertial reference frames}}$ & special relativity can be applied to two local inertial reference frames using the instantaneous relative velocity

Applications #1 Bending of light

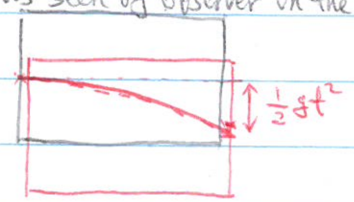
Remember only freely falling labs are inertial reference frames

as seen by observer inside lab



freely falling, thus a local inertial ref.

as seen by observer on the ground

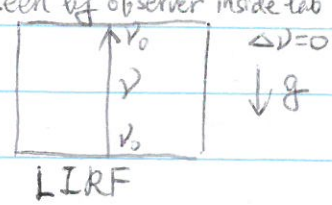


$t = l/c$

$$\Rightarrow r_c = \frac{c^2}{g} = 9\times10^{15}\,m \simeq 1\,ly$$

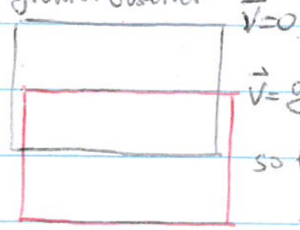
Application #2 Gravitational Redshift

seen by observer inside lab



LIRF

ground observer



blueshift: $\frac{\Delta\nu}{\nu_0} = \frac{v}{c} = \frac{gh}{c^2}$

so there must be a redshift to compensate it

$$\frac{\Delta\nu}{\nu_0} = -\frac{gh}{c^2}$$

When the photon travels to infinity distance from r_0, energy conservation states:

$$h\nu_\infty = h\nu_0 - \frac{GM h\nu_0}{r_0 c^2} \Rightarrow \frac{\nu_\infty}{\nu_0} = 1 - \frac{GM}{r_0 c^2} < 1$$

full gr result: $$\frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0 c^2}\right)^{1/2} \quad [17.10]$$

time dilation: $$\frac{\Delta t_0}{\Delta t_\infty} = \frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0c^2}\right)^{1/2} < 1 \Rightarrow \Delta t_0 < \Delta t_\infty$$

looks like time contraction:

time measured from a LIRF

time observed from infinite distance, observed time @ r_0

Δt_∞ = time running @ r_0 but observed @ r_∞

Δt_0 = time at infinite distance, r_∞

Einstein's Field Equation:

Cosmological constant

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

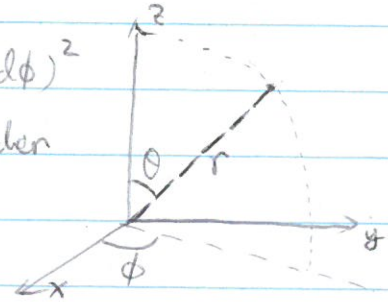
Rocci Curvature tensor — Rocci Scaler — Metric tensor — energy-momentum stress tensor

Schwarzschild metric (curved spacetime around a spherical mass M):

$$(ds)^2 = \left(c\,dt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

in comparison, for flat spacetime

$$(ds)^2 = (c\,dt)^2 - (dr)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$



where ds is the spacetime metric ~~interval~~, which is invariant under Lorentz transformation

Spacetime Interval:

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta l)^2$$

because Δs is invariant under Loretz transformation, we can define

$\Delta \tau \equiv \frac{\Delta s}{c}$ the proper time (time between two events that occur at same locati

$\Delta \mathcal{L} \equiv \sqrt{-\Delta s^2}$ proper distance (distance between two events that occur simultaneously

the former require $(\Delta s)^2 > 0$ timelike, the latter require $(\Delta s)^2 < 0$ spacelike

Maximum interval is achieved along a straight timelike ~~int~~ worldline between two events.

In curved spacetime, the "staightest possible worldlines" are called geodesics.

Δs along a timelike geodesic is an extremum, $\Delta S \equiv \int_A^B \sqrt{(ds)^2}$, $\frac{d}{dr}\Delta s = 0$

photons always follow a null geodesic, $\Delta s = \int \sqrt{(ds)^2} = 0$ because $(ds)^2 = 0$ for photons

freely-falling objects follow geodesics, $\Delta s > 0$

Proper distance & Proper time in Schwarzschild metric, dr & dt are coordicate radius & time

$$d\mathcal{L} = \sqrt{-(ds)^2} = \frac{dr}{\sqrt{1-2GM/rc^2}}, \quad d\tau = \frac{ds}{c} = dt\sqrt{1-2GM/rc^2}$$

$\Rightarrow d\mathcal{L} = dr$ @ $r \to \infty$

$d\mathcal{L} > dr$ @ finite r

$\Rightarrow d\tau = dt$ @ $r \to \infty$

$d\tau < dt$ @ finite r

since $\frac{\Delta t_0}{\Delta t_\infty} = \sqrt{1-2GM/rc^2}$, $dt = \Delta t_\infty$

coordinate time is observed time @ ∞ distance

Application of Schwarzschild metric : the orbit of a satellite

assume circular orbit, use metric to find relation between v & r

$$(ds)^2 = \left[\left(c\sqrt{1-2GM/rc^2}\right)^2 - r^2\left(\frac{d\theta}{dt}\right)^2\right] dt^2$$
$$= \left(c^2 - \frac{2GM}{r} - r^2\omega^2\right) dt^2$$

$$\Delta S = \int_0^{2\pi/\omega} \sqrt{ds^2} = \int_0^{2\pi/\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\, dt \sim f(r)$$

$$\frac{d}{dr}(\Delta S) = 0 \Rightarrow \frac{d}{dr}\sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2} = 0 \Rightarrow \frac{2GM}{r^2} - 2r\omega^2 = 0$$

$$\Rightarrow v^2 = \omega^2 r^2 = \frac{GM}{r}$$

Black Holes :

$1 - 2GM/rc^2 = 0 \Rightarrow R_s = \frac{2GM}{c^2}$ | the Schwarzschild radius of a non-rotating black hole |

proper time @ $r = R_s$, $d\tau = dt\sqrt{1-2GM/rc^2}$

goes at infinite speed! $d\tau = 0$ everything happens simultaneously!

To an observer @ $r = \infty$, $\Delta t_\infty = \Delta t_0 / \sqrt{1-2GM/rc^2} = \infty$, i.e., time stopped running

For photons at the Schwarzschild radius, $(ds)^2 = 0 \Rightarrow$ light is frozen

$$0 = \left(c\,dt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2$$
$$\frac{dr}{dt} = c\cdot\left(1 - \frac{2GM}{rc^2}\right) = c\left(1 - \frac{R_s}{r}\right) = 0 \quad @\ r = R_s$$

A photon's trip into a black hole

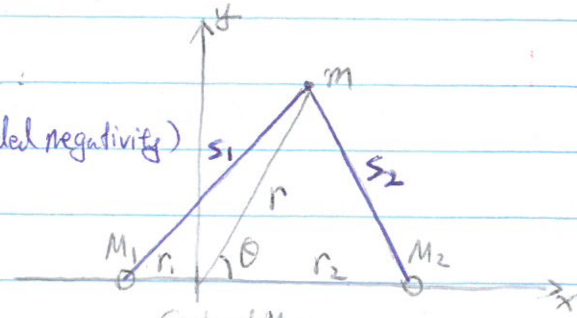
$$\Delta t = \int_{r_1}^{r_2} \frac{dr}{dr/dt} = \int_{r_1}^{r_2} \frac{dr}{c\left(1 - \frac{2GM}{rc^2}\right)} = \frac{r_2 - r_1}{c} + \frac{R_s}{c}\ln\left(\frac{r_2 - R_s}{r_1 - R_s}\right)$$

$\Delta t = \infty$ when $r_1 = R_s$ so to an observer at $r = \infty$, he never sees the photon reaches the event horizon.

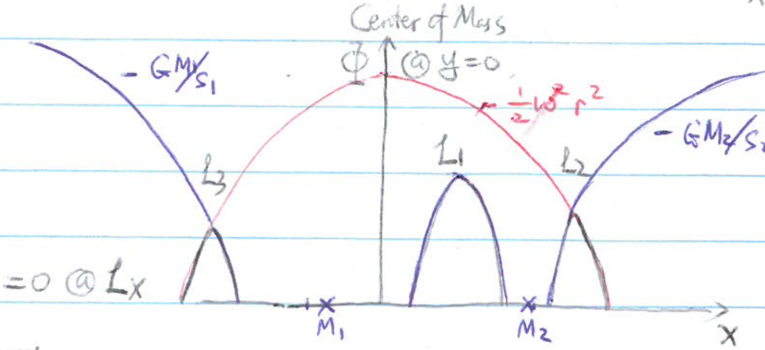
Chap 18 Close Binary Star Systems

Effective Gravitational Potential in a corotating frame:

$$\Phi = -G\left(\frac{M_1}{s_1} + \frac{M_2}{s_2}\right) - \frac{1}{2}\omega^2 r^2 \quad \text{(added negativity)}$$



$$\begin{cases} s_1^2 = r_1^2 + r^2 + 2r_1 r\cos\theta \\ s_2^2 = r_2^2 + r^2 - 2r_2 r\cos\theta \\ \omega^2 = \left(\frac{2\pi}{P}\right)^2 = \frac{G(M_1+M_2)}{a^3} \\ r_1 + r_2 = a \\ M_1 r_1 = M_2 r_2 \end{cases}$$



$$\vec{F} = -m\nabla\Phi \iff \vec{a} = -\nabla\Phi = 0 \ @ \ L_x$$

Centripedal force → centrifugal potential energy

$$U_f - U_i = \Delta U_c = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_c \cdot d\vec{r} = -\int_{r_i}^{r_f} m\omega^2 r\,dr = -\frac{1}{2}m\omega^2(r_f^2 - r_i^2)$$

define $U_c = 0$ at $r=0$, $\Rightarrow U_c = -\frac{1}{2}m\omega^2 r^2$

$$\Phi_c = -\frac{1}{2}\omega^2 r^2$$

Earth-Sun L_2 Point: 1.5×10^6 km from the Earth

Geosynchronous orbit: 4.2×10^4 km from the Earth

Earth-Moon distance: 3.8×10^5 km (1/4 the distance to L_2)

Equipotential Surface: Φ = const.

$-\nabla\Phi = \vec{a}$ is always perpendicular to the surface, i.e., transverse $\vec{a}_{\parallel} = 0$

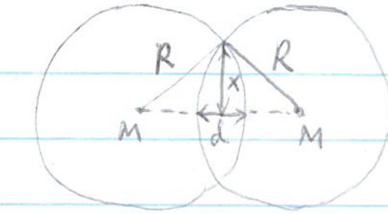
P is constant along the surface because $\nabla P = -\rho\vec{a} \Rightarrow \nabla P_{\parallel} = 0$

ρ is constant along the surface because $P = \frac{\rho k T}{\mu m_p}$

Types of Binaries: detached, semidetached, contact binaries

depending on if the Roche Lobes are filled

Mass transfer rate for equal mass binaries



$$\dot{M} = \rho\, v_{rms} \cdot \pi x^2 \simeq \pi \rho R d \sqrt{\frac{3kT}{\mu m_p}}$$

Solar wind $10^{-14}\, M_\odot/yr$

$$\simeq 10^{-11} - 10^{-7}\, M_\odot/yr$$

$x = \sqrt{Rd}$ when $d \ll R$, because $x^2 + (R - \frac{1}{2}d)^2 = R^2$

$\Rightarrow x^2 + R^2 \cdot (1 - \frac{1}{2}\frac{d}{R})^2 = R^2$ → small number approximation

$\Rightarrow x^2 + R^2(1 - d/R) = R^2 \Rightarrow x^2 = Rd$

Accretion Disks : define energy generation efficiency as $\eta = \frac{\Delta E}{mc^2}$

$\eta_{pp} = 0.007$ for nuclear fusion, p-p chain.

for graviational energy release, we have

$$\Delta E = -U = G\frac{Mm}{R} \Rightarrow \eta = \frac{GM}{Rc^2} = 0.148\left(\frac{M/1M_\odot}{R/10\,km}\right) \gg \eta_{pp}$$

T profile for an optically thick, steady state disk

Energy radiated away ≡ Potential Energy Loss

$$dE = \frac{dE_t}{dr}dr = \frac{d}{dr}\left[-G\frac{M(\dot{M}t)}{2r}\right]dr = G\frac{M\dot{M}t}{2r^2}dr \equiv dL\, t$$

where we used virial theorem $E = K + \Phi = \frac{1}{2}\Phi$ ⇐ how did the mass reach negative total energy? Hot spot!

$$dL = 4\pi r \sigma T^4 \cancel{dr} \equiv \frac{GM\dot{M}}{2r^2}\cancel{dr}$$

$$\Rightarrow T = \left(\frac{GM\dot{M}}{8\pi\sigma r^3}\right)^{1/4} \propto \left(\frac{R}{r}\right)^{3/4}$$

define $T_c = \left(\frac{GM\dot{M}}{8\pi\sigma R^3}\right)^{1/4} = \frac{1}{1.32} T_{disk}$ (Eq. 18.20) $= 1.58\, T_{max}$

$$= 10^7 K\, (M/1.4M_\odot)^{1/4}\, (\dot{M}/1.6\times10^{-9}M_\odot/yr)^{1/4} \cdot (R/10km)^{-3/4}$$

total luminosity $L = \int dL = \int_R^\infty \frac{GM\dot{M}}{2r^2}dr = \frac{GM\dot{M}}{2R}$ $\left[\int x^{-2}dx = -\frac{1}{x}\right]$

$$\Rightarrow \eta = \frac{GM}{2Rc^2} = 0.074\,\frac{M/1M_\odot}{R/10km}$$

Disk size: initial angular momentum $L = m\omega l_1^2 = m l_1^2\sqrt{\frac{G(M_1+M_2)}{a^3}}$ @ L1 point

once it reaches the outermost circular orbit $L = m r_c^2\sqrt{\frac{GM_1}{r_c^3}} = m\sqrt{GM_1 r_c}$

equating the two, $r_c = \frac{l_1^4}{a^3}\cdot\frac{M_1+M_2}{M_1} \simeq \frac{1}{2}R_{disk}$ due to disk spread

Effects of mass transfer on orbital dynamics (ignore accretion disk, just consider mass redistribution)

total angular momentum

$$L = M_1 \omega R_1^2 + M_2 \omega R_2^2 \quad , \quad R_1 + R_2 = a \, , \; \boxed{M_1 R_1 = M_2 R_2}$$

$$= M_1 R_1 \omega \cdot (R_1 + R_2) \qquad \omega = \sqrt{\frac{G(M_1+M_2)}{a^3}} \qquad \boxed{M_1 R_1 = \frac{M_1 M_2}{M_1+M_2} \cdot a}$$

$$= M_1 R_1 \sqrt{\frac{G(M_1+M_2)}{a^3}} \cdot a$$

$$= \frac{M_1 M_2 a}{M_1+M_2} \cdot a \cdot \sqrt{\frac{G(M_1+M_2)}{a^3}} = \mu \sqrt{GMa} \quad , \; \mu = \frac{M_1 M_2}{M_1 + M_2}$$

take time derivative, and we know $dL/dt = 0$

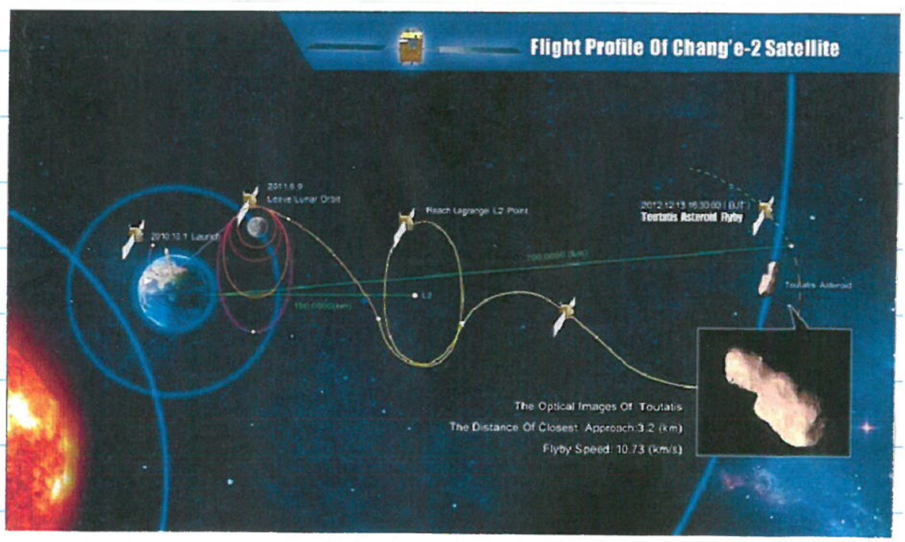
$$\frac{dL}{dt} = \sqrt{GM} \cdot \left(\frac{d\mu}{dt} \sqrt{a} + \frac{\mu}{2\sqrt{a}} \frac{da}{dt} \right) = 0$$

$$\Rightarrow \quad \frac{1}{a}\frac{da}{dt} = -\frac{2}{\mu}\frac{d\mu}{dt}$$

$$\frac{d\mu}{dt} = \frac{1}{M}\left(\frac{dM_1}{dt} M_2 + \frac{dM_2}{dt} M_1 \right) \quad , \text{ given that } \dot{M}_1 = -\dot{M}_2$$

$$\Rightarrow \quad \frac{1}{a}\frac{da}{dt} = 2\dot{M}_1 \frac{M_1 - M_2}{M_1 M_2} \begin{cases} < 0 \text{ when } \dot{M}_1 < 0 \, , \; M_1 > M_2 \; \boxed{\text{more likely}} \\ > 0 \text{ when } \dot{M}_1 < 0 \, , \; M_1 < M_2 \end{cases}$$

$$\omega \propto a^{-3/2} \quad \Rightarrow \quad \frac{1}{\omega}\frac{d\omega}{dt} = -\frac{3}{2}\frac{1}{a}\frac{da}{dt} \, , \text{ since } \frac{d\omega}{dt} = -\frac{3}{2} a^{-\frac{5}{2}} \frac{da}{dt}$$



WDs in semidetached binaries

① Cataclysmic Variables ② SN Ia

i.e. $\boxed{\text{Dwarf novae}}$ or $\boxed{\text{classical novae}}$

$\sim 0.86 M_\odot$ WD + less massive secondary

$P \sim$ hour to a week

Bursts due to $\dot{M} \uparrow$, brightening of accretion disk

DN: 2–6 mag brightening

CN: 7–20 mag → H-burning in CNO cycle at the WD surface $M_{layer} \sim 10^{-4} M_\odot$

CN: Ejection (Hydrodynamic, $> L_{Edd}$) → burning (Hydrostatic, L_{Edd}) → fireball expansion phase ($T \sim 9000$ K, $R \sim 1/3$ AU) → optically thin → dust formation ($T \sim 1000$ K)

Neutron star & BHs in Binaries

X-ray binaries

① Core collapse SNe in binaries
- ① unbound system if $\Delta M > \frac{1}{2} M_{total}$
- ② bound system if $\Delta M < \frac{1}{2} M_{total}$

② Tidally captured neutron stars, likely in globular clusters

③ Three body captured neutron stars

④ Thorne-Zytkow objects : penetration of NS into a giant star

Accreting Neutron Stars

Magnetic field channelling at $r < r_A$, the Alfvén radius

$$\frac{1}{2}\rho v^2 = \frac{B^2}{2\mu_0}$$, kinetic E ~ magnetic energy density

For free fall from ∞ distance

$$v^2 = 2GM/r$$

For spherical accretion: $\dot{M} = 4\pi r^2 \rho v$

Magnetic dipole strength: $B(r) = B_s\left(\frac{R}{r}\right)^3$, B_s is surface field strength, R is NS radius

Eleminate ρ & v from the first equation, we can solve for r

$$r = r_A = \left(\frac{8\pi^2 B_s^4 R^{12}}{\mu_0^2 G M \dot{M}^2}\right)^{1/7} = 3000 \text{ km for } 1.4M_\odot \text{ NS}, \ R = 10 \text{ km}$$

$$\dot{M} = 10^{-9} M_\odot/yr, \ B_s = 10^8 T$$

Double NS Binaries. Period decay due to gravitational quadruple radiation

$$\dot{P}_{orb} = -\frac{96}{5}\frac{G^3 M^2 \mu}{c^5}\left(\frac{4\pi^2}{GM}\right)^{4/3}\frac{f(e)}{P_{orb}^{5/3}} \sim -2.4 \times 10^{-12}$$

indirect GW detection in 1984! 30 years before direct LIGO detection.

Eddington Luminosity $$L_{Edd} = \frac{4\pi G M c}{\kappa} = \frac{4\pi G M m_p c}{\sigma_T} = 3\times10^4 L_\odot\left(\frac{M}{M_\odot}\right)$$

so the observed luminosity of X-ray binaries provides an lower limit on mass

ULXBs are thus normally considered ~~as~~ stellar mass BH candidates.

Fastest Growth Rate - Accreting at the Eddington Limit

① Energy conservation & definition of ϵ, the radiative efficiency

$$\dot{M}_{\cancel{tot}} c^2 = \dot{M} c^2 + \epsilon \dot{M}_{tot} c^2$$

$$L = \epsilon \dot{M}_{tot} c^2$$

$$\Rightarrow \dot{M} c^2 = \frac{1-\epsilon}{\epsilon} L \quad \Rightarrow L = \frac{\epsilon}{1-\epsilon} \dot{M} c^2$$

② Eddington luminosity

$$L_{Edd} = \frac{4\pi G M m_p c}{\sigma_T} = \frac{M c^2}{\sigma_T c / 4\pi G m_p} = \frac{M c^2}{t_{Edd}}$$

$$t_{Edd} = \frac{\sigma_T c}{4\pi G m_p} = 450 \text{ Myr}$$

, $\sigma_T = 6.65 \times 10^{-25}$ cm^2 Thomson scattering

$$\sigma_T = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0 m c^2}\right)^2 \propto \frac{q^4}{m^2}$$

$h\nu \ll mc^2$

③ If accreting at Eddington luminosity,

$$\eta = L / L_{Edd} = 1$$

$$\frac{\epsilon}{1-\epsilon} \dot{M} c^2 = \eta \cdot \frac{M c^2}{t_{Edd}}$$

$$\frac{dM}{M} = \eta \cdot \frac{(1-\epsilon)}{\epsilon} \frac{dt}{t_{Edd}}$$

④ Solving the differential Equation

$$M(t) = M_0 \cdot \exp\left[\eta \frac{(1-\epsilon)}{\epsilon} \frac{t}{t_{Edd}}\right]$$

so the e-folding timescale is

$$\tau = t_{Edd} \cdot \frac{\epsilon}{1-\epsilon} \cdot \frac{1}{\eta} = 50 \text{ Myr for } \epsilon = 0.1, \eta = 1$$

the 10-folding timescale is

$$\tau_{10} = \ln 10 \cdot \tau = 115 \text{ Myr}$$

derivation: $e^{t/\tau} = (e^{\ln 10})^{t/\tau \cdot \ln 10} = 10^{\frac{t}{\tau \cdot \ln 10}}$

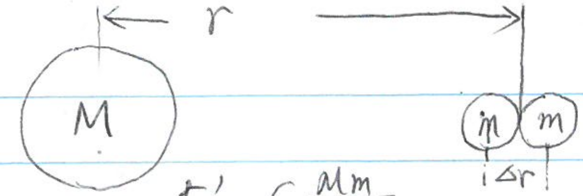
⑤ A black hole of $10^9 M_\odot$ just 1 Gyr after Big Bang, what's the seed BH mass?

$$M_0 = \frac{M(t)}{10^{t/\tau_{10}}} = 10^{9-8.7} = 10^{0.3} M_\odot = 2 M_\odot$$

What about a 12 billion $M_\odot$ BH? $M_0 = 24 M_\odot$

Wu et. al. (2015) reported a quasar at $z = 6.4$ (890 Myr after BB) with a BH mass of $12 \times 10^9 M_\odot \Rightarrow M_{seed} = 1.2 \times 10^{10} M_\odot \cdot 10^{-7.7} = 1.2 \times 10^{2.3} = 240 M_\odot$

Minimum Orbit Size : Roche limit



$$F' = G\frac{Mm}{r^2}$$

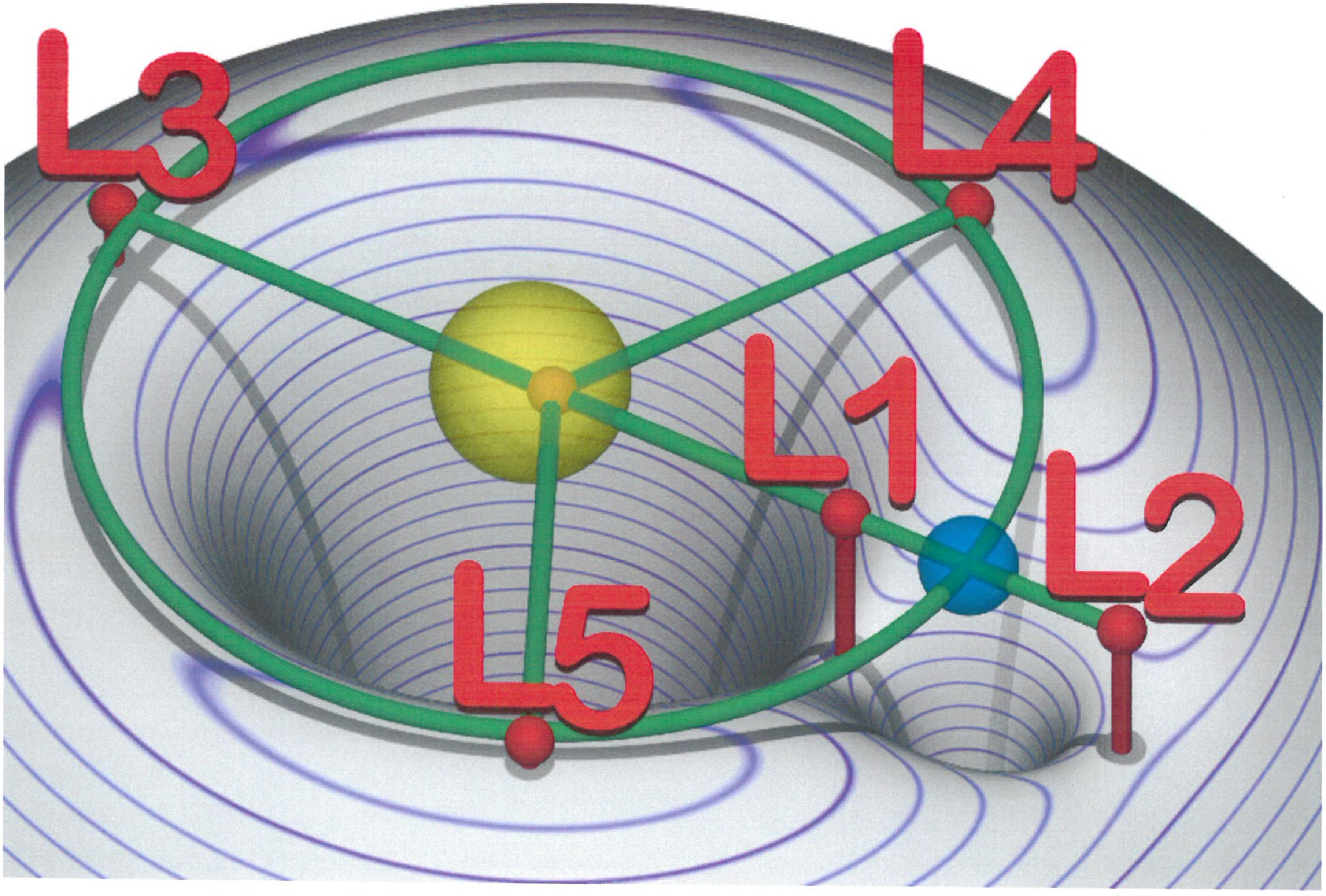
Tidal force from M to m

$$\Delta F' = \frac{dF'}{dr}\Delta r = \frac{2GMm}{r^3}\Delta r$$

Self gravity

$$F = -\frac{Gm^2}{(\Delta r)^2}$$

$$\Rightarrow r_R = \left(\frac{2M}{m}\right)^{1/3}\Delta r$$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L5

Chap 24 The Milky Way Galaxy

Morphology:

Star Count Method (William & Caroline Herschel 1738-1822)

$$m = M + 5\log \frac{d}{10\,pc} = M + 5\log d - 5$$

$$\Rightarrow d = 10^{0.2(m-M+5)}\ pc$$

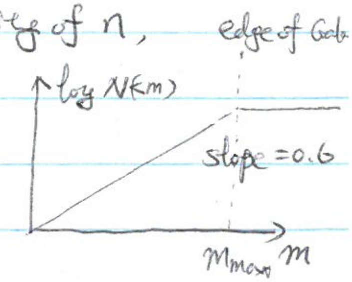
for a uniform distribution of stars with AM (absolute magnitude) of M and density of n,

$$N(<m) = \frac{\Omega}{3} d^3 \cdot n = \frac{\Omega}{3} 10^{0.6(m-M+5)} \cdot n$$

$$\Rightarrow \log N(<m) = 0.6\,m + const$$

until we reach the boundary of the Galaxy, d_{max}

$$d_{max} = 10^{0.2(m_{max}-M+5)}\ pc$$



Herschel's grindstone model. → Solution to Olber's paradox

Interstellar Extinction

$$m = M + 5\log \frac{d}{10\,pc} + A\ ,\quad A_V \sim 1\ mag/kpc$$

$$d = 10^{0.2(m-M-A+5)}\ pc = 10^{-0.2A} \cdot d_{apparent}$$

$$\Rightarrow d_{apparent} > d_{true}$$

DM Halo: $(1.26 \pm 0.24) \times 10^{12} M_\odot$ (McMillan+2011)

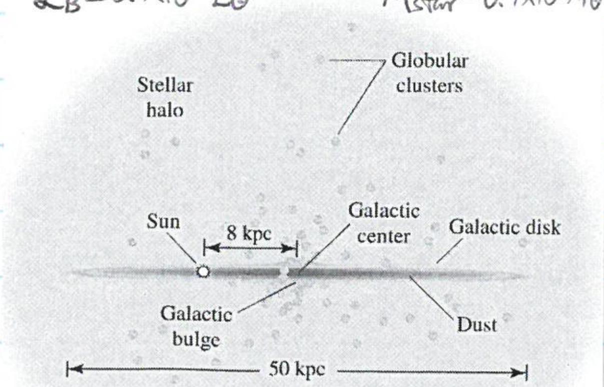
NFW: $R_{vir} = 280$ kpc, $\sigma = 165$ km/s, $T_{vir} = 7\times 10^5$ K

$L_B = 2\times 10^{10} L_\odot$ $\quad M_{star} = 6.4\times 10^{10} M_\odot$

Pop I: Young & Metal Rich

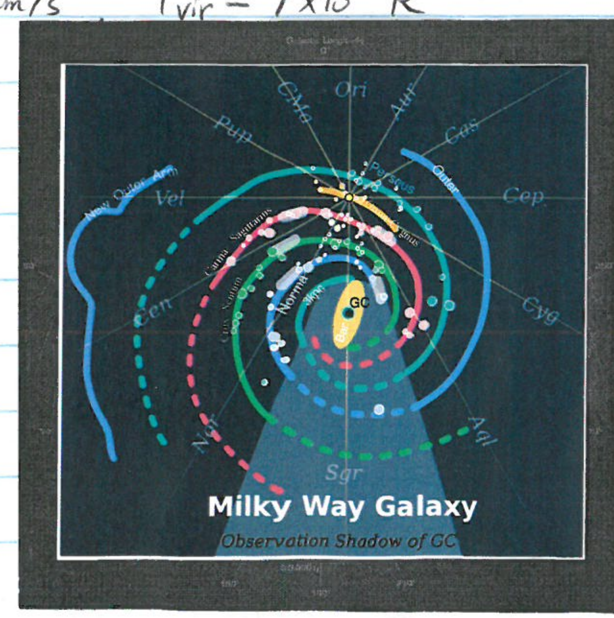
Pop II: Old & Metal poor

Pop III: First Stars



$h_{thin} = 0.3$ kpc $\quad R^{star}_{disk} = 15$ kpc

$h_{thick} = 1$ kpc $\quad R^{HI}_{disk} = 25$ kpc



Kinematics: How fast are we moving?

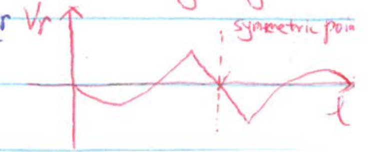
Coordinate system: Galactic latitude (b) & longitude (l)

Galactic center is at approximately $b=0$, $l=0$

Cylindrical Motion coordinates

How to find the GC? HI 21 cm survey along the disk

~~Sag A*~~ Sgr A*



$$\Pi = \frac{dR}{dt}, \quad \Theta = R\frac{d\theta}{dt}, \quad Z = \frac{dz}{dt}$$

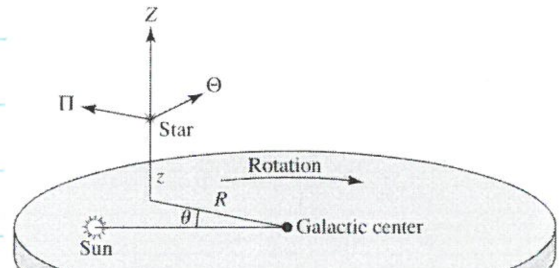
pi, radial, theta, tangential, zeta, vertical

Dynamical Local standard of rest (LSR)

centered on the Sun, circular motion

$\Pi_{LSR} = Z_{LSR} = 0 \qquad \Theta_{LSR} = \Theta(R_0)$

Galactic Year = 230 Myr



Peculiar velocity (velocity relative to LSR)

$\vec{V} = (V_R, V_\theta, V_z) = (u, v, w)$

$= (\Pi, \Theta - \Theta_0, Z)$ This only applies to stars in $\boxed{\text{solar neighborhood}}$ ($d < 1$ kpc)

For stars in the Solar neighborhood: $\langle u \rangle = \langle w \rangle = 0$, $\langle v \rangle < 0$

$\langle v \rangle < 0$ because there are more stars inside solar circle than outside

Observed Pecular velocity:

$\Delta\vec{V} = (\Delta u, \Delta v, \Delta w) = (u - u_\odot, v - v_\odot, w - w_\odot)$

$u_\odot = -10$ km/s, $v_\odot = 5.2$ km/s, $w_\odot = 7.2$ km/s

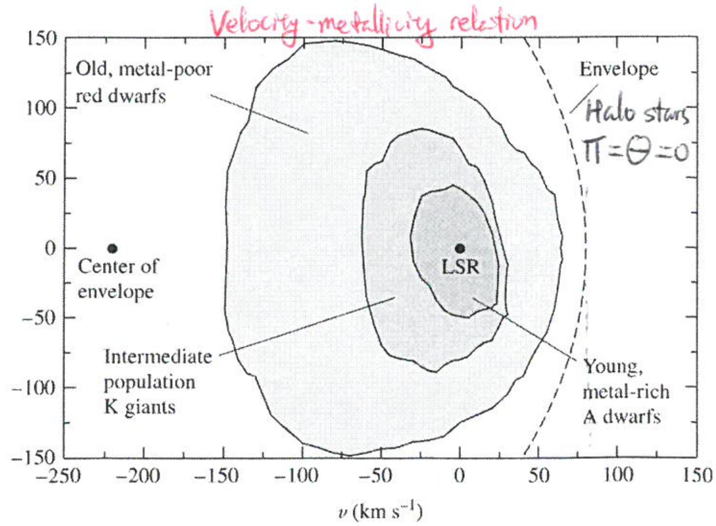
the sun is moving towards the solar apex @ 13.4 km/s

$u_\odot = -\langle \Delta u \rangle$, $v_\odot = \langle v \rangle - \langle \Delta v \rangle$, $w_\odot = -\langle \Delta w \rangle$, since $\langle u \rangle = \langle w \rangle = 0$

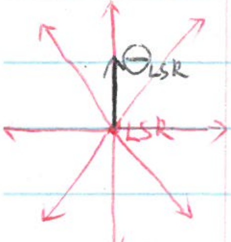
$\langle v \rangle = C \cdot \sigma_u^2$ where σ_u is the velocity dispersion in u

$\Rightarrow \langle \Delta v \rangle = C \cdot \sigma_u^2 - v_\odot$,

Velocity-metallicity relation



Once the solar peculiar velocity is corrected we can plot u vs. v and see the velocity ellipsoids, whose outer bound mark the random motion of halo stars



isotropic random motion Π_{halo} Θ_{halo}

$u = \Pi_{halo}$, $v = \Theta_{halo} - \Theta_{LSR}$

$\Rightarrow u^2 + (v + \Theta_{LSR})^2 = \Pi_{halo}^2 + \Theta_{halo}^2 = 300^2$ km/s

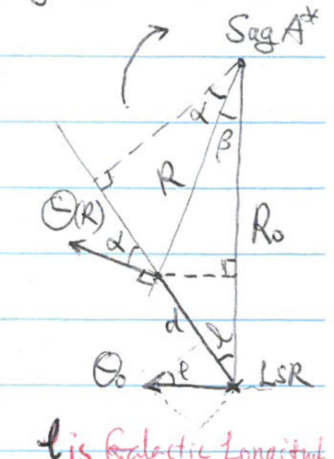
Going beyond Solar Neighborhood

Differential Galactic Rotation & Oort's Constants (A & B) 1920s analysis Jan Oort

The goal is to obtain expressions for radial & transverse velocity relative to LSR

$$\begin{cases} V_r = \Theta\cos\alpha - \Theta_0 \sin l = \Omega(R)\cdot R\cos\alpha - \Omega_0 R_0 \sin l \\ V_t = \Theta\sin\alpha - \Theta_0\cos l = \Omega(R)\, R \sin\alpha - \Omega_0 R_0 \cos l \end{cases}$$

assume circular motion



l is Galactic Longitud

The Oort Diagram

let's try to eliminate α

$$R\cos\alpha = R_0 \sin l \quad \& \quad R\sin\alpha = R_0\cdot\cos l - d$$

$$\Rightarrow \begin{cases} V_r = (\Omega - \Omega_0) R_0 \sin l \\ V_t = (\Omega - \Omega_0) R_0 \cos l - \Omega d \end{cases}$$

express $\Omega - \Omega_0$ as $\boxed{\text{Taylor expansion}}$ Solar neighborhood when $d \ll R_0$

$$\Omega - \Omega_0 \simeq \left.\frac{d\Omega}{dR}\right|_{R_0}\cdot(R - R_0), \quad R - R_0 \simeq -d\cos l$$

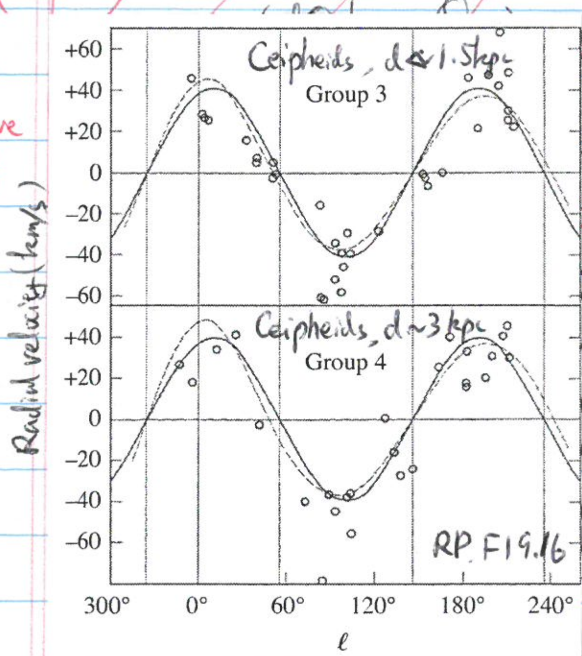
$$\Rightarrow \begin{cases} V_r = \left.\frac{d\Omega}{dR}\right|_{R_0}\cdot(R-R_0)\cdot R_0\cdot \sin l = -\frac{R_0}{2}\left.\frac{d\Omega}{dR}\right|_{R_0} 2d\cos l\cdot\sin l = -\left[\frac{R_0}{2}\left.\frac{d\Omega}{dR}\right|_{R_0}\right] d\cdot\sin 2l \\ \qquad = A\cdot d\cdot \sin 2l \\ V_t = \left.\frac{d\Omega}{dR}\right|_{R_0}(R-R_0)\cdot R_0\cos l - \Omega d = A d\cos 2l + B d, \quad B = A - \Omega_0 \end{cases}$$

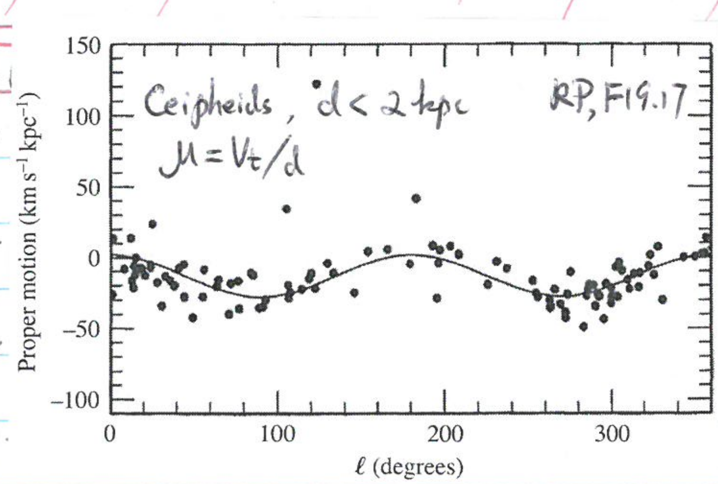
This part is optional see above for alternative definitions of A & B

given that $\Omega = \frac{\Theta}{R}$, we have $\left.\frac{d\Omega}{dR}\right|_{R_0} = \left.\frac{d\Theta}{dR}\frac{1}{R} - \frac{\Theta}{R^2}\right|_{R_0} = \frac{1}{R_0}\left(\left.\frac{d\Theta}{dR}\right|_{R_0} - \frac{\Theta_0}{R_0}\right)$

$-R_0)\sin l$

$-R_0)\cos l - \Omega_0 d$ assume $[\Omega \simeq \Omega_0]$





$l)]$

$l)]$

$$\Rightarrow \begin{cases} V_r = A d \sin 2l \\ V_t = A d\cos 2l + B d \end{cases} \qquad \begin{cases} A = -\frac{1}{2}\left(\left.\frac{d\Theta}{dR}\right|_{R_0} - \frac{\Theta_0}{R_0}\right) = 14.8\ \text{km/s/kpc} \\ B = -\frac{1}{2}\left(\left.\frac{d\Theta}{dR}\right|_{R_0} + \frac{\Theta_0}{R_0}\right) = -12.4\ \text{km/s/kpc} \end{cases}$$

Applications of Oort Constants

5.8 mas/yr

① $A = -\frac{1}{2}\left[\frac{d\Theta}{dR}\Big|_{R_0} - \frac{\Theta_0}{R_0}\right]$

$B = -\frac{1}{2}\left[\frac{d\Theta}{dR}\Big|_{R_0} + \frac{\Theta_0}{R_0}\right]$

$\Rightarrow \Omega_0 = \frac{\Theta_0}{R_0} = A - B = 27.2$ km/s/kpc

$\frac{d\Theta}{dR}\Big|_{R_0} = -(A+B) = -2.4$ km/s/kpc

$A = 14.8$ km/s/kpc

$B = -12.4$ km/s/kpc

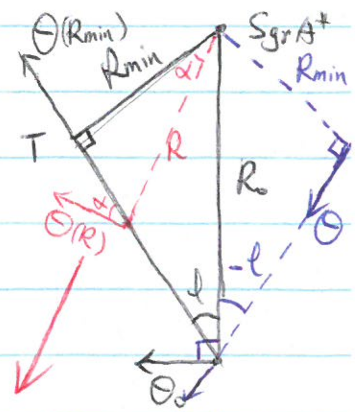
the small value of $\frac{d\Theta}{dR}\Big|_{R_0}$ hints at a flat rotation curve

② Point of maximum radial velocity is the tangent point, if $\Omega \downarrow$ as $R \uparrow$

[assuming $\Theta(R)/R \downarrow$ as $R \uparrow$, e.g., Keplerian motion $V/R = \sqrt{GM/R^3}$]

At the tangent point: $R_{min} = R_0 \cdot \sin l$

$\boxed{V_{r,max} = (\Omega - \Omega_0) \cdot R_{min}}$ $\quad V_{r,max} = \Theta(R_{min}) - \Theta_0(R_0) \cdot \sin l$



near solar neighborhood, $l \simeq 90°$ or $270°$, $d \ll R_0$,

$$\Theta(R_{min}) = \Theta_0(R_0) + \frac{d\Theta}{dR}\Big|_{R_0}(R_{min} - R_0)$$

$$\Rightarrow V_{r,max} = \Theta_0(R_0)(1-\sin l) + \frac{d\Theta}{dR}\Big|_{R_0}(R_{min} - R_0)$$

$$= \Theta_0(R_0)(1-\sin l) + \frac{d\Theta}{dR}\Big|_{R_0} R_0 (\sin l - 1)$$

$$= 2A \cdot R_0 (1 - \sin l) \approx 0$$

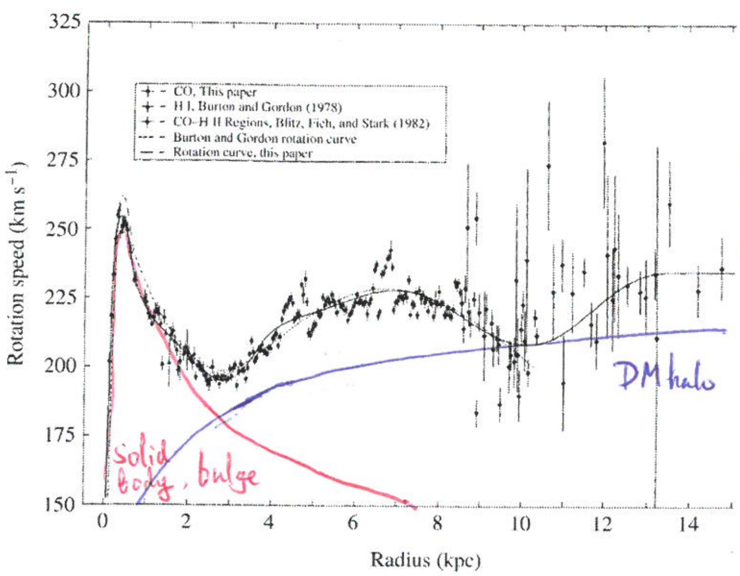
$\boxed{V_r(R) = \Theta(R) \cdot \cos\alpha - \Theta_0 \cdot \sin l = (\Omega(R) - \Omega_0) \cdot R_{min}}$

$V_{r,max} = \Theta(R_{min}) - \Theta_0 \sin l$

$\boxed{V_{r,max} \text{ occurs at } \Omega(R_{min}) \text{ if } \Omega \downarrow \text{ as } R\uparrow} \Leftarrow$

Galactic Rotation Curve from HI

Constant orbital speed, $\Theta = $ const, $M(R) \propto R$



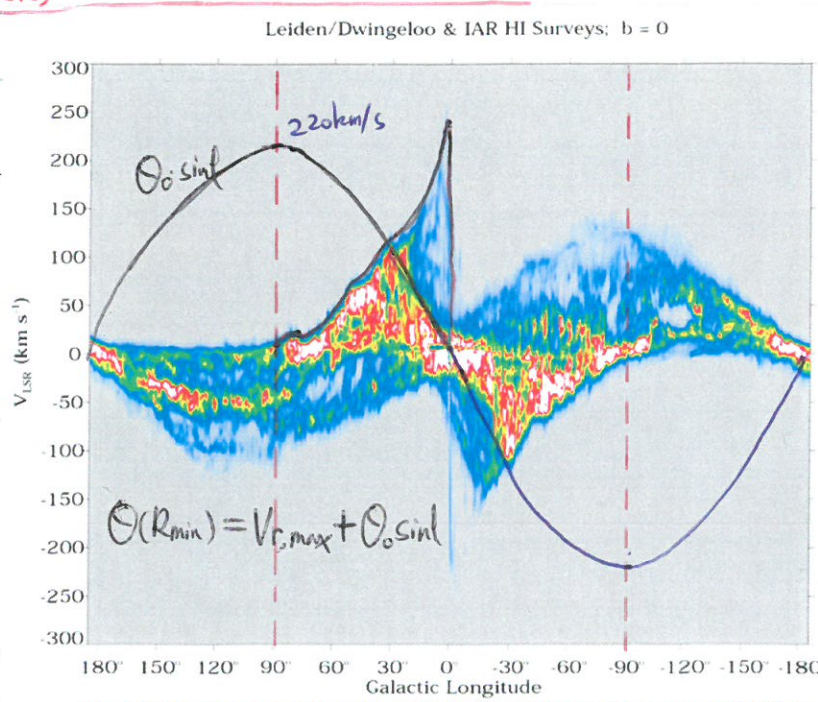


Fig 2.20 (D. Hartmann) 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Density Profile Required by Flat rotation curve ($\rho \propto 1/r^2$)

definition of circular velocity,

$$\frac{\Theta^2}{R} = \frac{G M_R}{R^2} \Rightarrow M_R = \frac{\Theta^2 R}{G} \Rightarrow \frac{dM_R}{dR} = \frac{\Theta^2}{G}$$

mass continuity (M_R is the enclosed mass within R)

$$dM_R = 4\pi R^2 \rho \, dR$$

combine the above two

$$4\pi R^2 \rho(R) = \Theta^2/G \Rightarrow \rho(R) = \frac{\Theta^2}{4\pi G R^2}$$

in comparison, the NFW (1996) profile

$$\rho_{NFW}(R) = \frac{\rho_0}{(R/a)(1+R/a)^2}$$

Density Profile required by rigid-body rotation [$\rho(R) = \text{const}$]

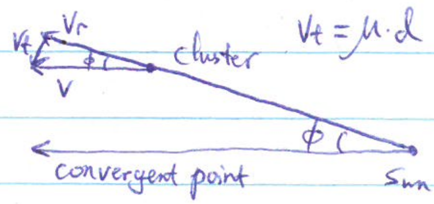
$$\Omega^2 \cdot R = \frac{G M_R}{R^2} \Rightarrow M(R) = \frac{\Omega^2 R^3}{G} \Rightarrow \frac{dM_R}{dR} = \frac{3\Omega^2 R^2}{G}$$

$$\Rightarrow 4\pi R^2 \rho(R) = \frac{3\Omega^2 R^2}{G} \Rightarrow \rho(R) = \frac{3\Omega^2}{4\pi G} = \text{const.}$$

Rotation Curve measurements outside of Solar Circle require distance measurements

① moving cluster method

$$d = \frac{\langle v_r \rangle \tan\phi}{\langle \mu \rangle} \Leftrightarrow d(pc) = \frac{\langle v_r/\text{km/s} \rangle \tan\phi}{4.74 \langle \mu/\text{arcsec/yr} \rangle}$$

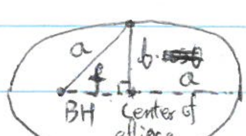


② main-sequence fitting : $m - M = 5\log(d/10pc)$

③ secular parallax : utilizing (a) solar peculiar motion : $u_\odot = -10$ km/s, $v_\odot = 5.2$ km/s

(b) circular motion around Galactic center [$R_0 = \Theta_0/\mu$]

④ Doppler shift + proper motion + Kepler's laws (distance to stars around Sgr A*)

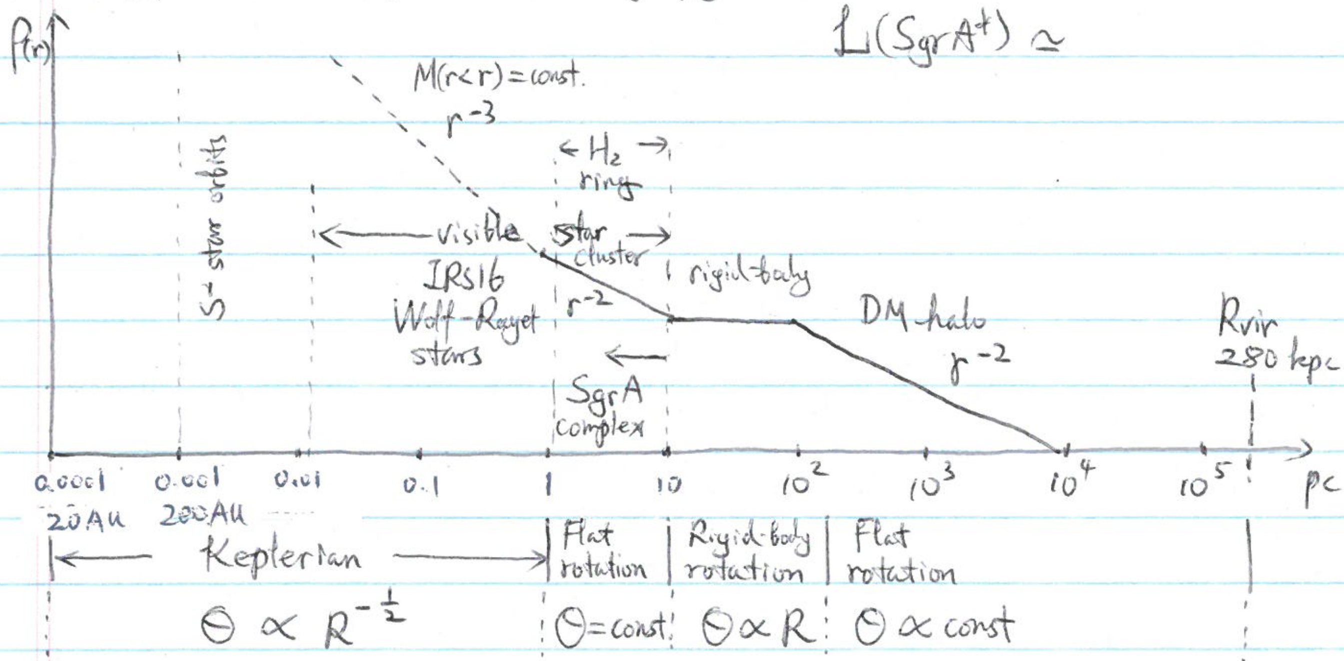


$a^2 = b^2 + f^2$, suppose the orbit is tilted along the major axis a & f would be preserved, while b_{app} will become $b\cos\theta$, thus we can solve for $\cos^2\theta = \dfrac{b_{app}^2}{a^2 - f^2}$ since $a^2 - f^2 = \dfrac{b_{app}^2}{\cos^2\theta}$

Galactic Center

$$M_{BH} = 4\times10^6 M_\odot \Rightarrow V_c = \sqrt{\frac{GM}{R}} = 6000\ \mathrm{km/s}\ @\ R = 100\ \mathrm{AU}$$

$$L(SgrA^*) \simeq$$



Scale height of isothermal atmosphere (See discussion in the next chapter)

$$\frac{dP}{dr} = -g\rho \quad , \quad P = \frac{\rho k T}{\mu m_p} \Rightarrow \frac{d\rho}{dr} = -\frac{\mu m_p g}{kT}\rho$$

$$\Rightarrow h = \frac{kT}{\mu m_p g} = 8.7\ \mathrm{km}\left(\frac{T}{300K}\right)\cdot\left(\frac{29}{\mu}\right)\cdot\left(\frac{9.8\ \mathrm{m s^{-2}}}{g}\right)$$

$$\Rightarrow h \propto \sigma^2 / g_{\mathrm{midplane}}$$ which explains the different scale heights of HI, thin, thick disks

Singular Isothermal sphere

Isothermal Collisionless System = Isothermal Collisional Gas $\left[\sigma^2 = \frac{kT}{m}\right]$

① Distribution Function of Collisionless System

Maxwellian Distribution

$$f(\mathcal{E}) \equiv \frac{\rho_1}{(2\pi\sigma^2)^{3/2}}\exp\left(\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}\right), \quad \mathcal{E} = \Psi - \frac{1}{2}v^2 \Rightarrow dn \propto \exp\left(-\frac{|\vec{v}|^2}{2\sigma^2}\right)d^3\vec{v}$$

Note that $\Psi = -\Phi + \Phi_0$ relative potential so there is a minus sign.

$$\Rightarrow \rho = \int f(\mathcal{E})\, d\mathcal{E} = \rho_1 \exp(\Psi/\sigma^2) \Rightarrow \frac{d\Psi}{dr} = \sigma^2 \cdot \frac{d\ln\rho}{dr}$$

$\boxed{\text{Poisson's Eq.}}$ $\nabla^2\Psi = -4\pi G\rho \Rightarrow \boxed{\frac{d}{dr}\left(r^2\frac{d\ln\rho}{dr}\right) = -\frac{4\pi G}{\sigma^2}r^2\rho}$

② Hydrostatic Equilibrium, $\frac{dP}{dr} = -\rho\frac{d\Phi}{dr}$

$$\frac{kT}{m}\frac{d\rho}{dr} = -\rho\frac{GM(r)}{r^2} \Rightarrow r^2\frac{d\ln\rho}{dr} = -\frac{Gm}{kT}M(r)$$

$$\Rightarrow \boxed{\frac{d}{dr}\left(r^2\frac{d\ln\rho}{dr}\right) = -\frac{Gm}{kT}\frac{dM}{dr} = -\frac{4\pi Gm}{kT}r^2\rho}$$

$$\Rightarrow \begin{cases} \rho(r) = \dfrac{\sigma^2}{2\pi G r^2} \\ M(r) = \dfrac{2\sigma^2 r}{G} \\ V_c(r) = \sqrt{2}\,\sigma \\ \Phi(r) = 2\sigma^2\ln(r) + \mathrm{const.} \end{cases}$$

Chap 25 The Nature of Galaxies

Morphologies of Nearby Galaxies

ellipticity $\epsilon = 1 - \beta/\alpha$, E7 means $\epsilon = 0.7$

Sa → Sc Bulge/Disk ↓ Tightness of Spiral arms ↓

Isophotes & the de Vaucouleurs Profile

$$\mu(r) = \mu_e + 8.3268\left[\left(\frac{r}{r_e}\right)^{1/4} - 1\right]$$ $\boxed{\text{Sérsic profile}}$

μ in mag arcsec^{-2}

$$\Leftrightarrow \log\left(\frac{I(r)}{I_e}\right) = -3.3307\left[\left(\frac{r}{r_e}\right)^{1/4} - 1\right]$$

$\boxed{r_e\text{, effective radius enclosing } 1/2 \text{ of light}}$

$$\Leftrightarrow \quad \Sigma(r) = \cancel{\Sigma_0 \exp\left[-\left(\frac{r}{r_0}\right)^{1/n} + 1\right]} = \Sigma_e \cdot \left[-K_n\left(\left(\frac{r}{r_e}\right)^{1/n} - 1\right)\right]$$

$K_n = 7.67$ when $n = 4$

Tully-Fisher Relation [for Spirals] : $L \propto V_{max}^4$ based on global 21 cm observations

(1977) Derivation

$$\underset{\uparrow \text{ Galaxy Mass}}{M} \overset{\text{Virial theorem}}{=} \frac{V_{max}^2 \cdot R}{G} \Rightarrow L = M\cdot\left(\frac{L}{M}\right) = \frac{V_{max}^2 R}{G}\cdot\left(\frac{L}{M}\right)$$

assuming constant surface brightness $L/R^2 = C_{SB}$

$\boxed{\Rightarrow \log R = -0.2M + \text{const} \quad \text{Radius-Luminosity relation}}$

assuming constant $M/L = C_{ML}$.

$$\Rightarrow L \cdot \frac{L}{R^2} = \frac{V_{max}^4}{G^2}\cdot\left(\frac{L}{M}\right)^2$$

$$\Rightarrow L = \left(\frac{L}{M}\right)^2\left(\frac{L}{R^2}\right)^{-1}\frac{1}{G^2} V_{max}^4 = C_{ML}^{-2} C_{SB}^{-1} \cdot G^{-2} V_{max}^4$$

$$\Rightarrow M = M_\odot - 2.5\log\left(\frac{L}{L_\odot}\right) = -10\log V_{max} + \text{const}$$

↑ This is absolute magnitude, not mass

$r_e \propto \sigma^{1.24} I_e^{-0.82}$ Faber-Jackson Relation

The Fundamental Plane [for Ellipticals] : $L \propto \sigma_0^{2.65} r_e^{0.65} \propto \sigma_0^4$

Virial Theorem $M \propto \frac{\sigma^2 R}{G} \Rightarrow L \propto \sigma^2 r_e$ if $M/L = \text{const}$.

Given $I_e = \frac{L}{r_e^2} \Rightarrow r_e = L^{1/2}/I_e^{\frac{1}{2}} \Rightarrow L \propto \sigma^4 I_e^{-1}$

Two-body Relaxation

① Strong close encounters

strong-encounter radius $\dfrac{Gm^2}{r_s} = \dfrac{mv^2}{2} \Rightarrow r_s = \dfrac{2Gm}{v^2} = 1\,AU \cdot \left(\dfrac{m}{0.5M_\odot}\right)\left(\dfrac{30\,km}{v}\right.$

mean free path between strong close encounters

$$l = \frac{1}{\pi r_s^2 \cdot n}, \quad n = N/R^3$$

mean free time

$$t_s = l/v = \frac{v^3}{4\pi G^2 m^2 n} = 4\times 10^{12}\,yr\left(\frac{v}{10\,km/s}\right)^3\left(\frac{m}{M_\odot}\right)^{-2}\left(\frac{n}{1\,pc^{-3}}\right)^{-1}$$

virial theorem $(2K + \Phi = 0)$ can be used to simplify t_s:

$$Nmv^2 = \frac{G(Nm)^2}{R} \Rightarrow v^2 = \frac{GNm}{R}$$

$$\Rightarrow t_s = \frac{v^4 R^2}{4\pi G^2 m^2 N} \cdot \frac{R}{v} = \frac{N}{4\pi}\, t_{cross}$$

$$t_{cross} = \frac{R}{v} = \sqrt{\frac{1}{G\rho}} \sim t_{ff}$$

For ISM, $l = 3\times 10^{-3}\,pc\left(\frac{\sigma}{\mathring{A}^2}\right)^{-1}\left(\frac{n}{1/c\ldots}\right.$

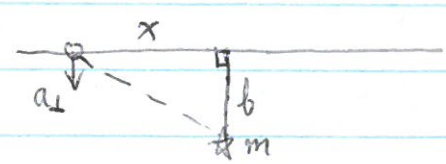
$t_s = \frac{l}{v} = 300\,yr\left(\frac{10\,km/s}{v}\right)$

$t_s = 300\,yr\left(\frac{10\,km/s}{v}\right)\left(\frac{1\mathring{A}^2}{\sigma}\right)\left(\frac{1\,cm^{-3}}{n}\right.$

so gas is collisional.

② Distant weak encounters

impulse approximation, $a_\perp = \dfrac{Gm}{b^2+x^2} \cdot \dfrac{b}{\sqrt{b^2+x^2}}$



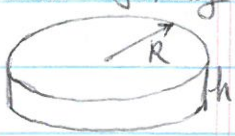
$$= \frac{Gm}{b^2} \cdot \frac{1}{[1+(\frac{x}{b})^2]^{3/2}}$$

$x' = \frac{vt}{b}$

$$\Delta v_\perp = \int_{-\infty}^{\infty} a \cdot dt = \frac{2Gm}{b^2}\int_0^\infty \frac{dt}{[1+(vt/b)^2]^{3/2}} = \frac{2Gm}{bv}\int_0^\infty \frac{dx'}{[1+x'^2]^{3/2}},$$

$\Delta v_\perp = \dfrac{2Gm}{bv}$ so when $b = r_s = \dfrac{2Gm}{v^2}$, $\Delta v_\perp = v$ for strong close encounter

assumed geometry



similar to random walk, $d = \sqrt{N}\,l$ or $d^2 = N l^2$, we find the total change in $\Delta v_\perp^2 = \displaystyle\int_{r_s}^{R}\left(\frac{2Gm}{bv}\right)^2 \cdot \frac{N}{\pi R^2 \cdot h} \cdot 2\pi b\, db \cdot h = \frac{8G^2m^2N}{v^2R^2}\ln\frac{R}{r_s}$

without assuming any geometry, we have

$$\Delta v_\perp^2 = \int_{b_{min}}^{b_{max}}\left(\frac{2Gm}{bv}\right)^2 \cdot n \cdot v \cdot t \cdot 2\pi b\, db = \frac{8\pi G^2 m^2 n t}{v}\ln\left(\frac{b_{max}}{b_{min}}\right)$$

define relaxation time as the time required for $\Delta v_\perp^2 = v^2$

$$t_{relax} = \frac{v^3}{8\pi G^2 m^2 n \ln\Lambda} = t_s/2\ln\Lambda \quad \text{where } \Lambda = \frac{b_{max}}{b_{min}}, \ \ln\Lambda \sim 18\text{-}22$$

Again we can use virial theorem to simplify the expression

$$NmV^2 = \frac{G(NM)^2}{R} \Rightarrow \Lambda = \frac{R}{r_s} = \frac{GNm}{v^2} \cdot \frac{v^2}{2Gm} = \frac{N}{2}$$

$$t_{relax} = \frac{t_s}{2\ln\Lambda} = \frac{N}{4\pi} t_{cross} \cdot \frac{1}{2\ln N/2} = \frac{N}{8\pi \ln N} \cdot t_{cross}$$

Effects of two-body relaxation — singular isothermal sphere

① Maxwellian distribution function, $f(\mathcal{E})$ where $\mathcal{E} = m\Phi(\vec{x}) + mv^2/2$

$$dN = f(\mathcal{E})\, 4\pi v^2 dv = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} \exp\left[-\frac{m\Phi(\vec{x}) + \frac{mv^2}{2}}{m\sigma^2}\right] 4\pi v^2 dv$$

$\Rightarrow \boxed{m\sigma^2 = kT}$, kinetic temperature, so if σ^2 is constant, this DF is singular isotherm

divergence theorem

$\oint_{\partial V} \vec{g} \cdot d\vec{A}$

$= \int_V \nabla \cdot g \, dV$

$= -4\pi GM$

$= -\int 4\pi G\rho \, dV$

$$\Rightarrow \rho(\vec{x}) = \int f(\mathcal{E})\, d\mathcal{E} = \rho_0 \exp\left[\frac{-\Phi(\vec{x})}{\sigma^2}\right] \Rightarrow \boxed{\frac{d\Phi}{dr} = -\sigma^2 \frac{d\ln\rho}{dr}}$$

use Poission's Equation $[-\nabla\Phi = \vec{g},\ \nabla\cdot\vec{g} = -4\pi G\rho]$

$$\nabla^2 \Phi = 4\pi G\rho \qquad \nabla^2 = \nabla\cdot\nabla \text{ is Laplacian (the divergence of gradient)}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 4\pi G\rho \Rightarrow \boxed{\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = -\frac{4\pi G}{\sigma^2} r^2\rho}$$

Compare with hydrostatic equilibrium for isothermal gas

$$\boxed{\frac{dP}{dr} = -\rho\frac{d\Phi}{dr}} \Rightarrow \frac{kT}{m}\frac{d\rho}{dr} = -\rho\frac{GM(r)}{r^2}$$

$$\Rightarrow \boxed{\frac{d}{dr}\left(r^2\frac{d\ln\rho}{dr}\right) = -\frac{Gm}{kT}\frac{dM(r)}{dr} = -\frac{4\pi Gm}{kT} r^2\rho}$$

For isothermal atmosphe

$$\frac{d\rho}{dr} = -\frac{\mu m_p g}{kT}\rho$$

$$h = \frac{kT}{\mu m_p g} = 8.7\,\text{km}\left(\frac{T}{300K}\right)\left(\frac{29}{\mu}\right)$$

$$h \propto \sigma^2/g\mu$$

NFW(1996)

$$\rho = \frac{\rho_0}{(r/a)(1+r/a)^2}$$

Solutions of Poission's Equation

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}, \quad M(r) = \frac{2\sigma^2 r}{G}, \quad V_c(r) = \sqrt{2}\sigma, \quad \Phi(r) = 2\sigma^2\ln(r)$$

② Evaporation time $t_{evap} \sim 136\, t_{relax}$

③ Mass segregation: massive stars or binaries move slower than less massive stars, sink to center

④ Core contraction: $t_{contract} \sim 10\text{–}20\, t_{relax}$

⑤ negative specific heat → removing energy makes the kinetic temperature hotter

$$E = K + \Phi = -K = -NkT \Rightarrow dE/dT = -Nk < 0$$

SMBH radius of influence

$$r_{inf} = \frac{G M_{BH}}{\sigma_*^2} \simeq 11\,pc \left(\frac{M_{BH}}{10^8 M_\odot}\right)\left(\frac{\sigma_*}{200\,km/s}\right)^{-2}$$ Radius of Influence

$$r_{Sch} = \frac{2 G M_{BH}}{c^2} = \frac{2\sigma_*^2}{c^2}\, r_{inf} = 2\,AU \left(\frac{M_{BH}}{10^8 M_\odot}\right)$$ Schwarzschild Radius

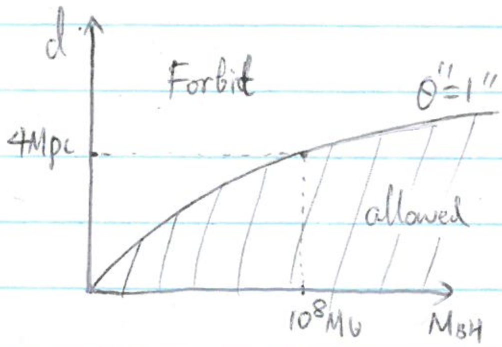
Observability: $M_{BH} \sim \sigma_*^4$ M-σ_* relation $\frac{M_{BH}}{10^9 M_\odot} = 0.310 \left(\frac{\sigma_*}{200\,km/s}\right)^{4.38}$ Kormendy & Ho 2013

$$r_{inf} \simeq 34\,pc \left(\frac{M_{BH}}{3.1\times10^8 M_\odot}\right)\cdot\left(\frac{\sigma_*}{200\,km/s}\right)^{-2}$$

$$= 34\,pc \cdot \left(\frac{\sigma_*}{200\,km/s}\right)^{2.38}$$

$$r_{inf} \simeq 11\,pc \left(\frac{M_{BH}}{10^8 M_\odot}\right)\left(\frac{M_{BH}}{3.1\times10^8 M_\odot}\right)^{-\frac{2}{4.38}}$$

$$= 18\,pc \left(\frac{M_{BH}}{10^8 M_\odot}\right)^{0.54}$$



if the observatory has a finite angular resolution θ''

$$r_{res} = d\cdot\theta''/206265$$

and we require $r_{res} \leq r_{inf}$

$$\frac{d\cdot\theta''}{2\times10^5} \leq 18\,pc\left(\frac{M_{BH}}{10^8 M_\odot}\right)^{0.54} \quad \text{or} \quad \frac{d\cdot\theta''}{2\times10^5} \leq 34\,pc \cdot \left(\frac{\sigma_*}{200\,km/s}\right)^{2.38}$$

Explanation of M_{BH}-σ_* relation (King 2003)

singular isothermal sphere with a gas fraction of f_g follows ($f_g = \frac{\Omega_b}{\Omega_M} \sim 16\%$)

$$\rho(r) = \frac{f_g \sigma^2}{2\pi G r^2} \quad \& \quad M(r) = \frac{2 f_g \sigma^2 r}{G} \quad \& \quad V_c(r) = \sqrt{2}\,\sigma$$

momentum deposition from radiation pressure of a SMBH radiating at Eddington limit

$$M(r)\cdot\frac{dr}{dt} = \frac{L_{Edd}}{c}\, t$$

integrate both sides

$$\frac{2 f_g \sigma^2}{G}\, r\,\frac{dr}{dt} = \frac{L_{Edd}}{c}\cdot t, \Rightarrow \quad r^2/t^2 = \frac{G L_{Edd}}{2 f_g \sigma^2 c}$$

during snowplow phase, $r/t \simeq \frac{1}{2}\frac{dr}{dt}$, and it must reach $\frac{dr}{dt} \gtrsim 2\sigma$ to escape

thus $r/t \simeq \sigma \Rightarrow \sigma^2 = \frac{G L_{Edd}}{2 f_g \sigma^2 c}$ where $L_{Edd} = \frac{4\pi G M_{BH} c}{\kappa}$

$$\Rightarrow \boxed{M_{BH} = \frac{f_g}{2\pi}\frac{\kappa}{G^2}\sigma^4}$$

Orbits of disk stars : Epicycles [Sparke §3.3]

here we assume an axisymmetric potential that is not time variable : $\Phi(R, z)$

① $\dfrac{\partial \Phi}{\partial \phi} = 0 \Rightarrow L_z = R^2 \dot{\phi} = \text{const}$ $\quad \dot{\phi} = \omega$ is the angular velocity

② $\ddot{R} = R\dot{\phi}^2 - \dfrac{\partial \Phi}{\partial R} = -\dfrac{\partial}{\partial R}\left[\Phi(R,z) + \dfrac{L_z^2}{2R^2}\right] = -\dfrac{\partial \Phi_{eff}}{\partial R}$

multiply $\dot{R}$ to both sides & integrate over dt

$$\dot{R}\frac{d\dot{R}}{dt} = -\frac{dR}{dt}\cdot\frac{\partial \Phi_{eff}}{\partial R} \Rightarrow \frac{1}{2}\dot{R}^2 + \Phi_{eff} = \text{const} \quad \text{for stars in midplane}$$

③ vertical motion

$$\ddot{z} = -\frac{\partial \Phi}{\partial z} = -\frac{\partial \Phi_{eff}}{\partial z} \underset{\text{Taylor expansion}}{\simeq} -z\left[\frac{\partial^2 \Phi}{\partial z^2}(R_g, z)\right]_{z=0} \equiv -\nu^2(R_g)\, z$$

harmonic oscillator with angular frequency $\nu(R_g)$, $\dfrac{\partial \Phi_{eff}}{\partial R} = 0$ @ $R = R_g$

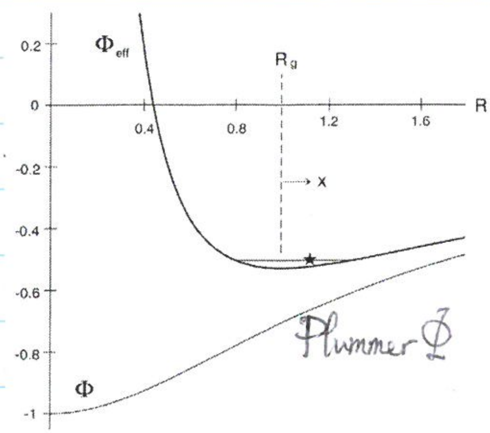
$$z(t) = Z_0 \cdot \cos[\nu(R_g)t + \theta_0]$$

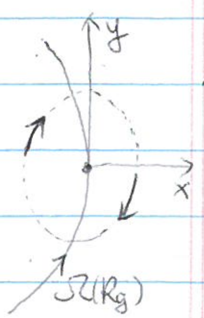
④ circular motion [i.e., guiding center of epicycles]

it occurs at $\dfrac{\partial \Phi_{eff}}{\partial R} = 0 \Leftrightarrow \dfrac{\partial \Phi}{\partial R} = -\dfrac{d}{dR}\left[\dfrac{L_z^2}{2R^2}\right] = \dfrac{L_z^2}{R^3}$

we also know $\dfrac{\partial \Phi}{\partial R} = R_g \Omega^2(R_g)$ for circular motion

$$\Rightarrow \Omega(R_g) = L_z / R_g^2$$





⑤ retrograde motion around guiding center

Mechanics Review :

Lagrangian : $\mathcal{L} = \frac{1}{2}\left[\dot{R}^2 + (R\dot{\phi})^2 + \dot{z}^2\right] - \Phi(R,z)$

momenta : $\vec{p} \equiv \left.\dfrac{\partial L}{\partial \dot{\vec{q}}}\right|_{q,t}$

$$p_R = \dot{R}, \quad p_\phi = R^2\dot{\phi}, \quad p_z = \dot{z}$$

Hamiltonian : $H = \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}(q, \dot{q}, t)$

$$H = \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{R^2} + p_z^2\right) + \Phi(R,z)$$

Hamiltonian Equations : $\dot{q} = \dfrac{\partial H}{\partial p}$, $\dot{p} = -\dfrac{\partial H}{\partial q}$

$$\dot{p}_R = \ddot{R} = p_\phi^2/R^3 - \frac{\partial \Phi}{\partial R}, \quad \dot{p}_\phi = \frac{d}{dt}(R^2\dot{\phi}) = -\frac{\partial \Phi}{\partial \phi} = 0, \quad \dot{p}_z = \ddot{z} = -\frac{\partial \Phi}{\partial z}$$

Chap 28 Active Galaxies

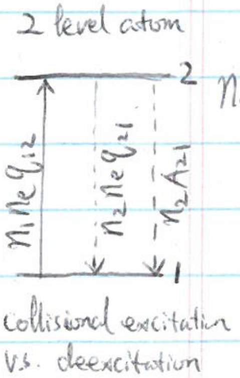
Forbidden lines : transition from $\boxed{\text{metastable levels}}$ with long ($\sim 1s$) spontaneous decay time

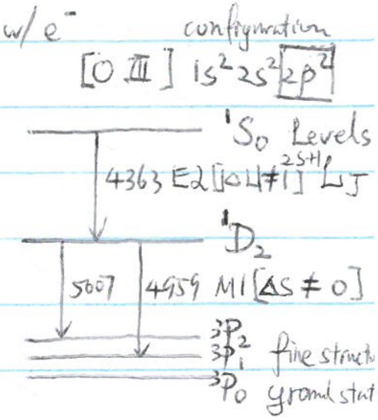
these transitions violate electric-dipole selection rules (small A)

the metastable levels are populated through collisions w/ e^-

[Osterbrock pg 51 to 54] $\quad q_{21} = \int_0^\infty u\,\sigma_{21}(u)\, f(u)\, du$

equilibrium between excitation & deexcitation





$$n_e n_1 q_{12} = n_e n_2 q_{21} + n_2 A_{21}$$

$$\Rightarrow \frac{n_2}{n_1} = \frac{n_e q_{12}}{A_{21}} \left[\frac{1}{1 + \frac{n_e q_{21}}{A_{21}}} \right]$$

line luminosity from spontaneous decay per volume :

q_{12}/q_{21} : velocity-averaged cross section

~~define collisional excitation rate~~

$\cancel{C_{12} = n_e n_1 q_{12}}$

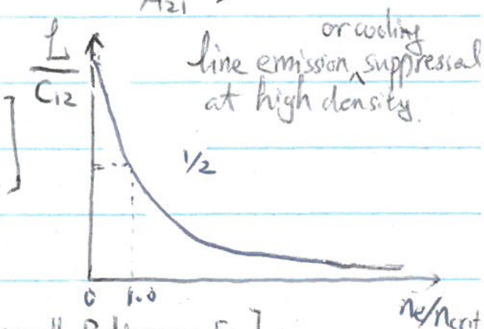
$$L = n_2 A_{21} h\nu_{21} = n_e \cdot n_1 q_{12} h\nu_{21} \left[\frac{1}{1 + \frac{n_e q_{21}}{A_{21}}} \right]$$

$q_{21} = \frac{8.6e{-6}}{\sqrt{T}} \cdot \frac{\Upsilon(1,2)}{\omega_2}$

$q_{12} = \frac{\omega_2}{\omega_1} q_{21} e^{-\chi/kT}$

define critical density , $\boxed{n_{crit} = A_{21}/q_{21}}$

$$L = n_e n_1 q_{12} h\nu_{21} \left[\frac{1}{1 + n_e/n_{crit}} \right]$$



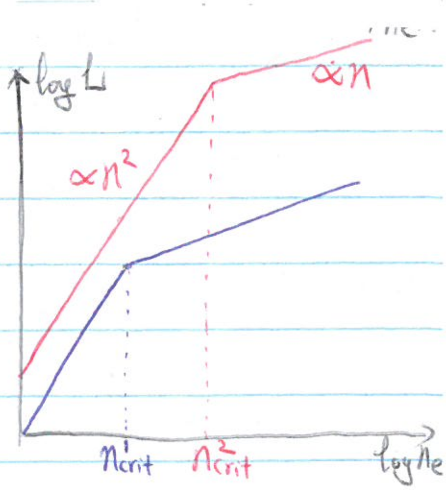
line luminosity severely suppressed at large n_e :

at large n_e, $\quad L \to \boxed{n_1 \frac{g_2}{g_1} \exp(-\chi/kT)} \cdot A_{21} h\nu_{21}$ [recall Boltzmann Eq.]

Now Let's show $\boxed{\text{Forbidden lines probe low density gas because of their low critical densities.}}$

$$L \propto \boxed{n_2 A_{21} = n_e n_1 q_{12} \left[\frac{1}{1 + \frac{n_e q_{21}}{A_{21}}} \right]}$$

$$\Rightarrow n_2 A_{21} = \begin{cases} n_1 n_e q_{12} & \text{when } n_e \ll n_{crit} \\ n_1 n_{crit} q_{12} & \text{when } n_e \gg n_{crit} \end{cases}$$



at large n_e, we can use Boltzmann equation to write

$$n_2 A_{21} = n_1 \cdot \frac{g_2}{g_1} e^{-h\nu_{21}/kT} \cdot A_{21}$$

thus $\quad \frac{q_{12}}{q_{21}} = \frac{g_2}{g_1} e^{-h\nu/kT}$

Configuration (n, l) e.g. Boron $1s^2 2s^2 2p$ Neon $1s^2 2s^2 2p^6$

Carbon $1s^2 2s^2 2p^2$

Terms (S, L) e.g. Boron's outer most e^- has $l = 1$, $s = 1/2$

so the term is ^2P where $2 = 2S+1$

Levels (S, L, J) e.g. $J = L+S, L+S-1, \cdots, \text{abs}(L-S)$ for $L > S$

(fine structure) doublet $\rightarrow$ $J = \frac{3}{2}, \frac{1}{2}$, multiplicity $= 2S+1$

so ^2P term has 2 levels $(2S+1)$: $^2P_{3/2}$, $^2P_{1/2}$

Fine Structure line: relativistic corrections (spin-orbit, etc) split degenerate levels

e.g. Na I doublet @ 5890, 5896 Å. $3^2P_{1/2} \rightarrow 3^2S_{1/2}$ & $3^2P_{3/2} \rightarrow 3^2S$

Hyperfine structure: coupling between J and the spin of nucleus I cause energy differences. $F = I + J = J-I, J-I+1, \cdots, J+I-1, J+I$

e.g. $1s\,^2S_{1/2}$ level of the Hydrogen ground state has $J = \frac{1}{2}$

$I = \frac{1}{2}$ for the spin of proton

we have $F = 0, 1$

the transition between $F=1$ to $F=0$ produces 21 cm photons

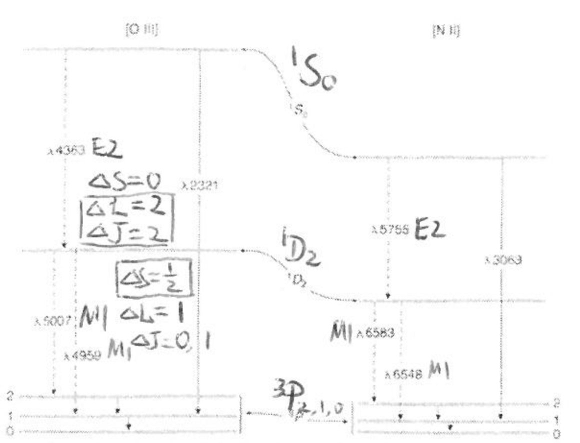
[O III], 6 electrons, configuration $1s^2 2s^2 2p^2$ C-like

For the outer two electrons @ $2p^2$ $L = 1+1, 1+1-1, 1-1 = 2, 1, 0$

we have ~~three~~ six terms $S = \frac{1}{2} - \frac{1}{2}, \frac{1}{2} + \frac{1}{2} = 0, 1$

^1S, ^1P, ^1D, ^3S, ^3P, ^3D

$J =$ 0 1 2 ~~012~~ 1 012 123



The three most commonly observed optical lines of [O III]

Partial energy level diagram

selection rules $\begin{cases} \Delta J = 0, \pm 1, \text{ but no } J=0 \rightarrow 0 \\ \Delta L = 0, \pm 1, \text{ but no } L=0 \rightarrow 0 \\ \Delta S = 0 \end{cases}$

M1: magnetic dipole forbidden transitions

E2: electric quadrupole forbidden

E1: allowed transitions

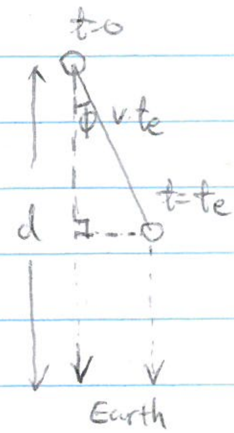
Superluminal velocities ($V_{app} > c$)

e.g. 3c273 knot proper motion : $\mu = 0.0008'' \, yr^{-1}$

distance to 3c273 : $d = c \cdot z / H_0 = 677$ Mpc [textbook uses $d = 440$ Mpc]

$D_A = 567$ Mpc , $D_L = 760$ Mpc

$V_{app} = d \cdot \mu \cdot 4.74 = 1.67 \times 10^6$ km/s $= 5.6\, c$



Explanation : photon arrival times $t_1 = d/c$

$$t_2 = t_e + \frac{d - v t_e \cos\phi}{c}$$

difference in arrival time

$$\Delta t = t_2 - t_1 = t_e \left(1 - \frac{v}{c} \cdot \cos\phi\right) < t_e$$

apparent velocity

$$V_{app}/c = \frac{v t_e \sin\phi}{\Delta t} \cdot \frac{1}{c} = \frac{(v/c) \sin\phi}{1 - (v/c)\cos\phi}$$

the superluminal condition implies

$$V_{app}/c > 1 \Rightarrow v/c\,(\sin\phi + \cos\phi) > 1$$

$$\cancel{\frac{v}{c} > \frac{1}{\sin\phi + \cos\phi}}$$

we know $1.0 < \sin\phi + \cos\phi < 1.414$, so it's not hard to get $V_{app}/c > 1$

further, we can solve for v/c to use V_{app}/c to constrain the true velocity:

$$\frac{v}{c} = \frac{V_{app}/c}{\sin\phi + (V_{app}/c) \cdot \cos\phi}$$

it has a minimum when $\cot(\phi) = V_{app}/c$ because $\frac{d}{d\phi}(\sin\phi + \beta\cos\phi) = 0 \Rightarrow \cot\phi = \beta$

$$\frac{V_{min}}{c} = \sqrt{\frac{V_{app}^2/c^2}{1 + V_{app}^2/c^2}} = 0.992 \text{ for 3c273}$$

$$\gamma_{min} = \frac{1}{\sqrt{1 - V_{min}^2/c^2}} = \sqrt{1 + V_{app}^2/c^2}$$

Einstein Coefficients [spontaneous decay A_{21}, absorption B_{12}, stimulated emission B_{21}] RP pg. 123-139

$$n_1 B_{12} J_\nu = n_2 B_{21} J_\nu + n_2 A_{21}$$

$A_{21} \sim 10^8\, s^{-1}$ for permitted transitions

$1\, s^{-1}$ for forbidden transitions

$J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega$ mean specific intensity

$B_{12} = g_2/g_1 \, B_{21}$, $A_{21} = \frac{2h\nu^3}{c^2} B_{21} = \frac{2h\nu^3}{c^2} \cdot \frac{g_1}{g_2} B_{12}$ [detailed balance]

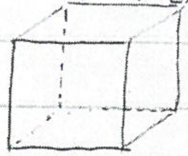
HII region: $\frac{d}{dt} n_e = n_{HI} \cdot \int_{\nu_i}^{\infty} \frac{4\pi J_\nu(r)}{h\nu} a_\nu \, d\nu - \alpha(H^0, T)\, n_e \cdot n_p$, $\alpha = V_{atom} \cdot A$

a_ν : cm^2 ; α : $cm^3 s^{-1}$; $= (10^{-10} m)^3 \cdot 10^9 s^{-1} \sim 10^{-15}$

$4\pi J_\nu(r) = \frac{L_\nu}{4\pi r^2} \exp\left[-\int_0^r n_{HI}(r') a_\nu \, dr'\right]$ define Stromgren sphere $\frac{4}{3}\pi R_s^3 \alpha_B n_H^2 = Q(H^0)$

$Q(H^0) = \int_{\nu_i}^{\infty} \frac{L_\nu}{h\nu} d\nu$

$V_p = \frac{V_c}{(1+z)^3}$



$\langle n_e \rangle \simeq \langle n_p \rangle$

$\boxed{\langle n_H \rangle = \langle n_{HI} \rangle + \langle n_p \rangle}$

Reionization Balance Equation

Consider part of the Universe at redshift z of proper volume V_p and comoving volume V_c, we know V_p expands with the Hubble expansion, while V_c is fixed.

The ionization balance is <u># of newly ionized H = # of ionizing photons − # of recombinations</u>

$$\frac{d}{dt}\left[\langle n_e \rangle V_p\right] = \boxed{\dot{n}_{ion} \cdot V_c - \alpha_B \langle n_e^2 \rangle \cdot V_p}$$

$\alpha_A = 4.2 \times 10^{-13} cm^3/s$

$\alpha_B = 2.6 \times 10^{-13} cm^3/s$

for 10^4 K Hydrogen

Since $\langle n_H \rangle \cdot V_p$ is invariable with time (conservation of Mass), we have

$$\frac{d}{dt}\left[\frac{\langle n_e \rangle V_p}{\langle n_H \rangle V_p}\right] = \boxed{\frac{\dot{n}_{ion} V_c}{\langle n_H \rangle V_p} - \frac{\alpha_B \langle n_e^2 \rangle V_p}{\langle n_H \rangle V_p}}$$

$\xi = \frac{n_p}{n_H}$, ionization fraction [ksai]

We know $V_p = V_c/(1+z)^3$

we define ① $\frac{\langle n_e \rangle}{\langle n_H \rangle} = Q_{HII}$ ionization fraction

② $C = \frac{\langle n_e^2 \rangle}{\langle n_e \rangle^2}$ clumpiness

$$\Rightarrow \frac{dQ_{HII}}{dt} = \boxed{\begin{aligned} &\frac{\dot{n}_{ion}}{\langle n_H \rangle (1+z)^{-3}} - \alpha_B \cdot \frac{\langle n_e^2 \rangle}{\langle n_e \rangle^2} \cdot \langle n_e \rangle \cdot Q_{HII} \\ &= \frac{\dot{n}_{ion}}{\langle n_H \rangle (1+z)^{-3}} - \frac{Q_{HII}}{(C \cdot \alpha_B(T) \langle n_e \rangle)^{-1}} \end{aligned}}$$

$X_H = 0.75$

$\Omega_{b,0} = 0.05$

$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = 1.8787 \times 10^{-29} h^2 \, g/cm^3$

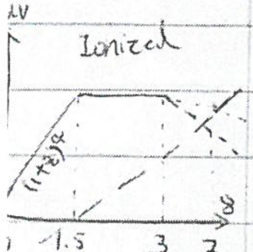
As the Universe expands, the matter density decreases as $\langle n_H \rangle \propto (1+z)^3$

so we can define the <u>comoving matter density</u>:

$$\boxed{\langle n_H \rangle_c = \langle n_H \rangle (1+z)^{-3}} = X_H \cdot \Omega_{b,0} \cdot \rho_{c,0} = \text{const.}$$

If when the Universe was reionized, most of the H is HII, i.e., $\langle n_e \rangle \simeq \langle n_H \rangle$

$$\frac{dQ_{HII}}{dt} = \boxed{\frac{\dot{n}_{ion}}{\langle n_H \rangle_c} - \frac{Q_{HII}}{[C \cdot \alpha_B(T) \langle n_H \rangle_c (1+z)^3]^{-1}}} \Leftarrow \tau_{recomb} \text{ timescale}$$



At ionization equilibrium, $\frac{dQ_{HII}}{dt} = 0$

$$Q_{HII} = \frac{\dot{n}_{ion}}{\langle n_H \rangle_c^2} \cdot \frac{1}{C \cdot \alpha_B(T)(1+z)^3} = 1 \text{ when Universe is fully ionized.}$$

$$\dot{n}_{ion} = \langle n_H \rangle_c^2 \cdot C \cdot \alpha_B(T) \cdot (1+z)^3$$

We also know that $\dot{n}_{ion} = \rho_{UV} \cdot \xi_{ion} \cdot f_{esc}$ ← UV escape fraction (~0.2)

ρ_{UV}: comoving UV (1500Å) luminosity density

ξ_{ion}: Rate of H-ionizing photons per unit UV luminosity

Chap 27 Structure of the Universe

Cosmological Distance Ladder

$$d = 10^{(m-M+5)/5}$$ distance modulus

Wilson-Bappu Effect, line width $\sim M_V$ for stars

Cepheids $M_V = -3.53 \log P_d - 2.13 + 2.13(B-V)$

Expanding photosphere $V_{exp} = V_\theta = \omega \cdot d \Rightarrow d = \frac{V_{Doppler}}{\omega}$

SNe Ia light curve $M_{B,peak} \sim$ rate of decline

Novae $M_V^{max} = -9.96 - 2.31 \log \frac{\Delta m}{\Delta t}$

Globular cluster LF $M_{turn\text{-}over} \simeq -6.5$

Universal LF shift observed LF in apparent magnitude to match universal LF

TF relation $M_H = -9.5(\log W/\mathrm{km/s} - 2.5) - 21.67$

D-σ relation [standard ruler] $\log D = 1.333 \log \sigma + c$, D - diameter out to 20.75 $B_{mag}/\mathrm{arcsec}^2$

Hubble's Law: empirical relation $v = \boxed{cz} = H_0 \cdot d = \boxed{H_0 \cdot c \cdot (t_0 - t)}$

Scale factor: $R_u = \frac{1}{1+z} = R_u(t)$ where t is the age of the Universe

$$z = \frac{1}{R_u} - 1 = H_0 \cdot (t_0 - t) \Rightarrow H_0 = \frac{1 - R_u}{R_u(t_0 - t)} = \frac{R_u(t_0) - R_u(t)}{R_u(t)(t_0 - t)}$$

$$H(t) = \frac{d \ln R_u}{dt} = \frac{1}{R_u(t)} \frac{dR_u(t)}{dt} \quad [29.8]$$

for constant expansion, $R_u(t) = a \cdot t$

$$H(t) = \frac{1}{at} \cdot a = \frac{1}{t}$$ Hubble constant is time variable

X-ray gas in Clusters

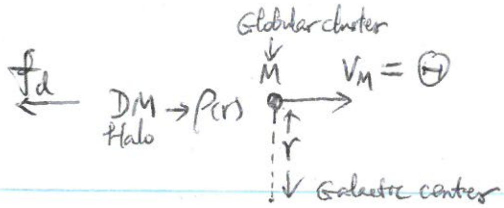
Thermal Bremsstrahlung, $l_\nu \, d\nu = 5.4 \times 10^{-52} (4\pi n_e^2) T^{-\frac{1}{2}} e^{-h\nu/kT} d\nu \; [\mathrm{W\,m^{-3}}]$

$$\mathcal{L}_{vol} = \int_0^\infty l_\nu \, d\nu = 1.4 \times 10^{-40} n_e^2 T^{\frac{1}{2}} \; [\mathrm{W\,m^{-3}}]$$

$$\mathcal{L}_X = \mathcal{L}_{vol} \times \frac{4}{3}\pi R^3 \propto n_e^2 T^{\frac{1}{2}} R^3$$

$$M_{gas} = \frac{4}{3}\pi R^3 n_e \cdot m_H \propto L_X \cdot n_e^{-1} \cdot T^{-\frac{1}{2}}$$

$\alpha \propto T^{-\frac{1}{2}}$ For recombination: $M_H = m_{\cancel{H}} \cdot n_e \cdot V f = m_p L_{H\beta} / (\alpha_{H\beta} n_e h\nu) \propto L_{H\beta} n_e^{-1} T^{+\frac{1}{2}}$



Dynamical Friction

① Minor merger: force on M: $f_d = C \cdot \dfrac{G^2 M^2 \rho(r)}{V_M^2}$ from a surrounding medium with density profile $\rho(r)$

DM density profile ~ SIE $\quad \rho(r) = \dfrac{V_M^2}{4\pi G r^2}$, given $V_M = \Theta$ the rotation curve

$$f_d = C\left(\frac{V_M}{\sigma}\right) \frac{GM^2}{4\pi r^2}$$

SIE: $\rho(r) = \dfrac{\sigma^2}{2\pi G r}$, $V_M = \sqrt{2}\,\sigma$

$M(r) = \dfrac{2\sigma^2 r}{G}$

torque due to dynamical friction

$$\tau = r \cdot f_d = \frac{dL}{dt}, \quad L = M V_M \cdot r$$

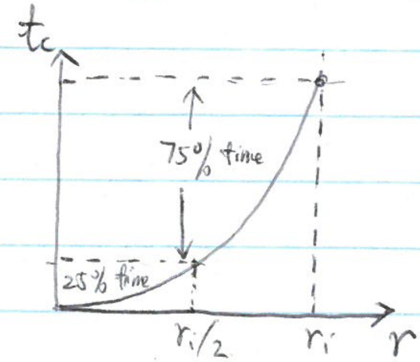
$$\Rightarrow \quad -r \cdot C \cdot \frac{GM^2}{4\pi r^2} = \frac{d}{dt}(M V_M \cdot r) = M V_M \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad r\,dr = -\frac{CGM}{4\pi V_M}\,dt$$

integrate both sides

$$\int_{r_i}^{0} r\,dr = -\frac{CGM}{4\pi V_M}\int_0^{t_c} dt$$

$$t_c = \frac{2\pi \boxed{V_M r_i^2}}{CG\boxed{M}}$$ where r_i is the initial captured position



② Major merger [Binney & Tremaine §8.1.1 b]

main galaxy $\quad \sigma_M \sim 200$ km/s

satellite galaxy $\quad \sigma_s \sim 100$ km/s

$$t_c = \frac{2.34}{\ln\Lambda}\boxed{\frac{\sigma_M^2}{\sigma_s^3}\, r_i} = \frac{2.7\,\text{Gyr}}{\ln\Lambda}\,\frac{r_i}{30\,\text{kpc}}\left(\frac{\sigma_M}{200\,\text{km/s}}\right)^2\left(\frac{100\,\text{km/s}}{\sigma_s}\right)^3$$

the Coulomb logarith

$$\ln\Lambda = \ln\frac{b_{max}}{b_{90}} \simeq 6 \quad \text{for } b_{max} = 5\,\text{kpc},\ M = 10^8 M_\odot,\ V_{typ} \simeq \sigma = 200\,\text{km/s}$$

$$= \ln\frac{b_{max} V_{typ}^2}{GM}, \quad b_{90} = \frac{GM}{V_{typ}^2} \quad \text{(impulse approximation)}$$

Line Excitation Mechanisms

① Photoionization → recombination → cascade down rapidly

Line luminosity ~ recombination rate ~ ionization rate

$$n_e n_{ion} \alpha_B(T) \sim Q/V \sim \int_{\nu_0}^{\infty} a_\nu n_{HI} \frac{F_\nu}{h\nu} d\nu$$

② collisional excitation → spontaneous decay

$$n_1 n_e q_{12} = n_2 n_e q_{21} + n_e A_{21} \quad , \quad n_c \equiv A_{21}/q_{21}$$

$$q_r \equiv \int_0^\infty u \, \sigma(u) f(u,T) \, du \propto 1/\sqrt{T}$$

③ Photon absorption → spontaneous decay / stimulated emission

$$n_1 B_{12} J_\nu = n_2 B_{21} J_\nu + n_2 A_{21}$$

Continuum Emission Mechanisms

① Stellar blackbody emission $F_\nu = \pi B_\nu = \frac{2\pi h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1}$

② Dust thermal emission $F_\nu = \pi B_\nu (1 - e^{-\tau})$, $\tau = (\nu/\nu_0)^\beta$

③ Free-Free / Thermal Bremsstrahlung $F_\nu \propto \nu^3$ @ low ν , $F_\nu \propto e^{-h\nu/kT}$ @ high ν

④ Synchrotron emission $F_\nu \propto \nu^{2.5}$ @ low ν , $F_\nu \propto \nu^{-0.7}$ @ high ν

⑤ Inverse Compton scattering $\nu' = \gamma^2 \nu_0$

Chap 29 Cosmology

|Expansion history| composition | critical density|

Goals: $R(t)$, $\Omega(R)$, $H(R)$

to connect with observables $R = \frac{1}{1+z}$

Newtonian Cosmology [Pressureless Dust Model]

scale factor $R(t) = \frac{1}{1+z}$ (needs proof) mass conservation: $R^3(t)\rho(t) = R_0^3\rho_0 = \rho_0$

comoving coordinate $r_0 = r(t)/R(t)$

Hubble parameter $H(t) = \frac{1}{R(t)}\frac{dR(t)}{dt}$

Hubble's law

$$\begin{cases} v(t) = H(t)\,r(t) = H(t)\,R(t)\,r_0 \\ v(t) = \frac{dr(t)}{dt} = \frac{dR(t)}{dt} r_0 \end{cases}$$

Friedmann Eqn: Energy conservation $K(t) + U(t) = E$

$$\frac{1}{2}v^2(t) - G\frac{M_r}{r(t)} = -\frac{1}{2}kc^2 r_0^2 \quad , \quad M_r = \frac{4}{3}\pi r^3(t)\rho(t) = \frac{4}{3}\pi R^3(t)\,r_0^3\cdot\rho(t)$$

$$\Rightarrow H^2(t)\cdot R^2(t)\,r_0^2 - \frac{8}{3}\pi G r^2(t)\cdot\rho(t) = -\cancel{\frac{1}{2}}kc^2 r_0^2$$

$$\Rightarrow \left(H^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2 \quad \cdots\ [29.9]$$

$\boxed{\text{Critical density}}$

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}$$ the density that results in $k=0$

rewrite energy conservation equation for $k=0$

$$\boxed{R^2\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho_0/R^3\right] = -kc^2}$$ Friedmann Equation

$$\text{or} \quad \left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho_0/R = -kc^2$$

Solution : boundary condition for current time

① $H(z)$ } for any k

② $\Omega(z)$

$$H(t_0) = \frac{1}{R_0}\frac{dR(t_0)}{dt} = 71\ \text{km/s/Mpc} \equiv H_0$$

$$R(t_0) = 1$$

③ $\underbrace{t(R) \Leftrightarrow R(t)}_{\text{for } k=0 \text{ only}}$ so the equation to solve is [with only ρ_0 as the free parameter]

$$\boxed{\left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho_0/R = -kc^2 = H_0^2 - \frac{8}{3}\pi G\rho_0} = H_0^2(1-\Omega_0)$$

$$\rho_0 = \Omega_0\cdot\rho_{c,0} = \Omega_0\frac{3H_0^2}{8\pi G} \quad , \quad \rho_{c,0} = 1.88\times10^{-26}\,h^2\ \text{kg m}^{-3}$$

density parameter: $\Omega(t) = \rho(t)/\rho_c(t)$, now we can rewrite Eq. 29.9

$$\begin{cases} \boxed{H^2(1-\Omega)R^2 = -kc^2 = H_0^2(1-\Omega_0)} \\ \text{combined with } \frac{\Omega}{\Omega_0} = \frac{\rho}{\rho_0}\frac{H_0^2}{H^2} = (1+z)^3\frac{H_0^2}{H^2} \Rightarrow \Omega H^2 = (1+z)^3\Omega_0 H_0^2 \end{cases}$$

$$\boxed{\Omega = 1\Big/\left(1+\left(\tfrac{1}{\Omega_0}-1\right)R\right)}$$ Flatness Problem

we can solve for $\boxed{H = H_0(1+z)(1+\Omega_0 z)^{1/2}}$ & $\boxed{\Omega = \left(\frac{1+z}{\Omega_0 z+1}\right)\Omega_0 = 1 + \frac{\Omega_0 - 1}{1+\Omega_0 z}}$

Solution of Friedman Equation - Expansion history in radiation/matter/Λ eras

① $\Omega_{m,0} = 1$, $\Omega_{rel,0} = 0$, $\Omega_{\Lambda,0} = 0$ $\quad \sum \Omega = 1.0 \Rightarrow k = 0$

matter-only

matter era

$$\begin{cases} \frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 - \frac{8\pi}{3} G \rho_m = 0 \\ \rho_{m,0} = \frac{3 H_0^2}{8\pi G} = \rho_m \cdot R^3 \end{cases}$$

$$\Rightarrow \left(\frac{dR}{dt}\right)^2 = \frac{H_0^2}{R} \quad \Rightarrow \int_0^R \sqrt{R'}\, dR = H_0 \int_0^t dt'$$

$$\Rightarrow R(t) = \left(\frac{3}{2}\right)^{2/3} \left(\frac{t}{t_H}\right)^{2/3}$$

② $\Omega_{m,0} = 0$, $\Omega_{rel,0} = 1$, $\Omega_{\Lambda,0} = 0$

photon-only

radiation era

$$\begin{cases} \frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 - \frac{8\pi}{3} G \rho_{rel} = 0 \\ \rho_{rel,0} = \frac{3 H_0^2}{8\pi G} = \rho_{rel} \cdot R^4 \end{cases}$$

$$\Rightarrow \frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 - \frac{H_0^2}{R^4} = 0 \quad \Rightarrow \int_0^R R'\, dR' = H_0 \int_0^t dt'$$

$$\Rightarrow R(t) = \sqrt{2 H_0 t} = \sqrt{2} \left(\frac{t}{t_H}\right)^{1/2}$$

③ $\Omega_{m,0} = 0$, $\Omega_{rel,0} = 0$, $\Omega_{\Lambda,0} = 1$

Λ-only

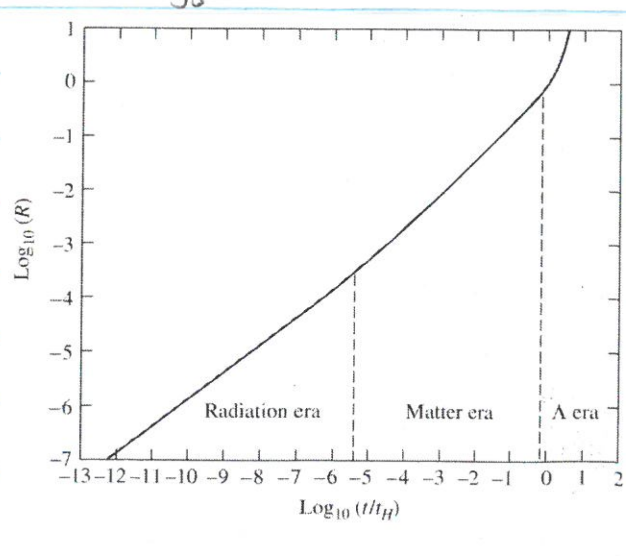
Λ era

$$\rho_{\Lambda,0} = \frac{3 H_0^2}{8\pi G} = \rho_\Lambda$$

$$\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 = H_0^2 \quad \Rightarrow \int_0^R \frac{1}{R'}\, dR' = H_0 \int_0^t dt'$$

$$\Rightarrow \ln R - \ln R_0 = H_0 t$$

$$\frac{R(t)}{R_0} = e^{t/t_H}$$



Including Pressure in Cosmology model

① $\boxed{\text{fluid equation}}$ from 1st law of thermodynamics

$$dQ = dE + PdV \quad \& \quad dQ = 0 \quad \Rightarrow \quad dE/dt = -P\,dV/dt$$

given that $E = \frac{4}{3}\pi r_0^3 R^3 \cdot u = \frac{4}{3}\pi r_0^3 R^3 \rho \cdot c^2$ & $V = \frac{4}{3}\pi r_0^3 R^3$

we have $$\frac{d(\rho R^3)}{dt} = -\frac{P}{c^2}\frac{d(R^3)}{dt}$$

② $\boxed{\text{Equation of state}}$

$$P = wu = w\rho c^2$$, plug this into the fluid equation ⇒

③ $\boxed{\rho\text{–}R \text{ relation}}$ $R^{3(1+w)}\rho = \rho_0 = \text{const.}$

$w_m = 0$, $w_\gamma = \frac{1}{3}$, $w_\Lambda = -1$ for matter, radiation, dark energy

① + Friedmann ⇒ ④ $\boxed{\text{acceleration equation}}$ $$\frac{d^2R}{dt^2} = -\frac{4}{3}\pi G\left(\rho + \frac{3P}{c^2}\right)R$$

Deceleration Parameter (positive values mean deceleration)

$$q(t) \equiv -\frac{R\, d^2R/dt^2}{(dR/dt)^2} = \frac{1}{2}\sum_i (1+3w_i)\,\Omega_i(t)$$
$$= \frac{1}{2}\Omega_m + \Omega_\gamma - \Omega_\Lambda$$

$$q_0 = q(t_0) = \frac{1}{2}\times 0.3 - 0.7 = -0.55$$

⑤ $\boxed{\text{Friedmann equation}}$

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right]R^2 = -kc^2 = H_0^2(1-\Omega_0) = \text{const.}$$

derivation of acceleration equation

multiply R to ⑤ and take $\frac{d}{dt}$ on both sides, plug in ① fluid equation

$$\left(\frac{dR}{dt}\right)^2 + 2R\frac{d^2R}{dt^2} + 8\pi G\frac{P}{c^2}R^2 = -kc^2 = \left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho R^2$$

rearrange $$\frac{d^2R}{dt^2} = -\frac{4}{3}\pi G R\left(\rho + \frac{3P}{c^2}\right)$$

Relativistic Cosmology

Schwarzschild Metric

$$(ds)^2 = \left(c\,dt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

[17.22]

Robertson-Walker Metric

$$(ds)^2 = (c\,dt)^2 - \left(\frac{dr}{\sqrt{1-Kr^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

where $K(t) = \frac{k}{R^2(t)}$ is the time-dependent curvature

$r(t) = R(t)\cdot\varpi = R(t)\cdot r_0$ is the coordinate radial distance

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{dr_0}{\sqrt{1-kr_0^2}}\right)^2 + (r_0\,d\theta)^2 + (r_0\sin\theta\,d\phi)^2\right] \quad [29.106]$$

Friedmann Equation

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G(\rho_m + \rho_{rel}) - \frac{1}{3}\Lambda c^2\right]\cdot R^2 = -kc^2$$

define $\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G}$, $\rho_c \equiv \frac{3H^2}{8\pi G}$, $\Omega \equiv \frac{\rho}{\rho_c}$, $H \equiv \frac{1}{R}\frac{dR}{dt}$

$$H^2(t)\left[1-(\Omega_m + \Omega_{rel} + \Omega_\Lambda)\right]R^2 = -kc^2$$

Solution of Friedmann Equation ① H vs z ② t vs. R or $(1+z)$ ③ Ω vs. z

Boundary condition $-kc^2 = H_0^2(1-\Omega_0)$ & cosmological z: $R \equiv \frac{1}{1+z}$

Ratios of Ω's

$$\frac{\Omega_m}{\Omega_{m,0}} = \frac{\rho_m H_0^2}{\rho_{m,0} H^2} \Rightarrow \Omega_m H^2 = (1+z)^3\,\Omega_{m,0} H_0^2$$

For photons

$$u = \frac{4\pi}{c}\int_0^\infty B_\nu(T)\,d\nu = \frac{4\sigma}{c}T^4 \equiv aT^4$$

$$\boxed{\rho_{rel} = \frac{u_{rel}}{c^2} = \frac{g_* a T^4}{2c^2} \qquad g_* = 3.363 \quad [29.77]}$$

for photons + neutrinos

$$\frac{\Omega_{rel}}{\Omega_{rel,0}} = \frac{\rho_{rel} H_0^2}{\rho_{rel,0} H^2} \Rightarrow \Omega_{rel} H^2 = (1+z)^4\,\Omega_{r,0} H_0^2$$

$$\frac{\Omega_\Lambda}{\Omega_{\Lambda,0}} = \frac{H_0^2}{H^2} \Rightarrow \Omega_\Lambda H^2 = \Omega_{\Lambda,0} H_0^2$$

Add all five equations together

$$H^2(z) = H_0^2(1+z)^2\left[\Omega_{m,0}(1+z) + \Omega_{rel,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2} + 1 - \Omega_0\right]$$

$\Rightarrow \Omega_m(z),\ \Omega_{rel}(z),\ \Omega_\Lambda(z)$

Solution of Friedmann Equation ② $t(R)$, t vs. R relation

for $k=0$, we have

$$\left(\frac{1}{R}\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G\left(\rho_m + \rho_{rel} + \rho_\Lambda\right)$$

$$\rho_m = \rho_{m,0}\frac{1}{R^3}, \quad \rho_{rel} = \rho_{rel,0}\frac{1}{R^4}, \quad \rho_\Lambda = \rho_{\Lambda,0}$$

$$\Rightarrow \left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G R^2\left(\rho_{m,0}\frac{1}{R^3} + \rho_{rel,0}\frac{1}{R^4} + \rho_{\Lambda,0}\right)$$

$$\Rightarrow (dt)^2 = \frac{3}{8\pi G}\frac{R^2 (dR)^2}{\rho_{m,0}R + \rho_{rel,0} + \rho_{\Lambda,0}R^4}$$

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G}, \quad t_H = \frac{1}{H_0}$$

$$\Rightarrow t = \int_0^t dt = \int_0^R \sqrt{\frac{3}{8\pi G}}\cdot\frac{R'dR'}{\sqrt{\rho_{m,0}R' + \rho_{rel,0} + \rho_{\Lambda,0}R'^4}} = t_H\int_0^R \frac{R'dR}{\sqrt{\Omega_{m,0}R' + \Omega_{rel,0} + \Omega_{\Lambda}}}$$

this is the age of the Universe when the scale factor is $R \equiv \frac{1}{1+z}$

Special Analytical Solution for $\Omega_0 = \Omega_{m,0} = 1$, $\Omega_{rel,0} = 0$, $\Omega_\Lambda = 0 \Rightarrow k=0$

① Expansion history (R vs. t)

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G\rho_0}{3R} = 0, \quad \rho_0 = \rho_{c,0} = \frac{3H_0^2}{8\pi G}, \quad \rho = \frac{\rho_0}{R^3}$$

$$\Rightarrow \left(\frac{dR}{dt}\right)^2 - \frac{H_0^2}{R} = 0 \quad \Rightarrow \int_0^R \sqrt{R'}\,dR' = H_0\int_0^t dt'$$

$$\Rightarrow R(t) = \left(\frac{3}{2}\right)^{2/3}\left(\frac{t}{t_H}\right)^{2/3} \quad \text{where } t_H = 1/H_0$$

② Age of the Universe t vs. z, $t_0 \equiv t(z=0)$

Rewrite R vs. t relation with $R(t) = \frac{1}{1+z}$

$$\frac{t(z)}{t_H} = \frac{2}{3}\frac{1}{(1+z)^{3/2}} \quad \Rightarrow t(z=\infty) = 0, \quad t(z=0) = \frac{2}{3}t_H$$

③ Lookback time t_L vs. z

$$t_L = t(z=0) - t(z) = \frac{2}{3}t_H\left(1 - \frac{1}{(1+z)^{3/2}}\right) = t_0\left(1 - \frac{1}{(1+z)^{3/2}}\right)$$

Cosmological Redshift : $\dfrac{1}{R(t_e)} = \dfrac{\lambda_o}{\lambda_e} \equiv 1+z$

start with R-W metric for a null geodesic ($ds=0$) along radial path

$$(c\,dt)^2 - R^2\left(\frac{dr_o}{\sqrt{1-kr_o^2}}\right)^2 = 0$$

take negative root so that r_o decreases with time

$$\frac{-c\,dt}{R(t)} = \frac{dr_o}{\sqrt{1-kr_o^2}}$$

for a photon emitted at t_e and received at t_o, we can integrate the path

$$\int_{t_e}^{t_o}\frac{c\,dt}{R(t)} = \int_{r_{o,o}}^{r_{o,e}}\frac{dr_o}{\sqrt{1-kr_o^2}} \quad \& \; r_{o,o}=0 \text{ comoving coordinate of Earth}$$

for the next photon emitted at $t_e+\Delta t_e$, and received at $t_o+\Delta t_o$

$$\int_{t_e+\Delta t_e}^{t_o+\Delta t_o}\frac{c\,dt}{R(t)} = \int_{r_{o,o}}^{r_{o,e}}\frac{dr_o}{\sqrt{1-kr_o^2}}$$

$$\int_{t_e+\Delta t_e}^{t_o+\Delta t_o} = \int_{t_e}^{t_o} - \int_{t_e}^{t_e+\Delta t_e} + \int_{t_o}^{t_o+\Delta t_o}$$

subtracting the two equations

$$\int_{t_o}^{t_o+\Delta t_o}\frac{c\,dt}{R(t)} - \int_{t_e}^{t_e+\Delta t_e}\frac{c\,dt}{R(t)} = 0$$

$$\Rightarrow \frac{c\cdot\Delta t_o}{R(t_o)} - \frac{c\cdot\Delta t_e}{R(t_e)} = 0 \Rightarrow \Delta t_o = \frac{\Delta t_e}{R(t_e)}$$

$$\Rightarrow \boxed{\frac{1}{R(t_e)} = \underbrace{\frac{\Delta t_o}{\Delta t_e}}_{\text{time dilation}} = \underbrace{\frac{\lambda_o}{\lambda_e}}_{\text{redshift}} \equiv 1+z}$$

Adiabatic Expansion of Photon Gas [RP 23.1 page 532]

$$u = aT^4,\quad P = \frac{1}{3}u = \frac{1}{3}aT^4 \Rightarrow E = u\cdot V = aT^4\cdot V \quad \boxed{V = r_o^3 R(t)}$$

1st law of Thermodynamics $dQ = dE + PdV$, adiabatic $dQ=0$

$$\frac{dE}{dt} = -P\frac{dV}{dt} \Rightarrow a\cdot\left(4T^3\frac{dT}{dt}V + T^4\frac{dV}{dt}\right) = -\frac{a}{3}T^4\frac{dV}{dt}$$

$$\Rightarrow \frac{1}{T}\frac{dT}{dt} = -\frac{1}{3V}\frac{dV}{dt} = -\frac{1}{R}\frac{dR}{dt} \Rightarrow \boxed{T \propto R^{-1} = 1+z}$$