ASTR3771

Chap 1 Celestial Sphere

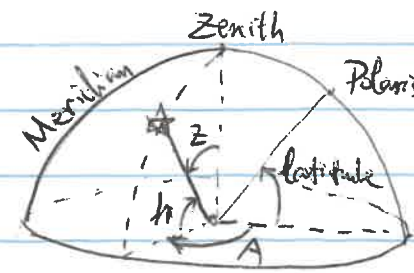
Readings: CO §1.3 Homework: Python exercise & Visibility Plot

Alt-Az System:

Altitude (h)

Zenith distance (z)

Azimuth (A)



Equatorial System

Declination (δ) → degrees N/S of celestial equator

Right Ascension (α) → degrees from vernal equinox (Υ) üpsilon (ram) to the hour circle

Time Solar vs Sidereal

Local Sidereal Time (LST) $\equiv$ hour angle of Υ (H)

Hour Angle (H): angle to the observer's meridian, similar to α

Solar Day: average interval between meridian crossing of the Sun

Sidereal Day: interval between consecutive meridian crossing of a star

$LST = H(\Upsilon) = RA(\text{meridian})$ HA = LST − RA

Precession

Period = 25,770 years

J2000 = noon on Jan 1, 2000 in Greenwich, England (UT)

$\Delta\alpha = M + N \sin\alpha \tan\delta$ M & N are 3rd order polynomials of

$\Delta\delta = N \cos\alpha$ $T = (t - 2000.0)/100$

$= (JD - 2451545)/36525$

Julian Date

zero time for JD: Jan 1, 4713 BC, noon UT

J2000.0 = JD 2451545.0 , 1 year = 365.25 solar days

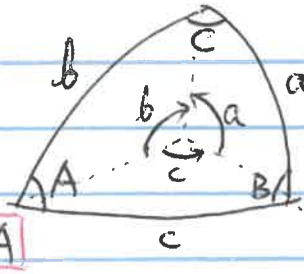
MJD = JD − 2400000.5 so that it begins at midnight.

Proper Motion

$\mu \equiv \dfrac{d\theta}{dt} = \dfrac{v_\theta}{r}$. transverse velocity divided by distance

Spherical Trigonometry

Law of sines $\boxed{\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}}$

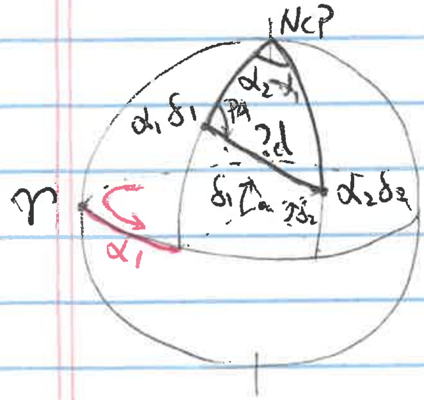


Law of cosines

$$\boxed{\cos a = \cos b \cos c + \sin b \sin c \cos A}$$

$$\cos A = -\cos B \cdot \cos C + \sin B \cdot \sin C \cdot \cos a$$

Application – distance between two coordinates & PA



d is the angular distance

$$\cos d = \cos(90^\circ - \delta_1) \cdot \cos(90^\circ - \delta_2) + \sin(90^\circ - \delta_1) \sin(90^\circ - \delta_2) \cdot \cos(\alpha_2 - \alpha_1)$$

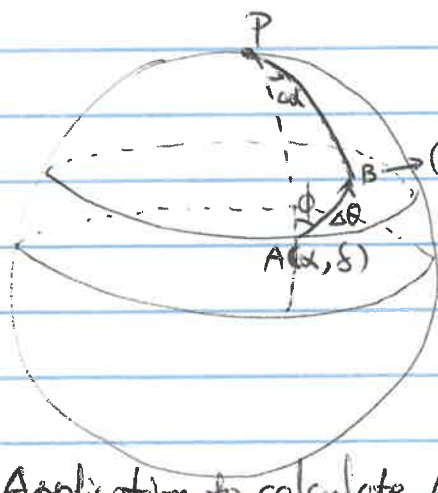
Position angle of (α_2, δ_2) relative to (α_1, δ_1), $PA = \phi$

$$\frac{\sin(90^\circ - \delta_2)}{\sin PA} = \frac{\sin d}{\sin(\alpha_2 - \alpha_1)}$$

Application to proper motion → small angle approximations

$$\Delta\alpha = \Delta\theta \cdot \frac{\sin\phi}{\cos\delta} , \quad \Delta\delta = \Delta\theta \cos\phi$$

$$\Rightarrow (\Delta\theta)^2 = (\Delta\alpha \cdot \cos\delta)^2 + (\Delta\delta)^2$$



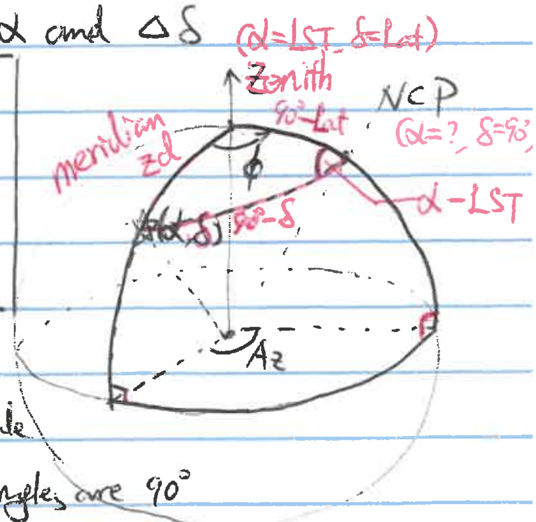
$B \rightarrow (\alpha + \Delta\alpha, \delta + \Delta\delta)$

Problem: from $\Delta\theta$, ϕ, α, δ, solve for $\Delta\alpha$ and $\Delta\delta$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$(\alpha = LST, \delta = Lat)$



Application to calculate Alt & Azimuth

Coordinates of Zenith $\alpha = LST$, $\delta = \text{latitude}$

$Az = PA = \phi$ because the other two angles are 90°

so this problem is the same as the first application

$$\cos zd = \cos(90^\circ - \delta) \cdot \cos(90^\circ - lat) + \sin(90^\circ - \delta) \cdot \sin(90^\circ - lat) \cdot \cos(\alpha - LST)$$

Chap 2 Celestial Mechanics

History: Hipparchus (~150 BC) - Greek - deferent & epicycle
Ptolemy (~100 AD) - equants

Copernicus (1473-1543) - Polish - heliocentric model
Tycho (1546-1601) - Denmark - 4′ positional accuracy, no parralax
Kepler (1571-1630) - German - Mars is on elliptical orbits, K1/2/3
Galileo (1564-1642) - Italian - phases of Venus, moos of Jupiter
Newton (1642-1727) - British - Laws of motion, Law of gravity
Theorems of gravity

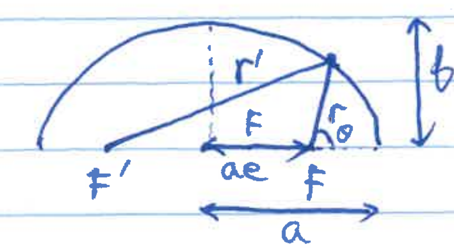
[sə'nädik]
Retrograde Motion & Synodic Period - Period between oppositions.

$$\frac{1}{S} = \frac{1}{P_\oplus} - \frac{1}{P} \quad \text{or} \quad \frac{1}{P} - \frac{1}{P_\oplus}$$

$[e_\oplus = 0.0167]$

for Mars: $S = 2.135$ yr, $P = 1.88$ yr, $e = 0.0935$, $a = 1.5$ AU

Geometry of Ellipse



definition: $r + r' = 2a$, $F = a \cdot e$

equalities: $b^2 = a^2(1-e^2)$

Area $= \pi \cdot a \cdot b$

polar coordinates (r, θ)

$$\begin{cases} r'^2 = (2a - r)^2 \\ r'^2 = (2ae + r\cdot\cos\theta)^2 + r^2\sin^2\theta \end{cases} \Rightarrow \quad a - ae^2 = r(e\cos\theta + 1)$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

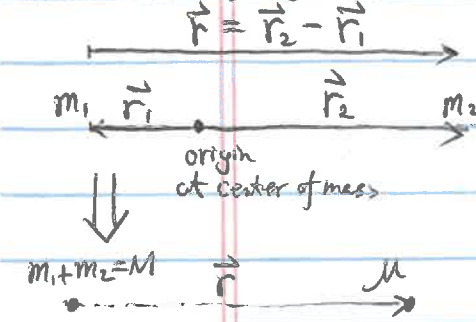
Parabola $(e=1)$

$$r = \frac{2P}{1+\cos\theta}$$

Hyperbola $(e>1)$

$$r = \frac{a(e^2-1)}{1+e\cos\theta}$$

Reduced mass & center-of-mass coordinate system for two body problem



definition of CoM: $\dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = 0$

$$\Rightarrow \vec{r}_1 = -\frac{m_2}{m_1+m_2}\vec{r} \quad , \quad \vec{r}_2 = \frac{m_1}{m_1+m_2}\vec{r}$$

$\vec{r}_2 - \vec{r}_1 = \vec{r}$ the displacement vector

define reduced mass $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ we have $\vec{r}_1 = -\dfrac{\mu}{m_1}\vec{r}$

$\vec{r}_2 = \dfrac{\mu}{m_2}\vec{r}$

this definition greatly simplifies the energy & angular momentum expression

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|} \quad , \quad \vec{v}_i = \frac{d\vec{r}_i}{dt}$$

$$= \underline{\frac{1}{2}\mu v^2} - G\frac{M\mu}{r} \quad , \quad M = m_1 + m_2 ,$$

$$m_1 v_1^2 + m_2 v_2^2 = m_1 \frac{\mu^2}{m_1^2}\left(\frac{dr}{dt}\right)^2 + m_2 \frac{\mu^2}{m_2^2}\left(\frac{dr}{dt}\right)^2$$

$$= \mu^2 v^2 \left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \mu v^2$$

$$\vec{L} = m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2 = m_1 \cdot \left(-\frac{\mu}{m_1}\vec{r}\right) \times \left(-\frac{\mu}{m_1}\frac{d\vec{r}}{dt}\right)$$

$$+ m_2\left(\frac{\mu}{m_2}\vec{r}\right) \times \left(\frac{\mu}{m_2}\frac{d\vec{r}}{dt}\right)$$

$$= \mu^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \cdot \vec{r} \times \vec{v} = \mu \vec{r} \times \vec{v}$$

for Coulomb forces: $U = -\dfrac{1}{4\pi\epsilon_0}\dfrac{q_1 q_2}{r}$

$K = \dfrac{1}{2}\mu v^2$

$\vec{L} = \mu \vec{r} \times \vec{v} \Rightarrow L = \mu r v$

Derivation of Kepler's Laws

Spherical symmetric systems

$$\mathcal{L}(r, \dot{r}, \varphi, \dot{\varphi}) = \frac{1}{2}[\dot{r}^2 + (r\dot{\varphi})^2] - \Phi(r)$$

Equations of motion:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad \text{for } q \in (r, \varphi)$$

EoM1 $\Rightarrow \frac{d}{dt}(r^2\dot{\varphi}) = 0 \Rightarrow r^2\dot{\varphi} = \text{const} \equiv \ell$ (1st IoM) K2

EoM2 $\Rightarrow \boxed{\frac{d^2r}{dt^2} - r\dot{\varphi}^2 + \frac{d\Phi}{dr} = 0}$ we need to remove t from this equation to obtain the relation between r & φ

replace dt with $dt = \frac{r^2}{\ell}d\varphi$ & $\dot{\varphi}^2 = \ell^2/r^4$

$$\frac{\ell}{r^2}\frac{d}{d\varphi}\left(\frac{\ell}{r^2}\frac{dr}{d\varphi}\right) - \frac{\ell^2}{r^3} = -\frac{d\Phi}{dr}$$

take ℓ^2 to the right side, and replace $u \equiv \frac{1}{r}$ & $dr = -\frac{1}{u^2}du$

$$u^2\frac{d}{d\varphi}\left[u^2\left(-\frac{1}{u^2}\frac{du}{d\varphi}\right)\right] - u^3 = -\left[-u^2\frac{d\Phi}{du}\right]\cdot\frac{1}{\ell^2}$$

EoM2u $\Rightarrow \boxed{\frac{d^2u}{d\varphi^2} + u = -\frac{1}{\ell^2}\frac{d\Phi}{du}}$

For point mass $\Phi(r) = -\frac{GM}{r} = -GMu$

$$\frac{d^2u}{d\varphi^2} + u = \frac{GM}{\ell^2} \quad \text{from EoM2u}$$

solve this differential equation

$$u(\varphi) = c\cdot\cos(\varphi - \varphi_0) + \frac{GM}{\ell^2}$$

K1

$$\Rightarrow \boxed{r(\varphi) = \frac{1}{u} = \frac{\ell^2/GM}{1 + \frac{c\ell^2}{GM}\cos(\varphi - \varphi_0)} \equiv \frac{a(1-e^2)}{1 + e\cos(\varphi - \varphi_0)}}$$

$$\Rightarrow e = \frac{c\ell^2}{GM} \quad \& \quad a = \frac{\ell^2}{GM(1-e^2)}$$

what is c? when $\varphi = \varphi_0$, $r = q = a(1-e)$, $\frac{1}{q} = c + \frac{GM}{\ell^2}$

Kepler's 3rd law

$$P = \frac{\pi a b}{\frac{1}{2} r^2 \dot{\varphi}} = \frac{2\pi a^2 \sqrt{1-e^2}}{\ell}$$

because $a = \dfrac{\ell^2}{GM(1-e^2)} \Rightarrow \ell = \sqrt{GMa(1-e^2)}$, we have

$$P = \frac{2\pi a^2 \sqrt{1-e^2}}{\sqrt{GMa(1-e^2)}} = 2\pi \sqrt{\frac{a^3}{GM}} \Rightarrow P^2 = \frac{4\pi^2}{GM} a^3$$

Proof of energy conservation

multiply $\dfrac{du}{d\varphi}$ to both side of EoM2u, and then integrate over $d\varphi$

$$\frac{du}{d\varphi}\frac{d^2u}{d\varphi^2} + \frac{du}{d\varphi} \cdot u = -\frac{1}{\ell^2}\frac{du}{d\varphi} \cdot \frac{d\Phi}{du}$$

$$\Rightarrow \frac{d}{d\varphi}\left[\frac{1}{2}\left(\frac{du}{d\varphi}\right)^2\right] + u\frac{du}{d\varphi} + \frac{1}{\ell^2}\frac{d\Phi}{d\varphi} = 0$$

integrate over $d\varphi$

$$\Rightarrow \left(\frac{du}{d\varphi}\right)^2 + \frac{2\Phi}{\ell^2} + u^2 = \text{const}$$ (what's the physical meaning of this IoM?)

multiply ℓ^2 to the left side

$$\ell^2\left(\frac{du}{d\varphi}\right)^2 + 2\Phi + \ell^2 u^2 \quad , \ell^2 = (r^2\dot{\varphi})^2$$

$$= r^4\left(\frac{du}{dt}\right)^2 + 2\Phi + r^4\dot{\varphi}^2 u^2$$

$$= \left[\left(\frac{dr}{dt}\right)^2 + r^2\dot{\varphi}^2\right] + 2\Phi \equiv 2 \cdot E = \text{const} \cdot \ell^2$$

because ℓ is a constant, E is a constant as well.

Vis Viva Equation

evaluate E at perihelion, $q = a(1-e)$, $\dot{r} = 0$

$$E = \frac{1}{2} r^2 \dot{\varphi}^2 + \Phi(r) \quad \text{where } r = q = a(1-e)$$

$$= \frac{L^2}{\cancel{2a^2}\, 2a^2(1-e)^2} - \frac{GM}{a(1-e)}$$

as $\boxed{a = \frac{L^2}{GM(1-e^2)}} \Rightarrow L^2 = GMa(1-e^2)$

$$E = \frac{a \cdot GM(1-e^2)}{2a^2(1-e)^2} - \frac{GM}{a(1-e)}$$

$$= \frac{GM}{a(1-e)}\left[\frac{1}{2}(1+e) - 1\right] = \frac{-GM}{2a}$$

therefore we have the vis viva equation

$$\frac{1}{2}v^2 - \frac{GM}{r} = -\frac{GM}{2a}$$

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) = \frac{GM}{a}\left(\frac{2a}{r} - 1\right)$$

Transfer Orbit (Hohmann)

an elliptical orbit whose pericenter is on an inner circular orbit & apocenter is on an outer circular orbit

$$a_t = (a_{low} + a_{high})/2$$

$$P(a_t) = a_t^{3/2} = 2 \cdot t_{transfer}$$

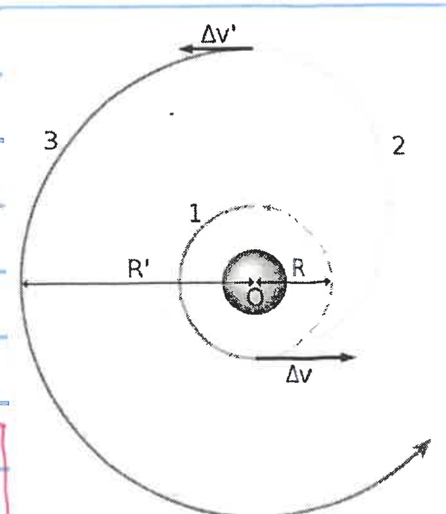
$$V_{peri} = \frac{2\pi a_t}{P_t}\left(\frac{2a_t}{a_{low}} - 1\right)^{1/2}$$

$$V_{ap} = \frac{2\pi a_t}{P_t} \cdot \left(\frac{2a_t}{a_{high}} - 1\right)^{1/2}$$

where we have used K3

$$P^2 = \frac{4\pi^2}{GM} a^3$$

$$\frac{GM}{a} = \frac{4\pi^2 a^2}{P^2}$$



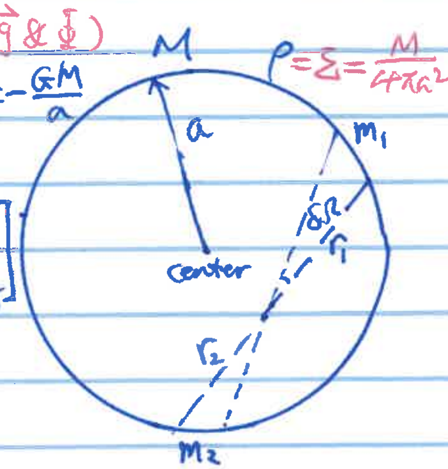
$\boxed{\text{What's the } e \text{ of the orbit?}}$

$$q = a(1-e) = a_{low} \Rightarrow 1 - e = \frac{2a_{low}}{a_l + a_h}$$

Newton's theorems & Potential of spherical systems.

1. objects inside a spherical shell (with constant density) feels ~~not~~ no net $\vec{g}$
2. gravity from a spherical shell (with constant density) is the same as if the shell's matter were in its center

Proof: using $\vec{g} = -\vec{\nabla}\Phi$ (this is based on the definitions of $\vec{g}$ & Φ)



Theorem #1: $\vec{g} = 0$ inside shell $(r < a)$ & $\Phi = -\frac{GM}{a}$

draw a small solid angle towards opposite directions

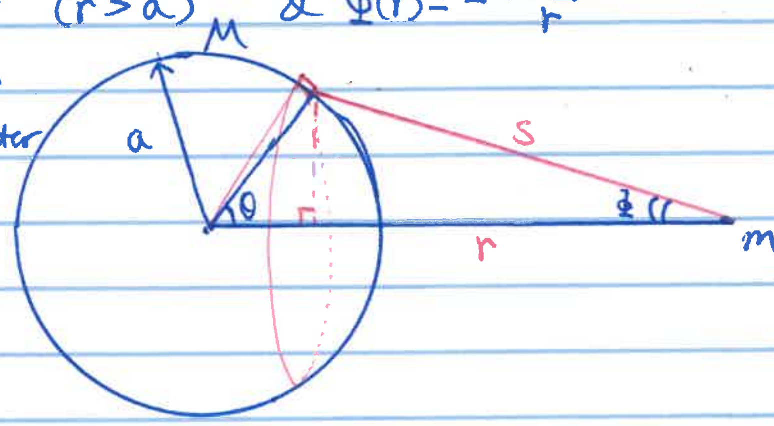
$$\left.\begin{aligned} m_1 &= r_1^2\,\delta\Omega\cdot \cancel{\rho}\Sigma \\ m_2 &= r_2^2\,\delta\Omega\,\cancel{\rho}\Sigma \end{aligned}\right\} \quad \text{where } \Sigma\cancel{\rho} = \frac{M}{4\pi a^2}\ \left[\text{surface mass density}\right]$$

the gravitational forces cancel each other $\Rightarrow \vec{g} = 0$

$$\Rightarrow \vec{\nabla}\Phi = 0 \quad \Rightarrow \Phi = \text{const} = -\frac{GM}{a}$$

Theorem #2: $\vec{g} = -\frac{GM}{r^2}\hat{r}$ $(r > a)$ & $\Phi(r) = -\frac{GM}{r}$

consider a ring perpendicular to the line between m & spherical center



$$\boxed{\begin{aligned} dM &= a\cdot d\theta \cdot 2\pi a\sin\theta \cdot \Sigma \\ s^2 &= a^2 + r^2 - 2ar\cos\theta \\ \cos\phi &= \frac{r - a\cdot\cos\theta}{s} \end{aligned}}$$

$$g = \int \frac{G}{s^2}\cos\phi\, dM$$

$$= \int_0^\pi \frac{G}{s^2}\left(\frac{r - a\cos\theta}{s}\right)\cdot a\cdot 2\pi a\sin\theta\,\Sigma\, d\theta$$

$$= 2\pi a^2 G\Sigma \int_0^\pi \frac{(r - a\cos\theta)\sin\theta}{s^3}\, d\theta = \frac{GM}{2}\cdot\int_0^\pi \frac{(r - a\cos\theta)\sin\theta}{s^3}\, d\theta$$

define $u \equiv s^2 = a^2 + r^2 - 2ar\cos\theta$ which range between $u_0 = (r-a)^2$ & $u_1 = (r+a)^2$

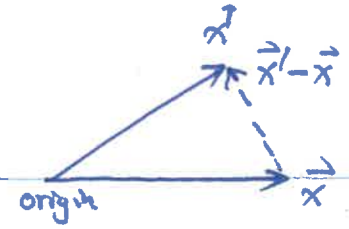
$$\cos\theta = \frac{r^2 + a^2 - u}{2ar}, \quad \sin\theta\, d\theta = -d\cos\theta = \frac{du}{2ar}$$

$$\int_0^\pi \frac{(r - a\cos\theta)}{s^3}\sin\theta\, d\theta = \int_{u_0}^{u_1} \frac{r - (r^2 + a^2 - u)/2r}{u^{3/2}}\cdot\frac{du}{2ar} = \int_{u_0}^{u_1} \frac{r^2 - a^2 + u}{4ar^2 u^{3/2}}\, du$$

$$= \frac{1}{4ar^2}\left[\int (r^2 - a^2)u^{-\frac{3}{2}}\, du + \int u^{-\frac{1}{2}}\, du\right] = \frac{r^2 - a^2}{4ar^2}\left(-2u^{-\frac{1}{2}}\Big|_{u_0}^{u_1}\right) + \frac{1}{4ar^2}\left(2u^{\frac{1}{2}}\right)\Big|_{u_0}^{u_1}$$

$$= \frac{2}{r^2}$$

Extended Distribution of Mass



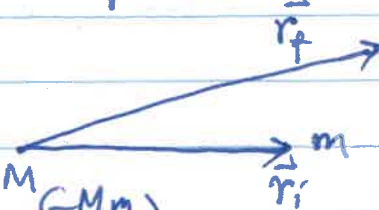
Gravitational field $\vec{g}(\vec{x}) \equiv G\int d^3x' \dfrac{\vec{x}'-\vec{x}}{|\vec{x}'-\vec{x}|^3}\rho(\vec{x}')$

Gravitational potential $\Phi(\vec{x}) \equiv -G\int d^3\vec{x}' \dfrac{\rho(\vec{x}')}{|\vec{x}'-\vec{x}|}$

$$\vec{g}(\vec{x}) = -\vec{\nabla}\Phi \quad \text{because} \quad \nabla_{\vec{x}}\frac{1}{|\vec{x}'-\vec{x}|} = \frac{\vec{x}'-\vec{x}}{|\vec{x}'-\vec{x}|^3}$$

Potential Energy: the work done against $\vec{g}$ in assembling a mass distribution $\rho(\vec{x})$

A single 2-body system ~~W=~~ $\cancel{\int d^3x\, \rho(\vec{x})\, \vec{g}(\vec{x})\cdot x}$ $\quad -\int_{\vec{r}_i}^{\vec{r}_f} m\vec{g}\cdot d\vec{r} = W_f - W_i$



$$\Rightarrow W_f - W_i = -\int_{\vec{r}_i}^{\vec{r}_f} \frac{GMm}{r^2}(-\hat{e}_r)\cdot d\vec{r}$$

$$= \int_{r_i}^{r_f} \frac{GMm}{r^2}\,dr = -\frac{GMm}{r_f} - \left(-\frac{GMm}{r_i}\right)$$

define $W(r=\infty)=0$, we have

$$W(r) = -\frac{GMm}{r} = m\cdot\Phi(r)$$

Potential of Point Mass or outside of a spherical body

$$\Phi(r) = -\frac{GM}{r}$$

Potential of homogeneous sphere (talk after proving Newton's theorems)

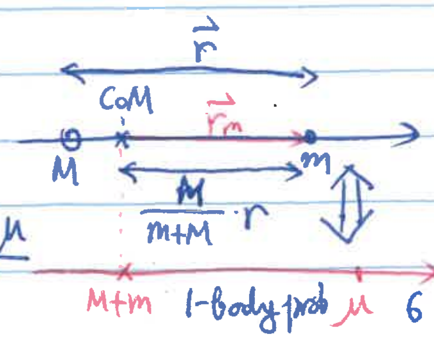
$M(r) = \frac{4}{3}\pi r^3\rho$ with a boundary at $r=a$

for $r<a$

$$\Phi(r) = -\frac{G}{r}\int_0^r 4\pi r'^2\rho\,dr' - G\int_r^a \frac{4\pi r'^2\rho\,dr'}{r'}$$

$$= -2\pi G\rho\left(a^2 - \frac{1}{3}r^2\right)$$

Reduced Mass of two body system



$\mu = \dfrac{Mm}{M+m}$, $\vec{r} \equiv \vec{r}_m - \vec{r}_M$, $\vec{v} \equiv d\vec{r}/dt$

$E = \frac{1}{2}mV_m^2 + \frac{1}{2}MV_M^2 - G\frac{mM}{r} = \frac{1}{2}\mu v^2 - G\frac{(M+m)\mu}{r}$

$\vec{L} = m\vec{r}_m\times\vec{v}_m + M\vec{r}_M\times\vec{V}_M = \mu\,\vec{r}\times\vec{v}$

Homogeneous Sphere (Application of Newton's Theorems)

$$M(r) = \frac{4}{3}\pi r^3 \rho \quad \text{for } r \le a \quad \Rightarrow dM = 4\pi r^2 \rho\, dr$$

inside the sphere, the potential has two components

$$\Phi(r) = -\frac{G}{r}\int_0^r 4\pi r'^2 \rho\, dr' - G\int_r^a \frac{4\pi r'^2 \rho\, dr'}{r'}$$

$$= -2\pi G\rho\left(a^2 - \frac{1}{3}r^2\right), \quad \rho = \frac{M}{\frac{4}{3}\pi a^3} \Rightarrow \Phi = \frac{3GM}{2a}\left[1 - \frac{r^2}{3a^2}\right]$$

write down the Lagrangian in Cartesian coordinates:

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{2}{3}\pi G\rho(x^2+y^2) + 2\pi G\rho a^2$$

the equations of motion are $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$ for $q \in (x, y)$

EoMs $\begin{cases} \ddot{x} + \frac{4}{3}\pi G\rho x = 0 \\ \ddot{y} + \frac{4}{3}\pi G\rho y = 0 \end{cases}$ solution $\begin{cases} x = A\cos(\Omega t + \varphi_x) \\ y = B\cos(\Omega t + \varphi_y) \end{cases}$

Azimuth period

$$P = \frac{2\pi}{\Omega} \quad \text{where } \Omega = \sqrt{\frac{4}{3}\pi G\rho} \quad \rightarrow \text{spherical harmonic oscillator}$$

elliptical orbits centered on the center of the sphere.

Virial Theorem: $2\langle K\rangle + \langle u\rangle = 0$ [regardless of geometry]

define $Q = \sum_{i=1}^{N} \vec{p}_i \cdot \vec{r}_i$ for a N-body self-gravitating system

$$\frac{dQ}{dt} = \frac{d}{dt}\left(\sum_i m_i \frac{d\vec{r}_i}{dt}\cdot\vec{r}_i\right) = \frac{d}{dt}\left[\sum_i \frac{1}{2}\frac{d}{dt}(m_i \vec{r}_i\cdot\vec{r}_i)\right] = \frac{1}{2}\frac{d^2 I}{dt^2}$$

where $I = \sum_i m_i r_i^2$ is the $\boxed{\text{moment of inertia}}$ & $Q = \frac{1}{2}\dot{I} + \text{const.}$

on the other hand, differentiate by parts

$$\frac{dQ}{dt} = \sum_i\left(m_i \frac{d\vec{r}_i}{dt}\cdot\frac{d\vec{r}_i}{dt} + m_i\frac{d^2\vec{r}_i}{dt^2}\cdot\vec{r}_i\right) = 2K + \sum_{i=1}^{N}\vec{F}_i\cdot\vec{r}_i$$

where $\vec{F}_i = \sum_{j\ne i}\vec{F}_{ij}$ is the force on i due to all other particles j

rewrite the $\boxed{\text{virial of Clausius}}$

$$\sum_i \vec{F}_i \vec{r}_i = \sum_i \left[\sum_{j\neq i} \vec{F}_{ij} \cdot \tfrac{1}{2}[(\vec{r}_i + \vec{r}_j) + (\vec{r}_i - \vec{r}_j)] \right]$$

$$= \frac{1}{2} \sum_i \left[\sum_{j\neq i} \vec{F}_{ij} (\vec{r}_i + \vec{r}_j) \right] + \frac{1}{2} \sum_i \left[\sum_{j\neq i} \vec{F}_{ij} \cdot (\vec{r}_i - \vec{r}_j) \right]$$

to prove that the 1st term is zero, think of a 3-body system

$$\sum_{i=1}^{3} \left[\sum_{j\neq i} \vec{F}_{ij} (\vec{r}_i + \vec{r}_j) \right] = F_{12}(r_1 + r_2) + F_{13}(r_1 + r_3)$$
$$+ F_{21}(r_2 + r_1) + F_{23}(r_2 + r_3)$$
$$+ F_{31}(r_3 + r_1) + F_{32}(r_3 + r_2) = 0$$

the 2nd term is the potential energy

$$\frac{1}{2} \sum_i \left[\sum_{j\neq i} \vec{F}_{ij} (\vec{r}_i - \vec{r}_j) \right] = -\frac{1}{2} \sum_i \sum_{j\neq i} \frac{G m_i m_j}{|r_j - r_i|} = \frac{1}{2} \sum_i \sum_{j\neq i} U_{ij} = U$$

$$\Rightarrow \frac{dQ}{dt} = \boxed{\frac{1}{2} \frac{d^2 I}{dt^2} = 2K + U}$$

take the time average on both side over one period τ

$$\left\langle \frac{d^2 I}{dt^2} \right\rangle = \frac{1}{\tau} \int_0^\tau \frac{d^2 I}{dt^2} dt = \frac{1}{\tau} \left(\frac{dI}{dt}\Big|_\tau - \frac{dI}{dt}\Big|_0 \right)$$

this should equal to zero for periodic systems of any equilibrium/steady state

$$2\langle K \rangle + \langle U \rangle = 0$$

Application of Virial Theorem

Virial Mass $\quad \sigma^2 + \left(-\frac{GM}{R} \right) = 0$

Zwicky 1933

Coma cluster

$$\Rightarrow \quad M = \frac{\sigma^2 R}{G}$$

$M_{dyn} \sim 10^{15} M_\odot$

$M^* \sim 10^{12} M_\odot$

Comparison with visiable mass in stars

$$M^* = L_K \cdot (M/L_K)$$

Inferred the existance of dark matter

Chap 3 Light

Parallax [p] & Proper Motion [μ] $d = \frac{1}{p''}$ pc $\min(p'') = 0.77''$ for Proxima Centauri

in 1836, Bessel measured $p = 0.316''$ for 61 Cygni

1pc = 206265AU

① distance to the moon measured by Hipparchus (Greek)

ecliptic plane

solar eclipse on Mar 14 189 BC [simultaneous parallax]

total eclipse @ Nicaea (40°N) 4/5 eclipse @ Alexandria (31°N)

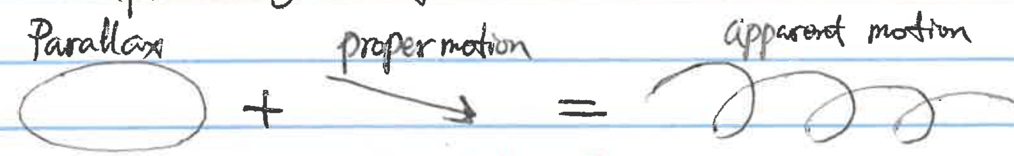
⇒ Dis to moon = 71–81 $R_\oplus$ vs 60.27 $R_\oplus$ from better observat

② Hipparchus satellite, ESA 1989–1993

$\sigma(p) = 1$ mas for 10^5 stars ⇒ $\max(d) = 1$ kpc

③ Gaia, ESA

$\sigma(p) = 10\ \mu$as for 10^9 stars ⇒ $\max(d) = 100$ kpc



monochromatic definition [vs. filtered definition]

Magnitude $m = -2.5 \log_{10}(F_\lambda / F_{\lambda,\mathrm{vega}})$ Vega magnitude system, A0V star

$m_{AB} = -2.5 \log_{10}(F_\nu / 3631\ \mathrm{Jy})$ AB system

$\nu f_\nu = \lambda f_\lambda$

$c = \nu\lambda$

$1\ \mathrm{Jy} = 10^{-23}\ \mathrm{erg\,s^{-1}\,Hz^{-1}\,cm^{-2}}$

① what's the magnitude of 10^4 stars? Each star is 14 mag

DM: $m - M = 5\log(d) - 5 = 5\log\frac{d}{10\mathrm{pc}}$

② what's the V mag of an A0V star with B = 3.5 mag? [Vega is A0V]

Abs Mag: M = apparent magnitude at d = 10pc

derivation: $M = -2.5\log\left[\frac{f\cdot d^2}{10^2} / f_{ref}\right] = m - 5\log(d/10\mathrm{pc})$

$$m - M = -2.5\log(F/F_{10}) = -2.5\log\left[\frac{L}{4\pi d^2} \Big/ \frac{L}{4\pi(10\mathrm{pc})^2}\right]$$

$$= 5\log\left(\frac{d}{10\mathrm{pc}}\right)$$

ΔM gives luminosity ratio: $L_2/L_1 = 10^{\frac{M_1 - M_2}{2.5}}$

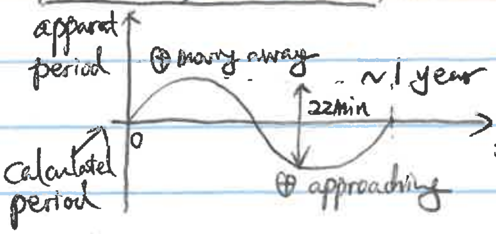
Bolometric luminosity of the Sun: $L_\odot = \int_0^\infty L_{\odot,\lambda}\, d\lambda = 3.839\times10^{33}$ erg/s

$M_\odot = +4.74$ magnitude, $m_\odot = -26.83$, $(m-M)_\odot = -31.57$

EM Wave

$c = 3\times10^5$ km/s , second is defined by the duration of N periods of transition of Cs-55. meter is defined by c in vacuum.

$\boxed{1675,\ \text{Roemer}}$ timed the eclipse of Jupiter's moons, esp. Io (P=42hr)

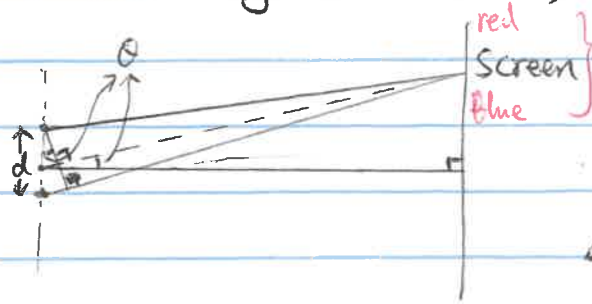


$P_{Jup} = 11.9 \simeq 12$ yrs $a_{Jupiter} = 5.2$ AU

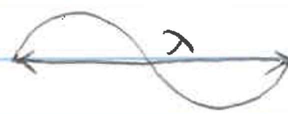
$c \cdot 22\text{ min} = 2$ AU

$\Rightarrow c = 2.2\times10^5$ km/s

Thomas Young (1773-1829) double slit experiment (after Newton died)



$$d\sin\theta = \begin{cases} n\lambda & \text{constructive} \\ (n-\frac{1}{2})\lambda & \text{destructive} \end{cases}$$



$c = \lambda\nu$

Maxwell (1831-1879)

discovered EM waves from his equations and noticed the similar properties with light. He then inferes that light is EM wave. (Polarization, speed, reflection, refraction)

Heinrich Hertz in 1889 produced radio EM waves in a lab

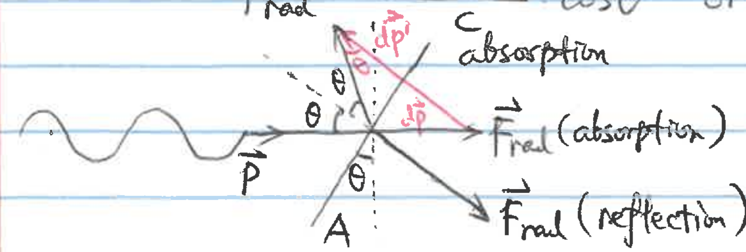
Poynting vector (describes the energy transfer rate of EM wave)

$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ unit: W/m^2 , μ_0, vacuum permeability
E: N/Coulomb, B: Tesla N/m/A

time averaged $\langle S\rangle = \frac{1}{2\mu_0}E_0B_0$ → E_0 & B_0 are the amplitudes & in vacuum $E_0 = c\cdot B_0$

Radiation Pressure (because EM wave also carries momentum) $\vec{F} = \frac{d\vec{p}}{dt}$

$F_{rad} = \frac{\langle S\rangle A}{c}\cos\theta$ or $\frac{2\langle S\rangle A}{c}\cos^2\theta = 2\cdot\cos\theta\, F_{abs}$
absorption reflection



$|d\vec{p}'| = 2\cdot\cos\theta|d\vec{p}|$ Special Relativity

$E^2 = (mc^2)^2 + p^2c^2$, $E = h\nu$, $m=0$

$\Rightarrow p = h\nu/c$ $E_{rest} = mc^2$ 2

Polarization of light (§2.3, Condon & Ransom)

Projection of the $\vec{E}$ field of a monochromatic EM wave traveling in $\hat{z}$ direction

$$\vec{E} = [\hat{x} E_x e^{i\phi_x} + \hat{y} E_y e^{i\phi_y}] e^{i(\vec{k}\cdot\hat{z} - \omega t)}$$

where $k = 2\pi/\lambda$ is the (wave number) magnitude of wave vector $\vec{k}$

$\omega = 2\pi\nu$ is the angular frequency

note Euler's formula, and the above is the plane wave solution of EM wave equation

$$e^{ix} = \cos x + i \sin x$$

define phase difference between E_x & E_y

$$\delta \equiv \phi_x - \phi_y$$

EM wave equations derived from Maxwell's equations

$$\frac{1}{c_0^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \nabla^2 \vec{E} = 0, \quad c_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3\times 10^8 \text{ m/s}$$

μ_0: vacuum permeability [mu]

ϵ_0: vacuum permittivity [epsilon]

when $\delta = 0$, the EM wave is linearly polarized

$\delta \neq 0$, the $\vec{E}$ vector traces an ellipse

Stokes Parameters $I = \langle E_x^2 + E_y^2 \rangle / (4\pi/c)$

linear polari. $\begin{cases} Q = \langle E_x^2 - E_y^2 \rangle / (4\pi/c) \\ U = \langle 2E_x E_y \cos\delta \rangle / (4\pi/c) \end{cases}$

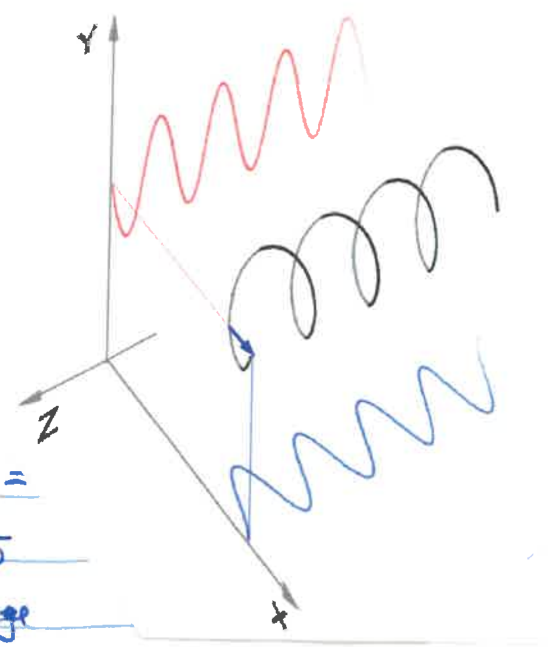
circular pol. $V = \langle 2E_x E_y \sin\delta \rangle / (4\pi/c)$

polarized flux density $I_p = \sqrt{Q^2 + U^2 + V^2}$

degree of polarization $P = I_p / I$

the above time average requires $\tau \gg 1/\Delta\omega = \frac{1}{2\pi\Delta\nu}$

where $\Delta\nu$ is the freq. range

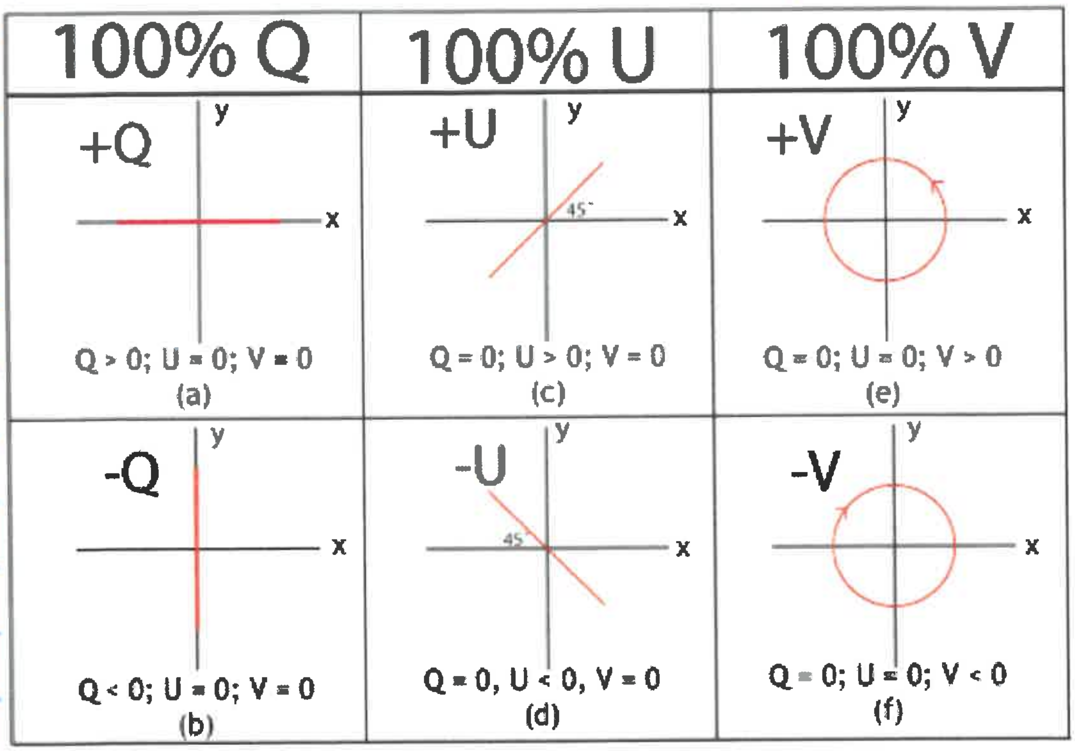


For linear polarization, $V=0$ because $\delta=0$

$$Q=\langle E_x^2 - E_y^2\rangle \qquad U=\langle 2E_xE_y\rangle$$

for 100% Q, $U=0 \Rightarrow E_x$ or $E_y=0$, so polarization along x or y axis

for 100% U, $Q=0 \Rightarrow E_x=E_y$, so polarization along 45° angle



[$\cos\theta$ is used to remove ~~angular~~ angle dependence]

Specific Intensity: $I_\nu \equiv \dfrac{dE}{dt\, dA \cos\theta\, d\Omega\, d\nu}$ $\qquad \langle I_\nu\rangle = \dfrac{1}{4\pi}\int I_\nu\, d\Omega$

Specific Flux: $F_\nu = \int I_\nu \cos\theta\, d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\nu \cos\theta \cdot \sin\theta\, d\theta\, d\phi$

Radiation Pressure $P_{rad,\nu} = \dfrac{1}{c}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\nu \cos^2\theta\, \underbrace{(\sin\theta\, d\theta\, d\phi)}_{d\Omega}$ $\left[P = \dfrac{F}{A} = \dfrac{dp/dt}{dA} = \dfrac{dE}{c\cdot dA\, dt}\right]$

$$= \frac{4\pi}{3c}\bar{I}_\nu \text{ for isotropic radiation field - } I_\nu = \bar{I}_\nu$$

Specific energy density $u_\nu \equiv \dfrac{1}{c}\int I_\nu\, d\Omega = \dfrac{4\pi}{c}\langle I_\nu\rangle$ $\left[u = \dfrac{dE}{dA\, dL} = \dfrac{dE}{dA\cdot c\cdot dt}\right]$

Special relativity $\qquad E^2 = (mc^2)^2 + p^2c^2$

for photons $(h\nu)^2 = 0 + p^2c^2 \Rightarrow p = \dfrac{h\nu}{c}$

Name ____________________ ASTR:3771 Intro to Astrophysics I

Boltzmann Statistics

Suppose that you have 3 coins that can be flipped and land either head (H) or tail (T) side up. Write down all of the possible outcomes for flipping the 3 coins. (For example: HHT)

HHH THH
HHT THT
HTH TTH
HTT TTT

Each of these possible outcomes is known as a **microstate**. The total number of microstates for 3 coins is 8.

A collection of microstates that satisfies some overall property is known as a **macrostate**. The number of microstates corresponding to a macrostate is known as the macrostate's **multiplicity** (Ω). Fill in the table below for the 3 coin example.

macrostate	Ω
3 Heads	1
2 Heads	3
1 Head	3
0 Heads	1

The probability of a macrostate can be calculated by dividing the multiplicity of the macrostate by Ω(all) = the total number of microstates.

(1) $P(\text{macrostate}) = \frac{\Omega(\text{macrostate})}{\Omega(\text{all})}$

Calculate the probability of getting 2 heads.

$$P(HH) = \frac{3}{8}$$

Relative probabilities of macrostates can be found by taking the ratio of the corresponding multiplicities.

(2) $\frac{P(\text{macrostate 1})}{P(\text{macrostate 2})} = \frac{\Omega(\text{macrostate 1})}{\Omega(\text{macrostate 2})}$

Calculate the following

$\frac{P(2\text{ heads})}{P(3\text{ heads})} = \frac{3}{1}$ $\frac{P(2\text{ heads})}{P(1\text{ head})} = \frac{3}{3}$

The entropy of a macrostate is defined as

(3) $S = k \ln \Omega$

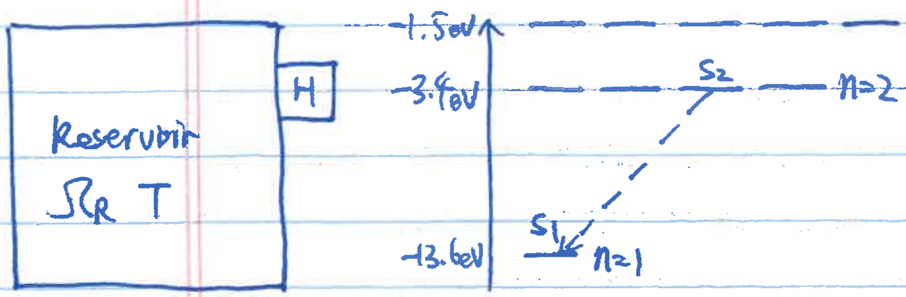
where $k = 1.381 \times 10^{-23}$ J/K $= 8.617 \times 10^{-5}$ eV/K is Boltzmann's constant. Thus the statement that entropy increases is the same as saying that systems head towards more probable states.

Boltzmann Statistics → Planck Function → Steffen-Boltzmann Equation

Key assuption: ① all microstates have equal probability of being occupied

② kT is the amount of energy required to e-fold the # of microstate

H atom in a thermal bath



Consider a small system made of a single H is attached to a large reservoir

initially the H atom is in state S_2, then it made a transition to state S_1 by losing energy ΔE to the Reservoir, which increases the multiplicity of the Reservoir from $\Omega_R(S_2)$ to $\Omega_R(S_1)$, so the relative probability of the H atom being in S_2 vs S_1 is:

$$\frac{P(S_2)}{P(S_1)} = \frac{\Omega_R(S_2)}{\Omega_R(S_1)} < 1$$

Entropy of a macrostate is $\quad S \equiv k \ln \Omega \quad \Rightarrow \Omega_R = e^{S_R/k}$

$$\Rightarrow \frac{P(S_2)}{P(S_1)} = e^{[S_R(S_2) - S_R(S_1)]/k}, \quad S_R(S_2) < S_R(S_1)$$

Given the thermal dynamic identity

$$dU = T\,dS - P\,dV$$

we have for the reservoir:

$$\Delta U_R = T \Delta S_R = -\Delta E = -[E(S_2) - E(S_1)]$$

$$\Rightarrow \frac{P(S_2)}{P(S_1)} = e^{-\frac{E_2 - E_1}{kT}}$$

for relative probability of macrostates of the H atom, we need to factor in the degeneracy g

$$\frac{P(n=2)}{P(n=1)} = \frac{g_2}{g_1} e^{-\frac{E_2 - E_1}{kT}}$$

e.g. $g_n = 2n^2$, $E_n = -13.6\,\mathrm{eV}/n^2$

$kT \sim 0.5\,\mathrm{eV}$ $[T \sim 5800\,\mathrm{K}]$

$$\frac{P_2}{P_1} = 4 \cdot e^{-10.2/0.5} = 5 \times 10^{-9}$$

Einstein Coefficients

For detailed balance $\quad n_1 B_{12} I_\nu = A_{21} n_2 + n_2 B_{21} I_\nu$

For thermodynamic equilibrium $\quad \frac{n_2}{n_1} = \frac{g_2}{g_1}\exp\left(-\frac{h\nu}{kT}\right)$

$$\Rightarrow \frac{n_2}{n_1} = \frac{B_{12} I_\nu}{A_{21} + B_{21} I_\nu} = \frac{g_2}{g_1} e^{-\frac{h\nu}{kT}}$$

where $h\nu = \Delta E = E_2 - E_1$

$I_\nu = B_\nu(T)$

$$\Rightarrow g_1 B_{12} I_\nu = e^{-\frac{h\nu}{kT}}\left(g_2 A_{21} + g_2 B_{21} I_\nu\right)$$

$$\Rightarrow \frac{g_2 A_{21}}{I_\nu} = g_1 B_{12} e^{h\nu/kT} - g_2 B_{21}$$

because Einstein coefficients are fixed probabilities of the atom, any relationships that we can derive at special conditions should be valid universally

① at the extreme that $T \to \infty$, the radiation field's brightness should $\to \infty$

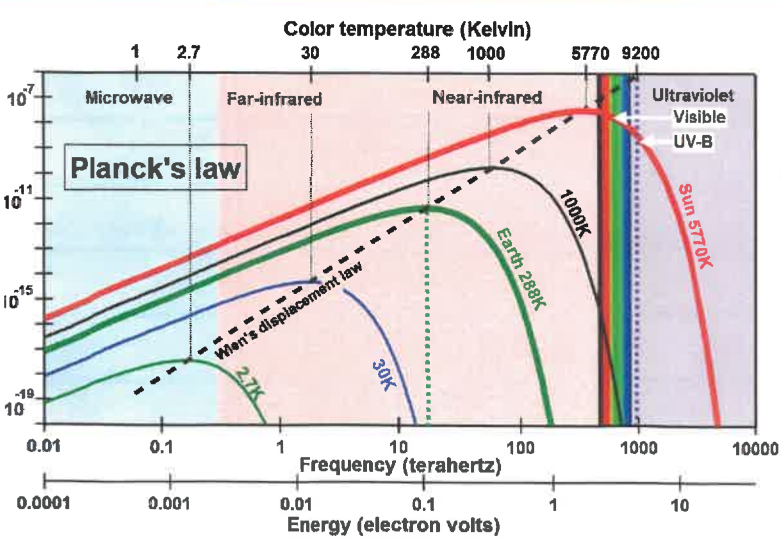
$$\Rightarrow \frac{g_2 A_{21}}{\infty} = g_1 B_{12} e^{h\nu/\infty} - g_2 B_{21}$$

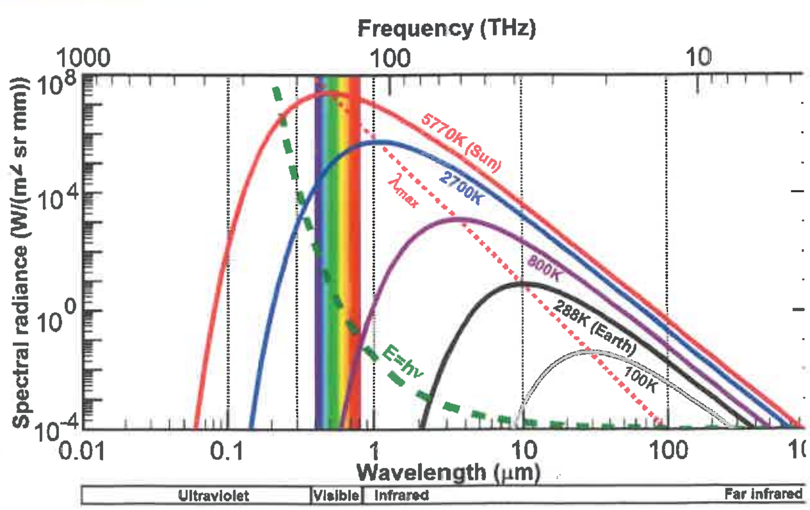
$$\Rightarrow g_1 B_{12} = g_2 B_{21}$$

② when $h\nu \ll kT$, Rayleigh-Jeans law, $\boxed{B_\nu \simeq \frac{2kT}{c^2}\nu^2}$

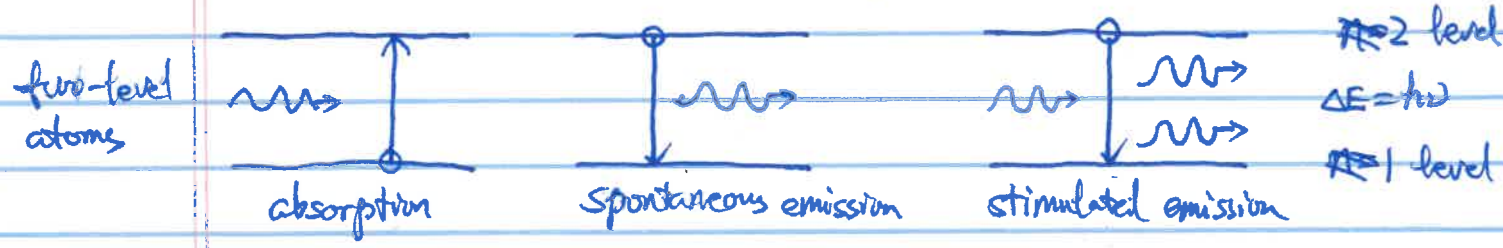
$$\Rightarrow g_2 A_{21} = g_1 B_{12}\left(e^{h\nu/kT} - 1\right)\cdot\frac{2kT}{c^2}\nu^2 \qquad \left[B_\lambda \simeq \frac{2ckT}{\lambda^4}\right]$$

$$= g_1 B_{12}\cdot\frac{h\nu}{kT}\cdot\frac{2kT}{c^2}\nu^2 = \frac{2h\nu^3}{c^2} g_1 B_{12}$$





Planck Function: $B_\nu(T) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT}-1}$



$$\frac{dn_1}{dt} = -n_1 \cdot B_{12} I_\nu \qquad \frac{dn_2}{dt} = -n_2 A_{21} - n_2 B_{21} I_\nu$$

at equilibrium $\frac{dn_1}{dt} = \frac{dn_2}{dt} \Rightarrow n_1 B_{12} I_\nu = n_2 A_{21} + n_2 B_{21} I_\nu$

$$\Rightarrow \frac{n_2}{n_1} = \frac{B_{12} I_\nu}{A_{21} + B_{21} I_\nu}$$

where I_ν is the specific intensity of the radiation field at frequency ν [$\Delta E = h\nu$]
and A & B are Einstein coefficients

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} \quad \& \quad B_{12} = \frac{g_2}{g_1} B_{21}$$

$\Rightarrow \frac{n_2}{n_1} = \frac{g_2/g_1 \cdot I_\nu}{2h\nu^3/c^2 + I_\nu}$ on the other hand, Boltzmann $\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu/kT}$

we have $I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT}-1}$ [Planck Function]

Rayleigh-Jeans tail : $h\nu \ll kT$

$$I_\nu = \frac{2kT\nu^2}{c^2} \Rightarrow T_B = \frac{c^2 I_\nu}{2k\nu^2}$$

Wein tail : $h\nu \gg kT$

$$I_\nu = \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$$

Wien's displacement law : $\lambda_{max} \cdot T \simeq 3\ mm \cdot K$

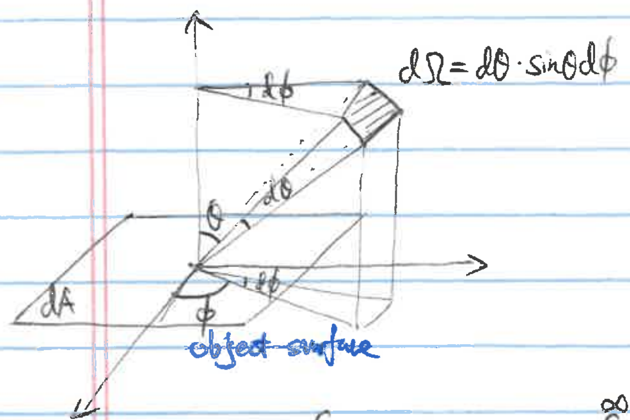
$\Big\}$ 1911 Nobel Prize Wilhelm Wien

λ_{max} is where $B_\lambda(T)$ peaks, i.e., $\frac{dB_\lambda}{d\lambda} = 0$

$$I_\nu \equiv \frac{\Delta E}{\Delta t \Delta A \Delta \nu \Delta\Omega} \cdot \frac{1}{\cos\theta}$$

Stefan-Boltzmann Equation (derivation)

specific intensity



specific flux

$$F_\nu = \frac{\Delta E}{\Delta t \Delta A \Delta \nu} = \int B_\nu \cos\theta \cdot d\Omega$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} B_\nu \cos\theta \sin\theta \, d\theta \, d\phi$$

$$= 2\pi B_\nu \int_0^{\pi/2} \sin\theta \, d(\sin\theta) = \pi B_\nu$$

$$F = \int F_\nu d\nu = \pi \int_0^\infty B_\nu d\nu = \frac{2\pi h}{c^2} \int \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}$$

replace $x = h\nu/kT$

$$F = \frac{2\pi h}{c^2} \cdot \left(\frac{kT}{h}\right)^4 \cdot \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

the integral $= \frac{\pi^4}{15}$

$$\Rightarrow F = \frac{2\pi^5}{15} \cdot \frac{k^4}{c^2 h^3} \cdot T^4 = \sigma_{SB} T^4$$

$$L = 4\pi R^2 \sigma T^4 = \int F \cdot dA$$

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

For an isotropic radiation field

I_ν is const along all directions

~~but F_ν changes with angle~~

Doing the integral with expansion & integrate by parts.

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}} = \sum_{n=1}^{\infty} e^{-nx}$$

$$\int \frac{x^3}{e^x - 1} dx = \int x^3 \cdot \frac{e^{-x}}{1 - e^{-x}} dx = \sum_{n=1}^{\infty} \int_0^\infty x^3 e^{-nx} dx$$

$$= \sum_{n=1}^{\infty} \frac{6}{n^4} = 6 \cdot \zeta(4) = 6 \cdot \frac{\pi^4}{90} = \pi^4/15$$

Reimann zeta function

$d(uv) = v\,du + u\,dv$ let $u = x^3$, $dv = e^{-x} dx$

$\Rightarrow du = 3x^2 dx$, $v = -e^{-x}$

$$\int x^3 e^{-x} dx = \int u\,dv = \int d(uv) - \int v\,du = -x^3 e^{-x} + 3\int x^2 e^{-x} dx$$

then let $u = x^2$, $dv = e^{-x} dx$ $\Rightarrow \int x^2 e^{-x} dx = -x^2 e^{-x} + 2\int x e^{-x} dx$

Cooling by radiation

amount of heat : $H = \overset{\text{specific heat capacity}}{S} \cdot M \cdot T$: m is the mass, T is temperature in K

heat radiation rate : $L = \sigma T^4 \cdot A$ - Stefan-Boltzmann law

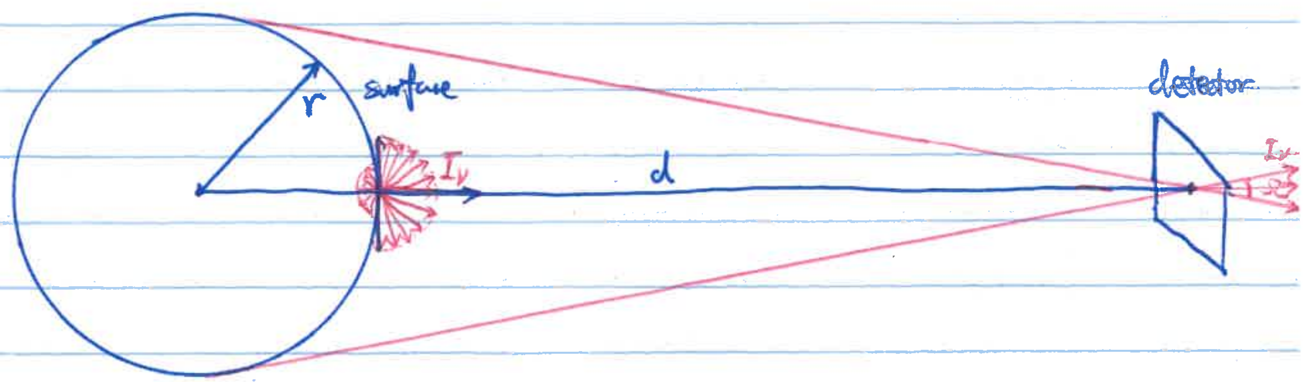
$$-\frac{dH}{dt} = L \Rightarrow t \simeq \frac{H}{L} = \frac{S \cdot \rho \frac{4}{3}\pi r^3 \cdot T}{\sigma T^4 \cdot 4\pi r^2} = \frac{S\rho r}{3\sigma T^3}$$

for iron at $500K = 227°C = \overset{440}{\cancel{450}} F$, $r = 10$ cm , $s = 0.45$ J/g/K

$$t = 4.7 \text{hr} \, (r/10\text{cm})(500K/T)^3 \cdot (\rho/8 \text{g/cc})(s/0.45 \text{J/g/K})$$

Your grill cools much faster because of convection of air

Proof of distance square law



$$f_\nu = \int I_\nu \cos\theta \, d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\nu \cos\theta \sin\theta \, d\theta \, d\phi$$

[Eddington Approximation]

at the surface, $I_\nu = 0$ when $\theta > \pi/2$, & we assume isotropic radiation field

$$f_\nu = 2\pi I_\nu \cdot \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = \pi I_\nu$$

at distance d , $I_\nu = 0$ outside of the solid angle Ω, and $\theta = 0$ in Ω

$$f_\nu = I_\nu \cdot \Omega = \pi I_\nu \cdot \frac{r^2}{d^2}$$

because I_ν is conserved along any light ray , $f_\nu \propto 1/d^2$

Chap 5 : Interaction of light & matter

★ Bohr model ★ atomic processes ★ M-B distribution ★ line broadening
★ spectroscopic notation ★ Kirchhoff's laws ★ critical density

Bohr Model - History

① 1890s, discovery of e^- by Thomson

② 1911, Rutherford discovery of positively charged, tiny, massive nuclei

③ 1885, Balmer found an empirical formula to reproduce H lines in optical

$$\frac{1}{\lambda_{nm}} = R_H\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$, for Lyman/Balmer/Paschen series, $m = 1, 2, 3$

$$R_H = \frac{\mu e^4}{64\pi^3 \epsilon_0^2 \hbar^3 c} \simeq 1.1\times10^7\ m^{-1}$$ Rydberg constant

this corresponds to 13.6 eV

- Derivation

assumptions: e^- p^+ system is controlled by Coulomb's law

e^- doesn't radiate EM wave at quantized angular momentum

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = \mu\frac{v^2}{r}, \quad L = \mu v r = n h/2\pi = n\hbar \Rightarrow v = \frac{\hbar\cdot n}{\mu r}$$

$$\Rightarrow \frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{r} = \mu\cdot\frac{\hbar^2 n^2}{\mu^2 r^2} \Rightarrow r_n = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}\cdot n^2 \equiv a_B\cdot n^2$$ [Bohr radius] 0.0529 nm

Energy of each state: virial theorem

$$E = K + U = \underset{\downarrow}{K - 2K} = -K = \frac{-1}{2}\mu v^2 = -\frac{1}{8\pi\epsilon_0}\frac{e^2}{r_n}$$

$$E_n = -\frac{\mu e^4}{32\pi^2\epsilon_0^2\hbar^2}\cdot n^{-2} = -13.6\text{eV}\cdot\frac{1}{n^2}$$

$$h\nu_{nm} = h\frac{c}{\lambda_{nm}} = E_n - E_m \Rightarrow \frac{1}{\lambda_{nm}} = \frac{\mu e^4}{64\pi^3\epsilon_0^2\hbar^3 c}\left[\frac{1}{m^2} - \frac{1}{n^2}\right]$$

Kirchhoff's laws (1860)

1. Hot, dense gas or solid emit continuum spectrum
2. Hot, diffuse gas against dark background emit emission lines
3. Cool, diffuse gas in front of hot continuum source produce absorption lines

$$\#\text{ of collisions/volume/time} = \frac{n_1 \cdot u\,\sigma_{12}\,dt}{dt}\cdot n_e f(u)\,du = n_1 n_e u\,\sigma_{12} f(u)\,du$$

Atomic Processes

Bound-Bound transition

① photo-excitation ② collisional excitation

① spontaneous emission ② stimulated emission ③ collisional deexcitation

$$\boxed{\begin{aligned} A_{21} &= \frac{2h\nu^3}{c^2}B_{21} \\ B_{12} &= \frac{g_2}{g_1}B_{21} \\ \frac{q_{12}}{q_{21}} &= \frac{g_2}{g_1}e^{-\Delta E/kT}\end{aligned}}$$

$$n_1 B_{12} I_\nu + n_1 n_e q_{12} = n_2 A_{21} + n_2 B_{21} I_\nu + n_2 n_e q_{21}$$

where q_{ij} is the velocity-averaged collisional cross-sections

$$q_{ij} = \int_0^\infty u\,\sigma_{ij} f(u)\,du \quad , \text{ where } f(u) = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} 4\pi u^2 \exp\left(-\frac{mu^2}{2kT}\right)$$

Maxwell-Boltzmann velocity distribution

given detailed balance:

$$\begin{cases} n_e n_1 u_1 \sigma_{12}(u_1) f(u_1)\,du_1 = n_e n_2 u_2 \sigma_{21}(u_2) f(u_2)\,du_2 \\ \frac{1}{2}mu_1^2 = \frac{1}{2}mu_2^2 + \Delta E \quad (\text{the above is valid when the K difference is } \Delta E) \end{cases}$$

we have $\quad \dfrac{q_{12}}{q_{21}} = \dfrac{n_2}{n_1} = \dfrac{g_2}{g_1} e^{-\Delta E/kT}$

For collision dominated regime, the I_ν terms drop out

$$\cancel{n_1 B_{12} I_\nu} + n_1 n_e q_{12} = n_2 A_{21} + \cancel{n_2 B_{21} I_\nu} + n_2 n_e q_{21}$$

$$\Rightarrow \cancel{\frac{n_2 A_{21}}{n_1} = n_e q_{12} - \frac{n_2 n_e q_{21}}{n_1}}$$

$$\frac{n_2}{n_1} = \frac{n_e q_{12}}{A_{21} + n_e q_{21}} = \frac{n_e q_{12}}{A_{21}}\left(\frac{1}{1 + n_e q_{21}/A_{21}}\right) = \frac{q_{12}/q_{21}}{1 + n_c/n_e}$$

define critical density: $\quad n_c = A_{21}/q_{21}$

$$\frac{n_2}{n_1} = \frac{n_e q_{12}}{A_{21}}\left(\frac{1}{1 + n_e/n_c}\right) \Rightarrow \mathcal{L} = n_2 A_{21} = n_1 n_e q_{12}\left(\frac{1}{1 + n_e/n_c}\right)$$

$$\frac{n_2}{n_1} = \frac{q_{12}/q_{21}}{1 + n_c/n_e} = \frac{g_2}{g_1} e^{-\Delta E/kT}\left(\frac{1}{1 + n_c/n_e}\right)$$

$$\Rightarrow \begin{cases} \mathcal{L} = n_1 n_e q_{12} & \text{when } n_e \ll n_c \\ \mathcal{L} = n_1 n_c q_{12} & \text{when } n_e \gg n_c \end{cases} \qquad \begin{cases} \frac{n_2}{n_1} \ll \frac{g_2}{g_1} e^{-\Delta E/kT} \\ \frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-\Delta E/kT} \quad [\text{thermalized}] \end{cases}$$

Maxwell-Boltzmann Dist.

$$\frac{N_i}{N_j} = \frac{g_i}{g_j}\exp\left(-\frac{E_i-E_j}{kT}\right) \qquad \text{Boltzmann Statistics}$$

$$\Rightarrow \frac{N_i}{N} = \frac{\exp(-E_i/kT)}{\sum_j \exp(-E_j/kT)} \qquad \text{when } g_i = g_j$$

For pure kinetic energy $E_i = \frac{1}{2}m(V_x^2+V_y^2+V_z^2)$

$$\frac{N_i}{N} \propto \exp\left[-\frac{m(V_x^2+V_y^2+V_z^2)}{2kT}\right]$$

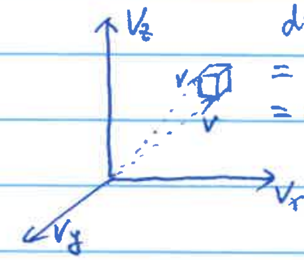
The probability density function of the velocity vector:

$$f_{\vec{v}}(V_x, V_y, V_z) = \left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left[-\frac{m(V_x^2+V_y^2+V_z^2)}{2kT}\right]$$

$$= f_{\vec{v}}(V_x)\cdot f_{\vec{v}}(V_y)\cdot f_{\vec{v}}(V_z)$$

Compare with Gaussian $\quad g(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$

$$\Rightarrow \sigma = \sqrt{\frac{kT}{m}}$$



$dV_x dV_y dV_z$
$= v^2 dv\, d\Omega$
$= v^2 \sin\theta\, dv\, d\theta\, d\phi$

The PDF of speed

$$v = \sqrt{V_x^2+V_y^2+V_z^2}$$

$$\cancel{f_{\vec{v}}(V_x,V_y,V_z)\,dV_x\,dV_y\,dV_z = f_u(u)\,u^2\,du\,d\Omega}$$

$$\cancel{= f_{\vec{v}}\cdot v^2\,dv\,d\Omega = f_v(v)\,dv\,d\Omega}$$

$$\Rightarrow f_v(v) = \left(\frac{\cancel{2}m}{2\pi kT}\right)^{3/2} 4\pi v^2 \exp\left[-\frac{mv^2}{2kT}\right]dv$$

because: $f_v(v)\,dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} f_{\vec{v}}(V_x,V_y,V_z)\cdot v^2 \sin\theta\, dv\, d\theta\, d\phi = 4\pi v^2\cdot f_{\vec{v}}(V_x,V_y,V_z)$

Properties: ① most probable speed, $V_{mp} = \sqrt{\frac{2kT}{m}} \simeq 422\,m/s$ for N_2 at 300K

② mean squared speed, $V_{rms} = \left(\int_0^\infty v^2 f(v)\,dv\right)^{1/2} = \sqrt{\frac{3kT}{m}}$

③ mean kinetic energy per particle

$$\langle E\rangle = \int_0^\infty E\, f(E)\, dE = \frac{3}{2}kT$$

④ $\sigma_z = \sqrt{\frac{kT}{m}}$

Broadening of emission/absorption lines [Bound-Bound transition]

$\Delta t \Delta E = \hbar$

$\Delta t \cdot h \Delta\nu = \hbar$

$\Rightarrow \Delta\nu = \frac{1}{2\pi \Delta t}$

$= A/2\pi$

$\sim 10^8$ Hz

for permitted lines

Heisenberg's uncertainty principle

① natural broadening $\quad \Delta x \cdot \Delta p = \frac{\Delta x}{c} \cdot c \cdot \Delta p = \Delta t \cdot \Delta E \gtrsim \hbar$

$\Delta t \sim$ lifetime of the upper level $\sim A_{21}^{-1} \simeq 10^{-8}$ s for permitted lines, $\Delta E \sim 10^{-18}$ erg

define damping constant $\quad \gamma_n = \sum_{n'<n} A_{nn'} \sim 10^8 \, s^{-1}$

Doppler shift

$$\left[\frac{\nu - \nu_0}{\nu_0} \simeq -\frac{u}{c} \qquad \frac{\lambda - \lambda_0}{\lambda_0} \simeq \frac{u}{c} \right]$$

Lorentz line profile

$$\phi(\nu) d\nu = \frac{\gamma_n / 4\pi}{(\nu - \nu_0)^2 + (\gamma_n/4\pi)^2} \frac{d\nu}{\pi}$$

② thermal broadening $\quad f(u)du \propto u^2 \exp\left[-\frac{mu^2}{2kT}\right] du \quad$ M-B distribution

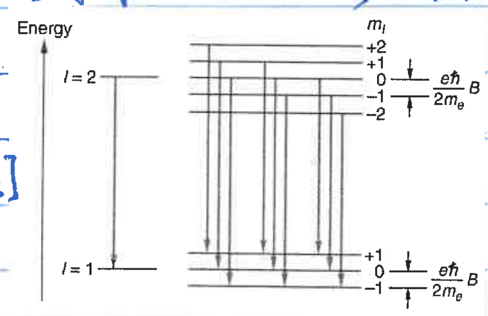
Gaussian line profile

$$\phi(u_z) du_z = \left(\frac{1}{2\pi \sigma_z^2}\right)^{\frac{1}{2}} \exp\left(-\frac{u_z^2}{2\sigma_z^2}\right) du_z$$

u_z is velocity along line of sight

$\sigma_z = \sqrt{\frac{kT}{m}} \sim 9$ km/s for protons at 10^4 K, 0.9 km/s @ 100K

Comparable to sound speed



③ Turbulent Doppler, [Larson's law of MCs]

④ Rotational Doppler, [rotation speed of Sun ~2 km/s]
rotation period 24 days

⑤ Pressure broadening

Lorentz profile $\quad \phi(\nu) d\nu = \frac{\Gamma/4\pi}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2} \frac{d\nu}{\pi}$

where $\Gamma = \gamma_n + 2N_{col}$, N_{col} is the collision rate

⑥ Zeeman splitting / broadening : $\quad \nu = \nu_0 + [-1, 0, 1] \frac{eB}{4\pi\mu}$, $\Delta\nu = 2.8 \, Hz \frac{B}{2e\text{-}10 \, T}$

Earth $B = 5e\text{-}5$ T

Doppler Shift : $\quad \lambda_{obs}/\lambda_{rest} = \sqrt{1+\beta / 1-\beta}$ where $\beta = v_r/c$

de Broglie λ & ν : $\quad \lambda_{dB} = h/p$, $\nu_{dB} = E/h$

Heisenberg uncertainty principle.

$\Delta x \Delta p \gtrsim \hbar \qquad \Delta E \Delta t \geq \hbar$

Spectroscopic Notation

Quantum Numbers for single electron

n : principle QN ~ energy, $n \in (1, 2, \cdots \quad)$ $\quad E = -13.6\,\text{eV}/n^2$

$l = 0, 1, 2, 3, 4, 5 \Leftrightarrow s\ p\ d\ f\ g\ h$

l : angular momentum QN $\quad l \in (0, 1, \ldots, n-1)$ $\quad L = \sqrt{l(l+1)}\,\hbar$

m_l : (magnetic QN) z-component of angular mom. $m_l \in (-l, \ldots, 0, \ldots, l)$ $\quad L_z = m_l \hbar$

m_s : spin angular momentum, $m_s \in (-\frac{1}{2}, \frac{1}{2})$, $S = \frac{1}{2}$, $S_z = m_s \hbar$

Selection rules for individual electron [talk about Zeeman Effect]

allowed transitions : $\Delta l = \pm 1$ & $\left\{\begin{array}{ll} \Delta m_l = 0 \text{ or } \pm 1 & \text{for } m_l \neq 0 \\ \Delta m_l = \pm 1 & \text{for } m_l = 0 \end{array}\right\}$ & $\Delta m_s = 0$

Xenon^{54}

Electron configuration, Na $1s^2 2s^2 2p^6 3s^1$, O_8 $1s^2 2s^2 2p^4$

Xe $6s^2 4f^1 5d^1$

Quantum Numbers for multiple electrons

$l_1 = 3 = f$

$l_2 = 2 = d$

$L = 3+2, \ldots, |3-2|$
$= 5, 4, 3, 2, 1$

$L = \sum_i l_i$ $\quad$ total orbital angular momentum, $L \in |l_1+l_2|, |l_1+l_2-1|, \ldots |l_1-$

$S = \cancel{\sum_i m_{s,i}}$ $\quad$ total electron spin angular momentum, $S = |S_1+S_2| \cdots |S_1-S$

$J \in L+S, L+S-1, \ldots, |L-S|$ $\quad$ total angular momentum

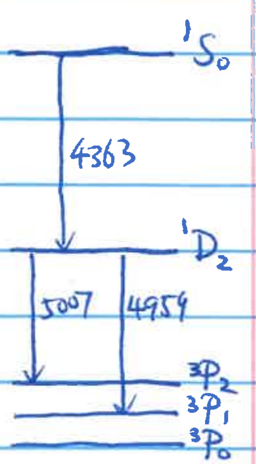
e.g. Filled shells of principle QN $= n = 2$ $[2s^2 2p^6]$

Pauli Exclusion Principle for Fermions

$n = 2$, $l = 0; 1$, $m_l = 0; -1, 0, 1$, $m_s = -\frac{1}{2}, \frac{1}{2}$

# of states $= g(n=2) = 2 \cdot \sum_{l=0}^{n-1} (2l+1) = 2n^2 = 8$ for $n=2$

$\Rightarrow \sum_{i=1}^{2n^2} m_{l,i} = 0$ & $\sum_{i=0}^{2n^2} m_{s,i} = 0 = S$ i.e. $L = 0$ & $S = 0$



e.g. O III - 6e$^-$ configuration $[n\,l^{\#}]$: $1s^2 2s^2 2p^2$

only the outermost 2 e$^-$ matter - $2p^2$, $n=2$, $l=1$, $m_l = -1, 0, 1$, $m_s = \pm$

$\Rightarrow L = 1+1, 1+1-1, 1-1 = 2, 1, 0 \Leftrightarrow D, P, S$

$S = \frac{1}{2} - \frac{1}{2}, \frac{1}{2} + \frac{1}{2} = 0, 1$

so we have six terms $[^{2S+1}L]$ and ten levels $[^{2S+1}L_J]$

	^1S	^1P	^1D	^3S	^3P	^3D	(6 terms)
$J =$	0	1	2	1	0, 1, 2	1, 2, 3	(10 levels)
$L =$	0	1	2	0	1	2	
$S =$	0	0	0	1	1	1	

Selection Rules of multiple electrons

$$\begin{cases} \Delta J = 0 \text{ or } \pm 1 & \text{but not } J=0 \to 0 \\ \Delta L = 0 \text{ or } \pm 1 & \text{but not } L=0 \to 0 \\ \Delta S = 0 \end{cases}$$

Other atomic processes

Bound-free : photoionization , collisional ionization

Free-Bound : recombination

Scattering : low energy - Thomson scattering
high energy - Compton scattering

Chap 8 Stellar Spectra

Harvard classification scheme : O B A F G K M | L T   Sun : G2 V

a temperature sequence $He\,I$ $H\,I$ $Ca\,II$ Brown Dwarfs $T < 2500$ K

Annie Cannon : 1901 , employed by Pickering

HD catalogue , named after Henry Draper.

Equivalent Widths : $W = \int \frac{F_c - F_\lambda}{F_c} d\lambda$, F_c is the continuum flux density

Maxwell-Boltzmann Velocity distribution :

The most likely populated electron levels/velocity at certain temperature

$$n_v\, dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, 4\pi v^2 dv$$

$$v_{mp} = \sqrt{\frac{2kT}{m}}$$, i.e. $\frac{1}{2} m v_{mp}^2 = kT$, mp → most probable speed

$$v_{rms} \equiv \sqrt{\overline{v^2}} = \sqrt{\frac{3kT}{m}}$$ root-mean-square ~~velocity~~ speed

Boltzmann Equation :

$$\frac{P(E_b)}{P(E_a)} = \frac{g_b}{g_a} \cdot \underbrace{e^{-(E_b - E_a)/kT}}_{\text{Boltzmann factor}}$$

g_b, g_a ← statistical weights of energy levels

1 eV = 1.6×10^{-19} Joules

$k = 8.62 \times 10^{-5}\ eV \cdot K^{-1}$

$kT = \frac{1}{40}$ eV @ 300 K

e.g. ground state of $H\,I$, $E_a = -13.6$ eV , $S_a = \{n=1, l=0, m_l=0, m_s=\frac{1}{2}\}$

$g(n) = 2n^2$ for $H\,I$, $E(n) = -13.6\,eV/n^2$ $= 1s\ ^2S_{\frac{1}{2}}$

If $E_b > E_a$, ① when $T \to 0$, $P(E_b)/P(E_a) = 0$

② when $T \to \infty$, $P(E_b)/P(E_a) = g_b/g_a$ (all levels equally accessible)

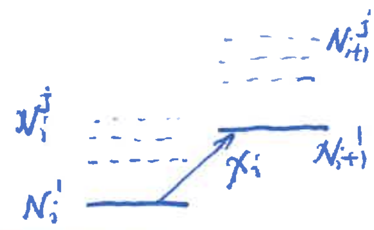
For large number of particles

$$N(E_b)/N(E_a) = P(E_b)/P(E_a) = \frac{g_b}{g_a} \cdot e^{-(E_b - E_a)/kT}$$

Example 8.1.3 At what temperature would there be equal #s of n=1 & n=2 $H\,I$?

$E_b = E(n=2) = -13.6\,eV/n^2$, $E_a = E(n=1) = -13.6\,eV$

$g_b = 2n^2 = 8$, $g_a = 2$, plug in BE => $\frac{10.2\,eV}{kT} = \ln 4$, $T = 85{,}400$ K



Saha Equation Derivation

Goal: $$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{Z_i}\frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT}$$

define partition function $$Z_i = \sum_{j=1}^{\infty} g_i^j \exp\left(-\frac{E_i^j - E_i^1}{kT}\right) \simeq g_i^1$$

Nomenclature: N_i & N_{i+1} number densities of two adjacent ionization states

N_i^j, g_i^j, E_i^j number density, statistical weight, energy of particles in ionization state i and excitation state j

n_e, g_e, E_e, p_e free electrons

For ionization state i

① $$\frac{N_i}{N_i^1} = \frac{\sum_{j=1}^{\infty} N_i^j}{N_i^1} \underset{\text{Boltzmann Eq.}}{\uparrow}= \frac{\sum_{j=1}^{\infty} g_i^j e^{-\frac{E_i^j}{kT}}}{g_i^1 e^{-E_i^1/kT}} = \frac{1}{g_i^1}\underbrace{\sum_{j=1}^{\infty} g_i^j e^{-(E_i^j - E_i^1)/kT}}_{Z_i}$$

For ionization state $i+1$ (ion + free electron)

$$\frac{N_{i+1}^{j,p_e}}{N_i^1} = \frac{g_{i+1}^j\, g_e(p_e)}{g_i^1} e^{-\frac{E_{i+1}^j + E_e - E_i^1}{kT}}, \quad \chi_i \equiv E_{i+1}^1 - E_i^1$$

$$= \frac{g_{i+1}^j}{g_i^1} e^{-\frac{E_{i+1}^j - E_{i+1}^1}{kT}} \cdot e^{-\frac{\chi_i}{kT}} \cdot g_e(p_e)\, e^{-\frac{p_e^2}{2kTm_e}}$$

$\Delta V = \frac{1}{n_e}$

$d^3p_e = 4\pi p_e^2 dp$

$g_e(p_e) = \frac{\Delta V \cdot d^3p_e}{h^3} \cdot 2$

$= \frac{1}{n_e}\frac{8\pi p_e^2 dp_e}{h^3}$

$$\int_0^{\infty} g_e(p_e) e^{-\frac{p_e^2}{2kTm_e}} dp_e = \int \frac{8\pi p_e^2}{n_e h^3} e^{-\frac{p_e^2}{2m_e kT}} dp_e, \quad x \equiv p_e^2/2m_e kT$$

$$= \frac{4\pi}{n_e h^3}(2m_e kT)^{3/2}\int_0^{\infty}\sqrt{x}\, e^{-x} dx, \quad \int = \Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$= \frac{2}{n_e h^3}(2m_e kT\pi)^{3/2}$$

② $\Rightarrow$ $$\frac{N_{i+1}}{N_i^1} = \sum_{j=1}^{\infty}\frac{N_{i+1}^j}{N_i^1} = \frac{1}{g_i^1}\Big[\underbrace{\sum_{j=1}^{\infty} g_{i+1}^j \cdot e^{-\frac{E_{i+1}^j - E_{i+1}^1}{kT}}}_{Z_{i+1}}\Big] e^{-\frac{\chi_i}{kT}} \cdot \frac{2}{n_e h^3}(2m_e kT\pi)^{3/2}$$

Saha Equation is the ratio of ②/①

Saha Equation: the relative number of atoms in different stages of ionization

Named after Meghnad Saha, derived in 1920

Partition function:

$$Z = \sum_{j=1}^{\infty} g_j \, e^{-(E_j - E_1)/kT}$$

Z_i & Z_{i+1} are the PF for atoms in its intial & final ionization stages

$$\frac{N_{i+1}}{N_i} = \frac{2 Z_{i+1}}{n_e Z_i} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_i / kT}$$

where χ_i is the ionization ~~sta~~ energy to remove an electron in the ground state.

Combine Saha & Boltzmann equations

Example 8.1.4 Pure H atmosphere with $5000\,K < T < 25{,}000\,K$

① calculate partition functions Z_I & Z_{II}.

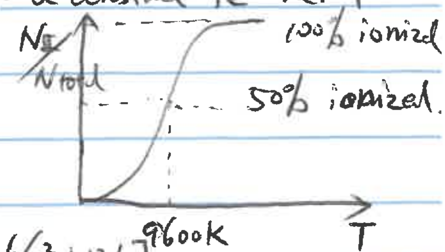
$Z_{II} = 1$ because H II is a single proton

$Z_I = g_1 = 2 \cdot n^2 = 2$ because almost all e^- are in ground state

$kT < 2.15\ eV$

② use Saha equation to evaluate N_{II}/N_I for a constant $P_e = n_e kT$

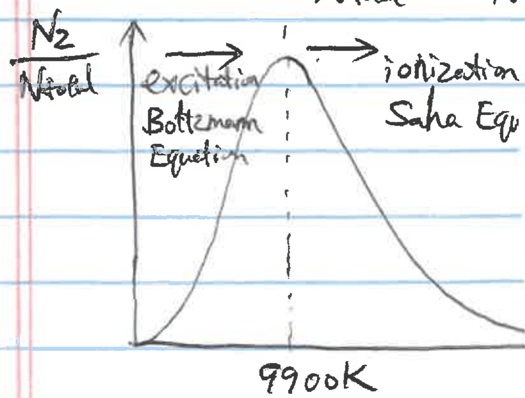
③ evaluate $\frac{N_{II}}{N_{total}} = \frac{N_{II}}{N_I + N_{II}} = \frac{N_{II}/N_I}{1 + N_{II}/N_I}$



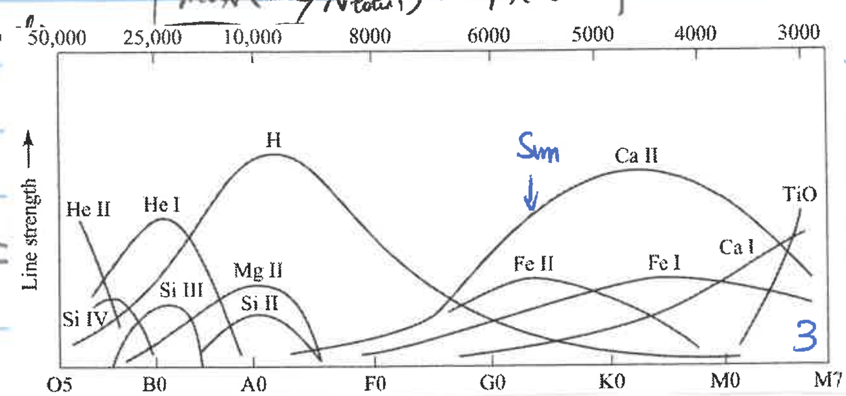
④ use Boltzmann equation to evaluate N_2/N_1

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/kT} = \frac{2(2)^2}{2(1)^2} \cdot \exp\left[- \frac{-13.6/2^2 + 13.6}{kT} \right]$$

⑤ calculate $\frac{N_2}{N_{total}} \simeq \left(\frac{N_2}{N_1 + N_2} \right) \left(\frac{N_I}{N_I + N_{II}} \right) = \frac{N_2/N_1}{1 + N_2/N_1} \cdot \frac{1}{1 + N_{II}/N_I}$



$\boxed{max(N_2/N_{total}) = 9 \times 10^{-6}}$



abundance: $\boxed{12 + \log(Ca/H) = 12 - 5.65 = 6.35}$

Mixed atmosphere. $N(He) : N(H) = 1 : 10$, $N(Ca) : N(H) = 1 : 500,000$

again we assume a constant electron pressure $P_e = n_e kT = 1.5\ N m^{-2}$ @ $T = 5800\ K$

$^{40}_{20}Ca$ — Balmer lines : H I $n=2$ to higher levels, $\chi_1 = 13.6\ eV$

e^- configuration — Ca H&K : Ca II $n=1$ ground state, $\chi_1 = 6.11\ eV$, $Z_I = 1.32$

[Ar]$4s^2$ — $\boxed{T = 5800K \Rightarrow kT = 0.5\ eV}$ $Z_{II} = 2.30$

$1s^2 2s^2 2p^6 3s^2 3p^6 4s^2$ — For the excited state of Ca II $E_2 - E_1 = 3.12\ eV$, $g_1 = 2$, $g_2 = 4$

Ca II $\lambda = hc/\Delta E = 398.8\ nm$

$3s^2 3p^6 4s^1$ — $\Rightarrow$ for H, $(N_2/N_{total})_{HI} = 5.06 \times 10^{-9}$

① $N_{II}/N_I = 1/13,000$ ② $\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\Delta E/kT} \simeq \frac{1}{198,000,000}$

$\Rightarrow$ for Ca, $(N_1/N_{total})_{CaII} = 0.995$

① $N_{II}/N_I = 918$ ② $N_2/N_1 = 1/264$

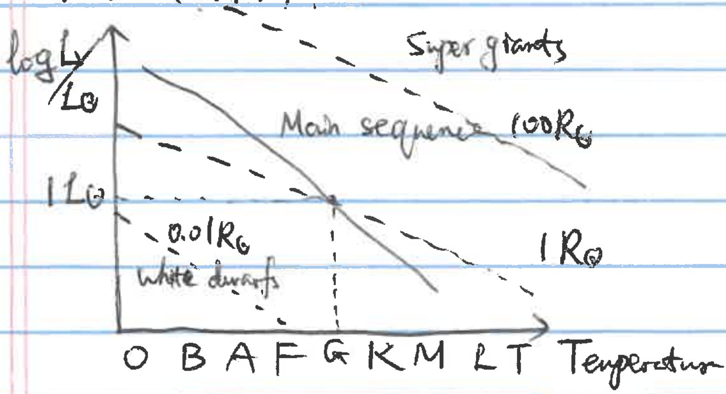
multiply the abundance ratio between H & Ca to $(N_2/N_{total})_{HI}$

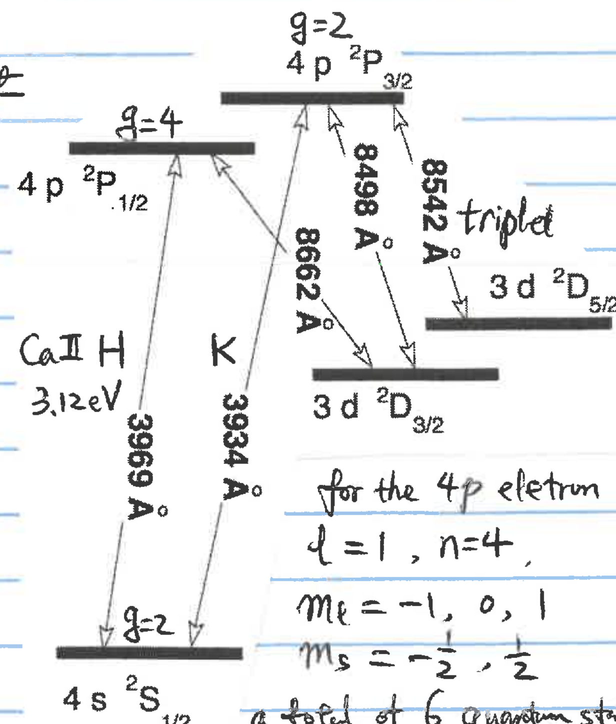
we have $500,000 \times 5.06 \times 10^{-9} = 1/395 \ll (N_1/N_{total})_{CaII} \simeq 1$

HR diagram: $L = 4\pi R^2 \cdot \sigma T^4$, $L = 4\pi d^2 \cdot f$

$\Rightarrow R = \frac{1}{T^2}\sqrt{\frac{L}{4\pi\sigma}}$

L vs T diagram first introduced by Russell 1914.





for the 4p electron

$l = 1$, $n = 4$

$m_l = -1, 0, 1$

$m_s = -\frac{1}{2}, \frac{1}{2}$

a total of 6 quantum state

4 give $J = \frac{1}{2}$, 2 $\Rightarrow J = \frac{3}{2}$

Morgan-Keenan Luminosity classes

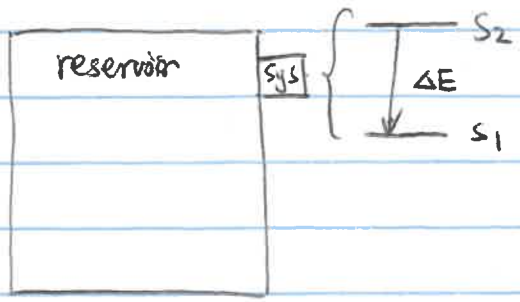
Ia-0, Ia, Ib	II	III	IV	V	D
super giants	bright giants	Normal Giants	Subgiants	MS	white Ds

Discussion on Boltzmann Equation & Saha Equation

Boltzmann Equation: $\frac{N_b}{N_a} = \frac{P_b}{P_a} = \frac{g_b}{g_a} \cdot e^{-(E_b-E_a)/kT}$

assumptions:
① all microstates have equal probability of being occupied
② kT is the energy required to increase Ω by a factor of e
③ when a reservior absorbs energy, Ω increases
④ no energy transfer between reseriors at the same temperature (0th law)

Derivation:



$\Omega_{total} = \Omega_R \cdot \Omega_{sys} = \Omega_R$ because $\Omega_{sys} = 1$
$\Omega(s_1) > \Omega(s_2)$ because reservior absorbs ΔE after the system makes the transition

(A) derivation of the Boltzmann factor:
① suppose the res has Ω numberof microstates before it receives ΔE, which made Ω to increase to $\Omega' = \Omega + f(\Delta E)$ or $\Omega' = \Omega \cdot f(\Delta E)$
② divide the res into two identical parts so that $\Omega = \Omega_1 \cdot \Omega_2 = \Omega_1^2$
and each part receives $\Delta E/2$, so that $\Omega_1' = \Omega_2' = \Omega_1 + f(\frac{\Delta E}{2})$ or $\Omega_1 f(\frac{\Delta E}{2})$
③ because the division is artificial, we have $\Omega' = \Omega_1' \cdot \Omega_2'$
so that $\Omega' = \Omega + f(\Delta E) = [\Omega_1 + f(\frac{\Delta E}{2})]^2 = \Omega_1^2 + f^2(\frac{\Delta E}{2}) + 2\Omega_1 f(\frac{\Delta E}{2})$
or $\Omega' = \Omega \cdot f(\Delta E) = \Omega_1^2 \cdot f^2(\frac{\Delta E}{2}) \Rightarrow f(\Delta E) = f^2(\frac{\Delta E}{2})$
④ only the latter works, so $f(\Delta E) = e^{\Delta E/b}$, where b is a constant

(B) derivation of the kT factor:
① from the ~~first theorem~~ zeroth law of ~~thermal dynamics~~ thermodynamics, systems in thermal equilibrium have the same temperature (T).
② imagine two different reserviors with the same T are in contact to each other they allow heat transfer but no heat flows between them.
③ each system takes or loses ΔE but the total Ω is conserved

$\Omega_0 = \Omega_a \cdot \Omega_b$ a loses ΔE to b

$\Omega_a' = \Omega_a \cdot e^{-\Delta E/f(a)}$, $\Omega_b' = \Omega_b \cdot e^{\Delta E/f(b)}$

$\Rightarrow \Omega_a' \cdot \Omega_b' = \Omega_a \cdot \Omega_b \cdot e^{-\Delta E\left(\frac{1}{f(a)} - \frac{1}{f(b)}\right)} = \Omega_0$

$\Rightarrow f(a) = f(b)$ now let's recall what is the same between the two their temperature, so that we can propose $\boxed{f = kT}$

Saha Equation. $\frac{N_{i+1}}{N_i} = \frac{2 Z_{i+1}}{Z_i} \left(\frac{2\pi m_e kT}{h^2 \cdot n_e^{2/3}} \right)^{3/2} e^{-\chi_i/kT}$

a useful formula, $\frac{h^2 n_e^{2/3}}{2\pi m_e} = 4.8 \times 10^{-19}\,\mathrm{eV} \left(\frac{n_e}{1\,m^{-3}} \right)^{2/3}$

$= 10^{-5}\,\mathrm{eV} \left(\frac{n_e}{10^{20}\,m^{-3}} \right)^{2/3}$

assumptions: ① thermal equilibrium among e^-, atoms, and ions (same T)

② Boltzmann equation

key realization → ③ Z_i is almost independent of T in the regime of interest

Saha's intuition

properties: ① it doesn't care about the detailed physical process that established the thermal equilibrium

② the atomic weight of the ions are irrelevant, only the ionization energy χ_i matters., e.g. $\chi_i = -13.6\,\mathrm{eV} \frac{Z^2}{1^2} = -13.6 Z^2\,\mathrm{eV}$ for Z proton, $1e^-$ sys

③ $\boxed{\text{it requires other equations to be solvable}}$

e.g. for pure H, $N_{II} = n_e$, $n_I + n_{II} = n_{total} \simeq 10^{20}\,m^{-3}$

④ $\boxed{\text{H reaches full ionization @ } T \sim 15{,}000\,K \sim 1.5\,\mathrm{eV}} \ll \chi_i = 13.6\,\mathrm{eV}$

why? Because recombination is equally difficult/unlikely as ionization

Planck Function for Blackbody emission. $B_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$.

$x = h\nu/kT$, $x_{max} = 2.821 \Rightarrow h\nu_{max} = 2.821 \cdot kT = 4.5\,\mathrm{eV} \ll \chi_i$

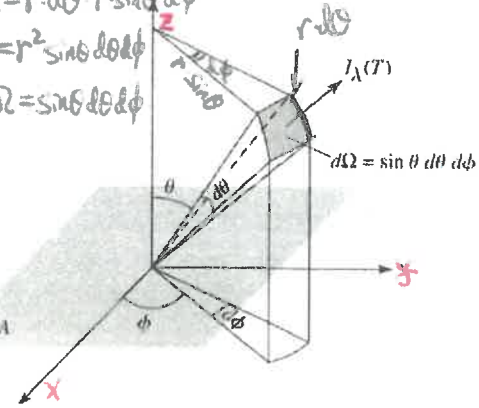
Maxwell-Boltzmann velocity distribution:

$n(v) = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} 4\pi v^2 dv \Rightarrow \frac{1}{2} V_{mp}^2 \cdot m = kT$ (most probable KE)

Chap 1: Introduction - Stellar Atmospheres / Spectra

① The Radiation Field

Area $= r\,d\theta \cdot r\sin\theta\, d\phi$
$= r^2 \sin\theta\, d\theta\, d\phi$
$\Rightarrow d\Omega = \sin\theta\, d\theta\, d\phi$



Specific Intensity [aka, surface brightness]: $\boxed{I_\nu \equiv \dfrac{dE}{dt\, dA\cos\theta \cdot d\Omega\, d\nu}}$

mean intensity: $\boxed{\langle I_\nu \rangle \equiv \dfrac{1}{4\pi}\int I_\nu\, d\Omega}$

specific flux [aka, flux density]: $\boxed{F_\nu \cancel{d\nu} = \int I_\nu \cancel{d\nu} \cos\theta\, d\Omega}$

$$F_\nu = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\nu \cos\theta \sin\theta\, d\theta\, d\phi$$

total flux: $F = \int F_\nu\, d\nu = \int B_\nu(T)\cos\theta\, d\Omega\, d\nu$

Radiation Pressure: $P_{rad,\nu} = \dfrac{1}{c}\displaystyle\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\nu \cos^2\theta\, d\Omega$ (transmission)

$\boxed{P_{rad,\nu} = \dfrac{4\pi}{3c} I_\nu}$, (isotropic radiation field - $I_\nu = \langle I_\nu \rangle$)

BB Radiation Pressure: $P_{rad} = \int P_{rad,\nu}\, d\nu = \dfrac{4\pi}{3c}\displaystyle\int_0^\infty B_\nu(T)\, d\nu = \dfrac{4\sigma_{SB}T^4}{3c} = \dfrac{1}{3}u$

Specific energy density: $\boxed{u_\nu = \dfrac{1}{c}\int I_\nu\, d\Omega = \dfrac{4\pi}{c}\langle I_\nu \rangle}$

total energy density: $u = \int u_\nu d\nu \underset{\text{for BB}}{=} \dfrac{4\pi}{c}\int B_\nu(T)\, d\nu = \dfrac{4\pi}{c}\cdot\dfrac{\sigma_{SB}T^4}{\pi} = \dfrac{4\sigma_{SB}T^4}{c}$

② Stefan-Boltzmann Equation

For an isotropic radiation field, $F_\nu = 0$ because there is no net transfer of energy. However, imagine the surface of a star, the radiation field is isotropic above the surface, but $I_\nu = 0$ below the surface, therefore the specific flux is not zero:

$$F_\nu = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\nu \cos\theta \sin\theta\, d\theta\, d\phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} B_\nu(T)\cos\theta\sin\theta\, d\theta\, d\phi \boxed{= \pi B_\nu(T)}$$

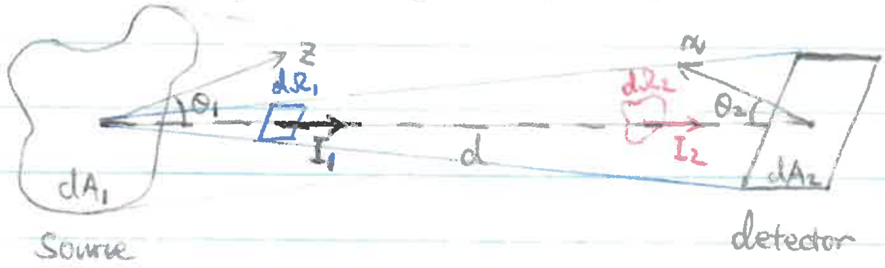
And the total flux (net energy that passes through dA per unit time along z-direction)

$$F = \int F_\nu\, d\nu = \frac{2\pi h}{c^2}\int \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1} = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3}T^4 \equiv \sigma_{SB}T^4$$

Total luminosity of the star is conserved, thus flux decrease as distance squared:

$$L = 4\pi R^2 F \Rightarrow \boxed{F \propto 1/R^2}$$

③ Conservation of specific intensity / surface brightness



Question:
① Why stars don't blind our eyes?
② Why the Sun show limb darkening?

$$\begin{cases} \dfrac{dE_1}{dt_1} = I_1\, dA_1 \cos\theta_1\, d\Omega_1\, d\nu \rightarrow \text{energy carried into the cone } d\Omega_1 \text{, emitted by } dA_1 \\ \dfrac{dE_2}{dt_2} = I_2\, dA_2 \cos\theta_2\, d\Omega_2\, d\nu \rightarrow \text{energy carried into the cone } d\Omega_2 \text{, received by } dA_2 \end{cases}$$

due to energy conservation, $dE_1/dt_1 = dE_2/dt_2$

we also have

$$d\Omega_1 = \frac{dA_2 \cdot \cos\theta_2}{d^2} \qquad d\Omega_2 = \frac{dA_1 \cos\theta_1}{d^2}$$

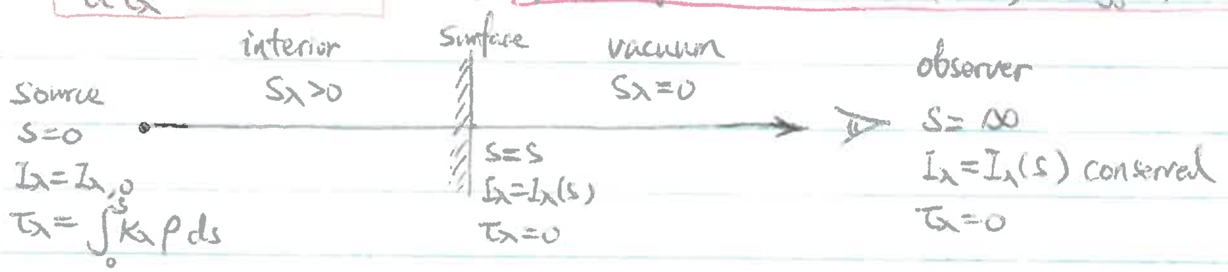
plug them in, we can easily show $I_1 = I_2$

④ Radiative Transfer

$$dI_\lambda = -\underset{\text{opacity}}{\kappa_\lambda}\rho I_\lambda\, ds + \underset{\text{emissivity}}{j_\lambda}\rho\, ds$$

$$-\frac{1}{\kappa_\lambda\rho}\frac{dI_\lambda}{ds} = I_\lambda - j_\lambda/\kappa_\lambda \equiv I_\lambda - \underset{\text{source function}}{S_\lambda}$$

$$\boxed{\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda}, \quad \boxed{\text{optical depth } d\tau_\lambda = -\kappa_\lambda\rho\, ds,\ \tau_\lambda = \int_0^s \kappa_\lambda \rho\, ds}$$



$$\begin{cases} I_\lambda = S_\lambda \Rightarrow dI_\lambda/ds = 0 \Rightarrow I_\lambda = \text{const} \quad (\text{LTE}: S_\lambda = \frac{j_\lambda}{\kappa_\lambda} = B_\lambda(T)) \\ I_\lambda > S_\lambda \Rightarrow dI_\lambda/ds < 0 \Rightarrow I_\lambda \searrow \text{ as beam travels (absorption)} \\ I_\lambda < S_\lambda \Rightarrow dI_\lambda/ds > 0 \Rightarrow I_\lambda \nearrow \text{ as beam travels (emission)} \end{cases}$$

when $\boxed{S_\lambda \text{ is constant (at all s)}}$ we have a simple solution of the transfer equation

$$\int_{I_\lambda(0)}^{I_\lambda(s)} \frac{dI_\lambda}{I_\lambda - S_\lambda} = \int_{s=0}^{s} d\tau_\lambda = -\tau_\lambda \Rightarrow \ln\frac{I_\lambda(s) - S_\lambda}{I_\lambda(0) - S_\lambda} = -\tau_\lambda, \quad \boxed{I_\lambda(s) = I_\lambda(0)e^{-\tau_\lambda} + S_\lambda(1 - e^{-\tau_\lambda})}$$

⑤ Optical depth & Opacity (κ_λ)

solution of transfer equation when $j_\lambda = 0$ (i.e. Absorption Only)

$$I_\lambda(s) = I_{\lambda,0}\, e^{-\int_0^s \kappa_\lambda \rho\, ds} = I_{\lambda,0}\, e^{-\tau_\lambda(s)}$$

for constant ρ & κ_λ (opacity)

opacity: cross-section per unit mass $\kappa_\lambda = \frac{n\sigma_\lambda}{\rho} = \frac{\sigma_\lambda}{m}$

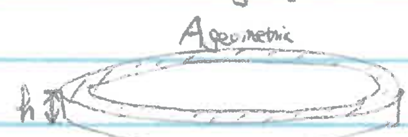
$$\tau_\lambda = \boxed{\kappa_\lambda \rho}\, s = \boxed{n\, \sigma_\lambda}\, s = \frac{s}{m.f.p}$$

$$m.f.p. = \frac{1}{n \cdot \sigma_\lambda} \equiv l_\lambda \quad \text{(mean free path)}$$

$$= \frac{1}{\kappa_\lambda \rho} \equiv l_\lambda$$

ⓐ How to estimate the optical depth of Saturn's ring? Based on an image of Saturn?

$$\tau_\lambda = n \cdot \sigma_\lambda\, s = n\, \sigma_\lambda\, h$$



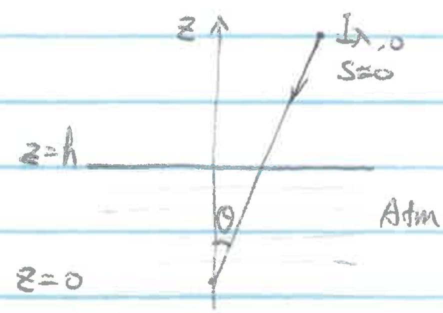
effective reflecting surface area:

$$A_{eff} = n \cdot (A_{geo} \cdot h)\, \sigma_\lambda = \tau\, A_{geo}$$

observed ring surface brightness

$$\Sigma_{obs} = \frac{\Sigma_{int} \cdot A_{eff}}{A_{geo}} = \tau \cdot \Sigma_{int}$$

ⓑ How to measure atmosphere extinction? How to correct it?



$$ds = -dz \cdot \sec\theta$$

$$\tau_\lambda = +\int_0^s \kappa_\lambda \rho\, ds = -\int_h^0 \kappa_\lambda \rho \sec\theta\, dz$$

$$= \tau_{\lambda,v} \sec\theta$$

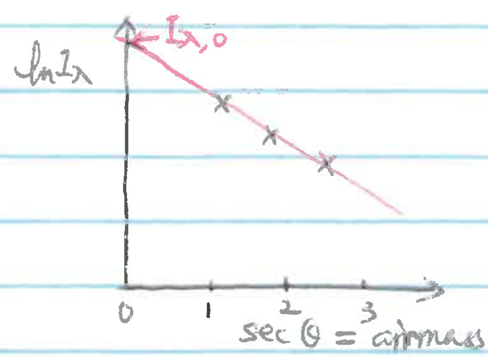
where we defined the vertical optical depth

$$\tau_{\lambda,v} = \int_0^h \kappa_\lambda \rho\, dz$$

solution of the transfer equation

$$I_\lambda = I_{\lambda,0} \cdot e^{-\tau_{\lambda,v} \sec\theta}$$

$$\Rightarrow \ln I_\lambda = \ln I_{\lambda,0} - \tau_{\lambda,v} \sec\theta$$



ⓒ Sources of opacity

① Bound-Bound transition (excitation)

② Bound-Free (photoionization)

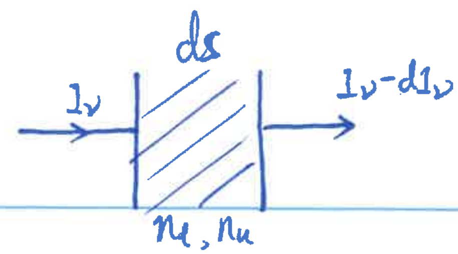
$$\lambda_{cutoff} = \frac{hc}{\chi_n} = \frac{hc}{13.6\,eV/n^2} = 912\text{Å} \cdot n^2$$

$$\sigma_{BF} \propto \nu^{-3}$$

③ Free-Free absorption (Bremsstrahlung)

④ e^- scattering (Thomson/Compton)

⑤ dust absorption & scattering



Microscopic radiative transfer

consider bound-bound transition:

$$n_l B_{lu}\langle I_\nu\rangle = n_u B_{ul}\langle I_\nu\rangle + A_{ul}\cdot n_u$$

\# of absorptions per unit volume per unit time

For a beam of light ($I_\nu = 0$ outside of $\delta\Omega$)

$$\langle I_\nu\rangle = I_\nu\cdot\frac{\delta\Omega}{4\pi}$$

$$\frac{n_{abs}}{n_\nu} = \frac{n_1 B_{12}\cancel{I_\nu}\cdot\cancel{\frac{\delta\Omega}{4\pi}}\cdot\frac{ds}{\cancel{c}}}{\frac{4\pi}{\cancel{c}}\cdot\cancel{I_\nu}\,\cancel{\frac{\delta\Omega}{4\pi}}\cdot\frac{1}{h\nu}} = \frac{h\nu}{4\pi}\cdot n_1 B_{12}\, ds$$

$$\frac{n_{em}}{n_\nu} = \frac{n_2 A_{21}\cdot\frac{ds}{\cancel{c}}\cdot\cancel{\frac{\delta\Omega}{4\pi}}}{\frac{4\pi}{\cancel{c}}\, I_\nu\,\cancel{\frac{\delta\Omega}{4\pi}}\,\frac{1}{h\nu}} = \frac{h\nu}{4\pi}\, n_2 A_{21}\cdot\frac{ds}{I_\nu}$$

microscopic radiative transfer equation

$$\frac{dI_\nu}{ds} = -\kappa_\nu\rho I_\nu + j_\nu\rho$$

Alternatively, think photoexcitation as a collisional process

$$n_1 B_{12}\langle I_\nu\rangle = n_1 n_\nu\int_0^\infty \sigma_\nu u f(u)\,du = n_1 n_\nu \sigma_\nu\cdot c$$

$$n_\nu = \frac{u_\nu}{h\nu} = \frac{4\pi}{c}\langle I_\nu\rangle / h\nu$$

$$\Rightarrow n_1\sigma_\nu = \frac{h\nu}{4\pi}n_1 B_{12}$$

$$\& \; n_1\sigma_\nu = \kappa_\nu\rho$$

how are they related?

① consider absorption only $\quad \frac{dI_\nu}{I_\nu} = -\kappa_\nu\rho\, ds$

this is the fraction of absorbed energy

absorbed photon energy per unit volume $= (n_l B_{lu} - n_u B_{ul})\langle I_\nu\rangle\cdot h\nu\cdot\frac{ds}{c}$

existing photon energy per unit volume $= u_\nu = \frac{1}{c}\int I_\nu\, d\Omega = \frac{4\pi}{c}\langle I_\nu\rangle$

$$\Rightarrow \frac{\text{absorbed}}{\text{existing}} = \boxed{\frac{h\nu}{4\pi}(n_l B_{lu} - n_u B_{ul})}\cancel{\varphi(\nu-\nu_0)}\,ds = -\frac{dI_\nu}{I_\nu} = \boxed{\kappa_\nu\rho}\,ds$$

For simplicity, do not consider line broadening

② consider emission only $\quad \frac{dI_\nu}{I_\nu} = j_\nu\rho\, ds/I_\nu$

emitted photon energy per volume $= A_{ul} n_u\cdot h\nu\cdot\frac{ds}{c}\cdot\cancel{\chi(\nu-\nu_0)}$

$$\Rightarrow \frac{\text{emitted}}{\text{existing}} = \frac{h\nu}{4\pi}A_{ul}n_u\chi(\nu-\nu_0)\,ds/I_\nu = \frac{dI_\nu}{I_\nu} = \frac{j_\nu\rho\, ds}{I_\nu}$$

$$\Rightarrow \kappa_\nu\rho = \frac{h\nu}{4\pi}(n_l B_{lu} - n_u B_{ul})\cancel{\varphi(\nu-\nu_0)}$$

$$j_\nu\rho = \frac{h\nu}{4\pi}A_{ul}n_u\cancel{\chi(\nu-\nu_0)}$$

define source function

$$S_\nu = \frac{j_\nu}{\kappa_\nu} = \frac{n_u A_{ul}}{n_l B_{lu} - n_u B_{ul}}$$

at LTE, $\quad n_u/n_l = \frac{g_u}{g_l}\exp(-\frac{h\nu}{kT})$

$$S_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} = B_\nu(T)$$

Relations between Einstein Coeffs

$$A_{ul} = \frac{2h\nu^3}{c^2}B_{ul}$$

$$B_{lu} = \frac{g_u}{g_l}B_{ul}$$

$$\Rightarrow S_\nu = \frac{A_{ul}/B_{ul}}{[(n_l/n_u)(B_{lu}/B_{ul}) - 1]} = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

Limb Darkening Effect (LDE) ~~dI(θ)~~ $I_\lambda(\theta)/I_\lambda(\theta=0)$ ~~$= 0.4 + 0.6\cos\theta$~~, $\theta \in (0, \frac{\pi}{2})$

$R_\odot = 7\times10^5$ km
photosphere 600 km

Plane-Parallel Atm. $\tau_\lambda = \tau_{\lambda,v} \cdot \sec\theta$ [$\tau_{\lambda,v}$: vertical optical depth]

$$\cos\theta \frac{dI_\lambda}{d\tau_{\lambda,v}} = I_\lambda - S_\lambda \quad (1)$$

Gray atm: $\kappa_\lambda = \bar{\kappa}$ mean opacity

$\Rightarrow \tau_{\lambda,v} = \bar{\tau}_v$ mean vertical optical depth

① integrat (1) over λ $\Rightarrow$

$$\cos\theta \frac{dI}{d\tau_v} = I - S$$

② integrate over $d\Omega$ $\Rightarrow$

$$\frac{d}{d\tau_v}\int I\cos\theta\, d\Omega = \int I\, d\Omega - S\int d\Omega$$

~~③×~~

$$\Rightarrow \frac{dF_{rad}}{d\tau_v} = 4\pi(\langle I\rangle - S)$$

③ $\times\cos\theta$ then integrate over $d\Omega$ $\Rightarrow$

$$\frac{d}{d\tau_v}\int I\cos^2\theta\, d\Omega = \int I\cos\theta\, d\Omega - S\int\cos\theta\, d\Omega$$

$$\Rightarrow \frac{dP_{rad}}{d\tau_v} = \frac{1}{c}\cdot F_{rad}$$ [Pressure gradient drives photon winds

Equilibrium atm: $F_{rad}(\tau_v) = \text{const.} = \sigma T_e^4$ [effective temperature]

$$\Rightarrow \begin{cases} \langle I\rangle = S \\ P_{rad} = \frac{1}{c}F_{rad}\,\tau_v + C \quad (2) \end{cases}$$

Eddington approximation:

$\langle I\rangle = \frac{1}{2}(I_{out} + I_{in})$, $F_{rad} = \pi(I_{out} - I_{in})$, $P_{rad} = \frac{2\pi}{3c}(I_{out} + I_{in}) = \frac{4\pi}{3c}\langle I$

plug the above into Eq.(2) & evaluate at surface: $\tau_v = 0$ & $I_{in} = 0$ & $F_{rad} = \pi I_o$

$$\frac{4\pi}{3}\langle I\rangle = F_{rad}(\tau_v + \tfrac{2}{3}) \Rightarrow \langle I\rangle = \frac{3\sigma}{4\pi}T_e^4(\tau_v + \tfrac{2}{3})$$

$\boxed{\text{LTE: } S_\lambda = B_\lambda \Rightarrow \int S_\lambda d\lambda = S = \frac{\sigma T^4}{\pi}}$

Equilibrium Atm: $S = \langle I\rangle \Rightarrow \boxed{T^4 = \frac{3}{4}T_e^4(\tau_v + \frac{2}{3})}$

& $S = \frac{3\sigma}{4\pi}T_e^4(\tau_v + \frac{2}{3})$

This step is unnecessary for LDE. But it provides a temperature profile

Given the source function: $S = \frac{3\sigma}{4\pi} T_e^4 (\tau_v + \frac{2}{3})$

what's the solution of $I(\tau_v = 0)$, the specific intensity at the surface.

① general solution of the transfer equation

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$$

$$\Rightarrow \frac{dI_\lambda}{d\tau_\lambda} e^{-\tau_\lambda} - I_\lambda e^{-\tau_\lambda} = - S_\lambda e^{-\tau_\lambda}$$

$$\Rightarrow d(I_\lambda e^{-\tau_\lambda}) = - S_\lambda e^{-\tau_\lambda} d\tau_\lambda$$

integrate from $\tau_{\lambda,0}$ to $\tau_\lambda = 0$

$$I_\lambda(\tau_\lambda=0) = I_{\lambda,0} e^{-\tau_{\lambda,0}} - \int_{\tau_{\lambda,0}}^{0} S_\lambda e^{-\tau_\lambda} d\tau_\lambda$$

[if S_λ is independent of τ_λ, $I_\lambda(\tau_\lambda=0) = I_{\lambda,0} e^{-\tau_{\lambda,0}} + S_\lambda(1-e^{-\tau_{\lambda,0}})$]

② for plane-parallel atm, replace τ_λ with $\tau_{\lambda,v} \cdot \sec\theta$

for gray atm, replace I_λ, τ_λ, S_λ with I, τ, S

for equilibrium atm using Eddington approximation: $S = \frac{3\sigma}{4\pi} T_e^4 (\tau_v + \frac{2}{3})$

for optically thick star, $\tau_{\lambda,0} = \infty$

$$I(\tau=0) = \int_0^\infty S \sec\theta \, e^{-\tau_v \sec\theta} d\tau_v$$

for $S = a + b\tau_v$, $I(\tau=0) = a + b\cos\theta = S(\tau_v = \cos\theta)$

where $a = \frac{\sigma}{2\pi} T_e^4$, $b = \frac{3\sigma}{4\pi} T_e^4$

$$\Rightarrow \frac{I(\tau=0, \theta=\theta)}{I(\tau=0, \theta=0)} = \frac{a + b\cos\theta}{a+b} = \frac{2}{5} + \frac{3}{5}\cos\theta$$

Conclusion: Limb darkening can be explained by a plane-parallel gray ATM in LTE using Eddington approximation

Intuitive Explanation: Observer sees the $\tau = 1$ surface of the atmosphere, because

$$I(\tau=0) = a + b\cos\theta = S(\tau_v = \cos\theta)$$

$$\tau = \tau_v \cdot \sec\theta \text{ when } \tau_v = \cos\theta, \ \tau = 1$$

Solution : $S = a + b\tau_v = \frac{3\sigma}{4\pi} T_e^4 \cdot \tau_v + \frac{\sigma}{2\pi} T_e^4$

$\Rightarrow a = \frac{\sigma T_e^4}{2\pi}$, $b = \frac{3\sigma}{4\pi} T_e^4$

Eq (4) becomes $I_\lambda(\theta) = \frac{\sigma T_e^4}{2\pi} + \frac{3\sigma}{4\pi} T_e^4 \cdot \cos\theta$

$$\frac{I_\lambda(\theta)}{I_\lambda(\theta=0)} = \frac{a + b\cdot\cos\theta}{a+b} = \frac{1 + \frac{3}{2}\cos\theta}{1 + \frac{3}{2}} = \frac{2}{5} + \frac{3}{5}\cos\theta$$

Eddington Approximation formula derived :

$$\langle I_\lambda \rangle = \frac{1}{4\pi}\int I_\lambda \, d\Omega = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi}\left[\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} I_{out}\cdot\sin\theta\, d\theta\, d\phi + \int_{\phi=0}^{2\pi}\int_{\theta=\pi/2}^{\pi} I_{in}\sin\theta\, d\theta\, d\phi\right]$$

$$= \frac{1}{4\pi}\left[2\pi\cdot I_{out} + 2\pi I_{in}\right] = \frac{1}{2}(I_{out} + I_{in})$$

$$F_{rad,\lambda} = \int I_\lambda \cdot \cos\theta \, d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda \sin\theta\cos\theta\, d\theta\, d\phi$$

$$= 2\pi\left(\int_{\theta=0}^{\pi/2} I_{out}\cdot\sin\theta\cos\theta\, d\theta + \int_{\theta=\pi/2}^{\pi} I_{in}\cdot\sin\theta\cos\theta\, d\theta\right)$$

$$= 2\pi\cdot\left(I_{out}\cdot\frac{\sin^2\theta}{2}\Big|_0^{\pi/2} + I_{in}\cdot\frac{\sin^2\theta}{2}\Big|_{\pi/2}^{\pi}\right)$$

$$= \pi\,(I_{out} - I_{in})$$

$$P_{rad,\lambda} = \frac{1}{c}\int I_\lambda \cos^2\theta\, d\Omega = \frac{1}{c}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda \cos^2\theta\sin\theta\, d\theta\, d\phi$$

$$= \frac{2\pi}{3c}(I_{out} + I_{in}) = \frac{4\pi}{3c}\langle I_\lambda\rangle \quad \left[\int_{\theta=0}^{} \cos^2\theta\sin\theta\, d\theta = -\frac{1}{3}\cos^3\theta\Big|_0^{\pi/2}\right]$$

Thermodynamic Equilibrium: $S_\lambda = B_\lambda$

Imagine an ideal reflective box of hot gas & photons at equilibrium, so there is no net flow of energy between matter & radiation, they share the same T.

In this case, $dI_\lambda/ds = 0$, intensity is constant throughout the box

Using the Transfer equation, we have $I_\lambda = S_\lambda = B_\lambda(T)$

Since $S_\lambda = j_\lambda/\kappa_\lambda$ is a property of the matter, we can generalize this $S_\lambda = B_\lambda(T)$ to LTE

Radiation Field

	general	isotropic	Eddington App.	small beam
$\langle I_\nu \rangle$	$\frac{1}{4\pi}\int I_\nu \, d\Omega$	I_ν	$\frac{1}{2}(I_{in} + I_{out})$	$I_\nu \cdot \frac{\delta\Omega}{4\pi}$
F_ν	$\int I_\nu \cos\theta \, d\Omega$	$\cancel{\pi I_\nu}$ 0	$\pi(I_{out} - I_{in})$	$I_\nu \, \delta\Omega \cdot \cos\theta$
P_ν	$\frac{1}{c}\int I_\nu \cos^2\theta \, d\Omega$	$\frac{4\pi}{3c} I_\nu$	$\frac{2\pi}{3c}(I_{out} + I_{in})$	$\frac{1}{c} I_\nu \, \delta\Omega \cdot \cos^2\theta$
u_ν	$\frac{1}{c}\int I_\nu \, d\Omega$	$\frac{4\pi}{c} \cdot I_\nu$	$\frac{2\pi}{c} \cdot (I_{in} + I_{out})$	$\frac{1}{c} I_\nu \, \delta\Omega$

$$\int_0^\infty B_\nu(T) \, d\nu = \frac{\sigma_{SB} T^4}{\pi} \quad \text{for planck function}$$

Osillator strength

f_{12} : the effective number of electrons per atom participating in a transition

e.g. $f_{H\alpha} = 0.637$, $f_{H\beta} = 0.119$

N_a : column density of absorbing atoms, e.g. for Balmer lines $N_a = N_2$

$\Rightarrow$ $f N_a$ is the column density of atoms actively participating the transition

$\sigma(\nu) \cancel{d\nu} = [2.655 \times 10^{-6} m^2 Hz] f_{12} \phi(\nu) \cancel{d\nu}$, $\phi(\nu)$: Hz^{-1} unit

f vs. σ $\quad \sigma_{12} \cancel{d\nu} = \frac{e^2}{4\epsilon_0 m_e c} f_{12} \phi_\nu \cancel{d\nu}$, where ϕ_ν is normalized distribution function

$\epsilon_0 = 1/\mu_0 c^2$ permittivity, μ_0: permeability

f vs Einstein coeffs. $\quad B_{12} = \frac{\cancel{4}\pi e^2}{\cancel{4}\epsilon_0 m_e h\nu c} f_{12} \phi_\nu \quad \left[B_{12} = \frac{4\pi}{h\nu} \sigma_{12} \right]$

$$= \frac{\pi e^2}{\epsilon_0 m_e c \cdot h\nu} \cdot f_{12} \phi_\nu$$

Note that here B_{12} is defined by $\langle I_\nu \rangle$, but sometimes it is defined by u_ν

$$B_{12} n_1 \langle I_\nu \rangle \equiv B'_{12} n_1 u_\nu$$

because $\quad u_\nu = \frac{4\pi}{c} \cdot \langle I_\nu \rangle$, $\quad B_{12} = B'_{12} \cdot \frac{4\pi}{c}$

therefore on Wikipedia

$$B'_{12} = \frac{e^2}{4\epsilon_0 m_e h\nu} f_{12} \phi_\nu$$

the other Einstein coeffs can also be expressed in f_{12}

Voigt Profile:

$\frac{u}{c} = \frac{\lambda - \lambda_0}{\lambda_0} = z$

Lorentz / Cauchy Profile:

$$\mathcal{L}(\nu) = \frac{1}{\pi} \frac{\Gamma_\nu}{(\nu - \nu_0)^2 + \Gamma_\nu^2} \Leftrightarrow L(u) = \frac{1}{\pi} \frac{\Gamma_u}{u^2 + \Gamma_u^2} \quad \boxed{\Gamma_u = \frac{\Gamma_\nu}{\nu_0} \cdot c}$$

the above are equivalent because of Doppler shift formula

$$\frac{\nu - \nu_0}{\nu_0} = -\frac{u}{c} \quad \text{[positive velocity means redshift]}$$

and the definition of PDF:

$$\mathcal{L}(\nu)\, d\nu = -L(u)\, du$$

Gaussian Profile:

$$\mathcal{G}(\nu) = \frac{1}{\sqrt{\pi}\, b_\nu} \exp\left[-\frac{(\nu - \nu_0)^2}{b_\nu^2}\right] \Leftrightarrow G(u) = \frac{1}{\sqrt{\pi}\, b_u} \exp\left[-\frac{u^2}{b_u^2}\right]$$

similarly, $\frac{b_u}{c} = \frac{b_\nu}{\nu_0}$, $\text{FWHM} = 2\sqrt{2\ln 2} \cdot \sigma = 2\sqrt{\ln 2} \cdot b$

The convolution of the two is a Voigt profile

$$V(u) = L \circledast G = \int_{-\infty}^{\infty} L(u - x) \cdot G(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{\Gamma_u}{(u - x)^2 + \Gamma_u^2} \cdot \frac{1}{\sqrt{\pi}\, b_u} \exp\left[-\frac{x^2}{b_u^2}\right] \cdot dx$$

Because both L & G are PDFs, their convolution is also a PDF:

$$\int_{-\infty}^{\infty} V(u)\, du = 1$$

Optical Depth:

$$\sigma(\nu) = \frac{q_e^2}{4 \varepsilon_0 m_e c} f \cdot \phi(\nu) \quad \text{where } f \text{ is oscillator strength, } \phi \text{ is a PDF}$$

$$= \frac{2.655 \times 10^{-6}\, \mathrm{m^2 Hz}}{\nu_0} \cdot f \cdot [\phi(\nu)\, \nu_0] \quad \phi(\nu)\nu_0 \text{ is dimensionless \& a preserved quantity}$$

$$\tau(\nu) = \sigma(\nu) \cdot N \propto fN/\nu_0 \, [\phi(\nu)\nu_0]$$

$$EW = \int_0^\infty [1 - e^{-\tau(\lambda)}]\, d\lambda \quad \cancel{\propto \lambda_0} = \int_0^\infty [1 - e^{-\tau(\nu)}] \frac{c}{\nu^2}\, d\nu$$

Absorption Clouds

$$W = \frac{1}{f_0}\int [f_0 - f(\lambda)]\,d\lambda = \int_0^{\infty} [1 - e^{-\tau(\lambda)}]\,d\lambda$$

Application: chemical abundances $\log(N/N_H)+12$ $\longrightarrow$ H, He, O, C, Ne, N, Mg, Si, Fe, S, Al, Ar, Ca, Na

$$\tau(\lambda) = N\sigma(\nu) = N\cdot\frac{1}{\sqrt{\pi}}\frac{c}{b}\frac{a_{ij}}{\nu_{ij}}\cdot \mathrm{Voigt}\Big(\underbrace{\frac{c}{b}\frac{\gamma}{\nu}}_{A_\nu},\ \underbrace{\frac{c}{b}\frac{\nu-\nu_{12}}{\nu}}_{B_\nu}\Big)$$ [Mo+2010]

b is the Doppler parameter for thermal + turbulent motion: $b = \sqrt{2}\,\sigma$

$$b^2 = \frac{2kT}{m} + b_{turb}^2$$

a_{ij} is another way of writing the oscillator strength f_{ij}

$$a_{ij} = \frac{\pi q_e^2}{m_e c} f_{ij}$$

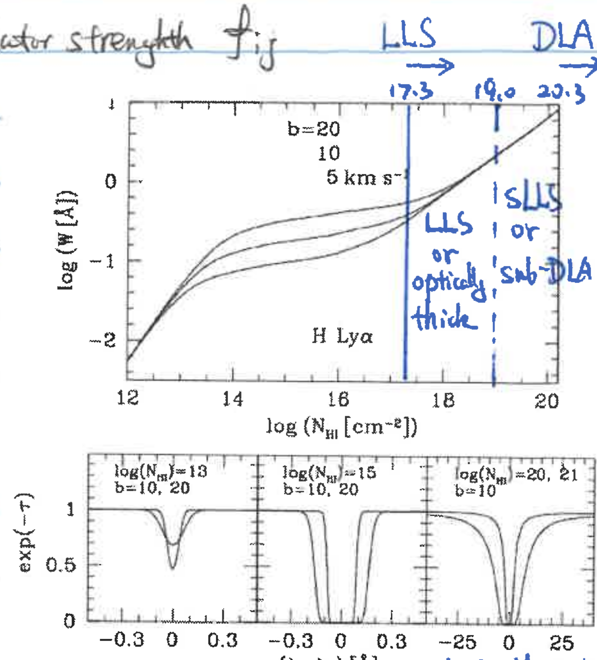


Fig 16.2 Mo 2010

Clearly $\tau(\lambda)$ depends on ① $f_{ij}\cdot N\cdot\lambda_{ij}$

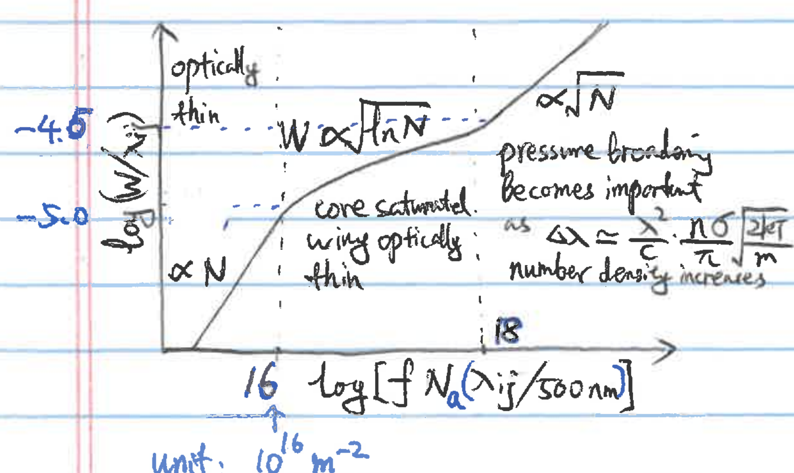
and ② the b parameter

Similarly EW depends on z, the redshift. $EW_{rest} = \frac{EW_{obs}}{1+z}$

Also W scales w/ wavelength because the absorption line show the same velocity profile at different wavelengths: $\frac{\Delta\lambda}{\lambda} \propto b/c$

so for the same velocity profile $W \propto \lambda_{ij}$

So we make a $\boxed{\text{general curve of growth}}$ by plotting W/λ_{ij} vs. $f N\cdot\lambda_{ij}$ (Fig 9.22)



Voigt Profile: $\phi(\nu)$ [Mo+2010]

$$\mathcal{L}(\nu) = \frac{1}{\pi}\frac{\gamma}{(\nu-\nu_{12})^2+\gamma^2}$$

$$\mathrm{FWHM}(\mathcal{L}) = 2\cdot\gamma \Rightarrow \gamma = \frac{\lambda^2}{2\pi c}\frac{1}{\Delta t_0}$$

$$\mathcal{D}(\nu) = \frac{1}{\sqrt{\pi}\,b}\exp\left(-\frac{\nu^2}{b^2}\right)$$

$$b = \sqrt{2}\,\sigma = \sqrt{\frac{2kT}{m} + b_{turb}^2}$$

$$\phi(\nu) = \mathcal{L}(\nu)\otimes\mathcal{D}(\nu) = \frac{1}{\sqrt{\pi}}\frac{c}{b}\frac{V(A,B)}{\nu}$$

where $V(A,B) = \frac{A}{\pi}\int_{-\infty}^{\infty}\frac{\exp(-y^2)\,dy}{(B-y)^2+A^2} \simeq \exp(-B^2) + \frac{1}{\sqrt{\pi}}\frac{A}{A^2+B^2}$

Voigt function

$$A = \frac{c}{b}\frac{\gamma}{\nu},\quad B = \frac{c}{b}\frac{\nu-\nu_{12}}{\nu}$$

Different Temperatures

more fundamental

T_b : Brightness Temperature : $I_\nu = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT_b}\right)-1\right]^{-1} \simeq \frac{2\nu^2 kT_b}{c^2}$ when $h\nu \ll kT_b$

T_k : Kinetic Temperature $\quad q_{Lu} = \frac{n_u}{n_L}\cdot q_{uL} = \frac{g_u}{g_L}\exp\left(-\frac{h\nu_0}{kT_k}\right)\cdot q_{uL}$

$$f(u)du = \left(\frac{m}{2\pi kT_k}\right)^{3/2} 4\pi u^2 \exp\left(-\frac{mu^2}{2kT_k}\right)du$$

T_x : Excitation Temperature (or Spin Temperature for HI) $\quad \frac{n_u}{n_L} = \frac{g_u}{g_L}\exp\left(-\frac{h\nu_0}{kT_x}\right)$

less fundamental

T_i : Ionization Temperature (= Excitation Temperature because TE is assumed)

$$\frac{N_{I+1}}{N_I} = \frac{2 Z_{I+1}}{Z_I}\cdot\left(\frac{2\pi m_e kT_i}{h^2 n_e^{2/3}}\right)^{3/2}\exp\left(-\frac{\chi_I}{kT_i}\right)$$

T_e : Effective Temperature (= T_b for Planck function) $\quad L_{bol} = 4\pi R^2 \sigma_{SB} T_e^4$

Two-Level system in balance

$$n_1 B_{12}\langle I_{\nu_0}\rangle + n_1 n_e q_{12} = n_2 A_{21} + n_2 B_{21}\langle I_{\nu_0}\rangle + n_2 n_e q_{21}$$

it can be shown (in §7.4 of Essential Radio Astronomy) that

$$\exp\left(-\frac{h\nu_0}{kT_x}\right) = \exp\left(-\frac{h\nu_0}{kT_b}\right)\cdot\frac{A_{21} + n_e q_{21}\exp\left(-\frac{h\nu_0}{kT_k}\right)\cdot\left[\exp\left(\frac{h\nu_0}{kT_b}\right)-1\right]}{A_{21} + n_e q_{21}\left[1-\exp\left(-\frac{h\nu_0}{kT_b}\right)\right]}$$

① when $A_{21} \gg n_e q_{21}$ (or $n_e \ll n_{crit}$), $T_x \to T_b$

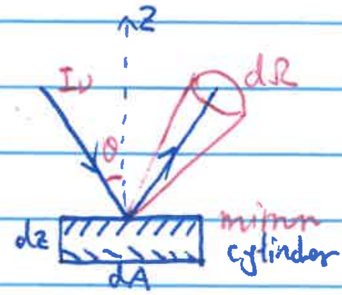
② when $A_{21} \ll n_e q_{21}$ (or $n_e \gg n_{crit}$), $T_x \to T_k$

③ any other conditions, T_x lies between T_k and T_b

Radiation Pressure [intrinsic vs. external]

① intrinsic = photon - photon interactions

consider a flat cylinder mirror in a radiation field on the top surface:



$$dp_\nu = \frac{2dE_\nu\cos\theta}{c} = \frac{2}{c} I_\nu\, dt\, dA \cos^2\theta\, d\Omega$$

because $I_\nu = \dfrac{dE/d\nu}{dt\, dA\cos\theta\, d\Omega} = \dfrac{E_\nu}{dt\, dA\cos\theta\, d\Omega}$

$$\Rightarrow P_{rad,\nu} = \left(\frac{dp_\nu}{dt}\right)\Big/dA = \frac{2}{c}\int_{hemisphere} I_\nu \cos^2\theta\, d\Omega \underset{\substack{\uparrow \\ \text{symmetric radiation field}}}{=} \frac{1}{c}\int_{sphere} I_\nu\cos^2\theta\, d\Omega$$

for blackbody radiation + isotropic

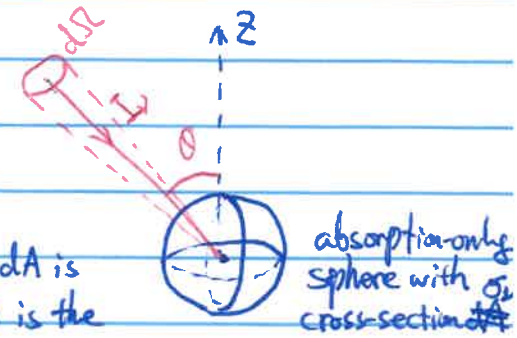
$$P_{rad} = \frac{4\pi}{3c}\int_0^\infty B_\nu(T)\,d\nu = \frac{4\,\sigma_{SB}T^4}{3c}$$

$$u = \frac{1}{c}\int_{\nu=0}^\infty \left(\int_{sph} I_\nu\, d\Omega\right) d\nu = \frac{4\pi}{c}\int_0^\infty B_\nu(T)\,d\nu = \frac{4\,\sigma_{SB}T^4}{c}$$

$$\Rightarrow P_{rad} = \frac{1}{3}u \quad \left[\text{for ideal gas, } P = nkT = \frac{2}{3}u,\ u = \frac{3}{2}nkT\right]$$

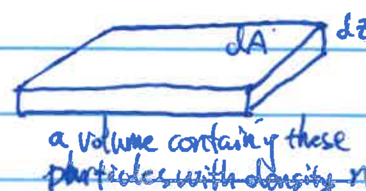
② external = photon - particle interactions

consider a sphere in a radiation field



$$dp_\nu = \frac{1}{c} I_\nu\, dt\, dA\cdot\cos\theta\, d\Omega$$

it's missing a $\cos\theta$ term because the cross-section dA is independent of the angle θ. The remaining $\cos\theta$ is the momentum projection along the z axis:



$$\Rightarrow dP^{ext}_{rad,\nu} = \frac{1}{c}\int_{sph} I_\nu\cos\theta\, d\Omega\cdot\frac{n\, dA\, dz\cdot\sigma_\nu}{dA}$$

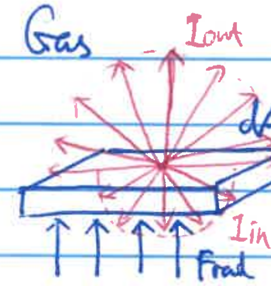
$$= \frac{1}{c} F_{rad,\nu}\, d\tau_\nu$$

$$\Rightarrow P^{ext}_{rad,\nu} = \frac{1}{c}\int F_{rad,\nu}\, d\tau_\nu \underset{\substack{\uparrow \\ \text{plane-parallel equilibrium atm.}}}{=} \frac{1}{c}F_{rad,\nu}\,\tau_\nu$$

Radiation Pressure of Photon Gas vs. Radiation Pressure on Gas

Net energy transfer per unit area per unit time is

$$F_{rad} = \int I \cos\theta \, d\Omega = \frac{dE}{dA\, dt}$$



⇒ interceptel energy transfer by gas particles with σ (cross section) & n (density)

$$F_{rad} \cdot \frac{n \cdot dA \cdot ds \cdot \sigma}{dA} = F_{rad} \cdot n \cdot \sigma \cdot ds$$

⇒ intercepted momentum per unit time per unit area, which is pressure

$$dP = \frac{dE/c}{dA\,dt} = \frac{F_{rad}}{c} n \cdot \sigma \, ds$$

⇒ total pressure from intercepted momentum

$$\boxed{P = \int \frac{F_{rad}}{c} n \sigma \, ds = \frac{F_{rad}}{c} \cdot \tau} \text{ [for plane-parallel equilibrium]}$$

$$\Rightarrow \boxed{\frac{dP}{d\tau} = \frac{F_{rad}}{c}} \quad \text{but } P \neq P_{rad}$$

On the other hand, the radiation pressure of photon gas is defined as

$$P_{rad} = \frac{1}{c} \int I \cos^2\theta \, d\Omega \quad \text{(transmission case)}$$

Using the RT equation for ~~gray~~ atmosphere plane-parallel

$$\cos\theta \cdot \frac{dI}{d\tau_\nu} = I - S \qquad \text{multiply} \times \cos\theta \text{ then integrate over } d\Omega$$

$$\frac{d}{d\tau_\nu} \int I \cos^2\theta \, d\Omega = \int I \cos\theta \, d\Omega - S \int \cos\theta \, d\Omega$$

$$\Rightarrow \boxed{\frac{dP_{rad}}{d\tau_\nu} = \frac{F_{rad}}{c}}$$

These two equations are not the same. Because P_{rad} exists independent of gas particles. But the existence of gas creates a gradient in P_{rad}, which drives radiative flux (F_{rad}), which then drives pressure on gas particles.

Chap 10 Stellar Interior

Equations of Stellar Structure (spherical symmetry)

Continuity: $\frac{dM}{dr} = 4\pi r^2 \rho\cancel{(r)}$

pressure E of State: $P = \frac{\rho k T}{\mu m_p}$, $\mu = \frac{\sum_i n_i A_i}{\sum_i n_i}$

Hydrostatic Equilibrium: $\frac{dP}{dr} = -\frac{GM\cancel{(r)}\rho\cancel{(r)}}{r^2}$

Energy Transport:

$\frac{dT}{dr} = -\frac{3\kappa\rho L\cancel{(r)}}{64\pi\sigma_{SB} r^2 T^3}$ radiative

$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$ convective $P \propto \rho^\gamma$ adiabatic process
or $P = K\rho^\gamma$, $\gamma = \frac{5}{3}$ for ideal, monatomic gas

Energy generation

$\frac{dL}{dr} = 4\pi r^2 \rho\, \epsilon\cancel{(r)}$

Constitutive relations

$\mu(\rho, T)$, $\kappa(\rho, T)$, $\epsilon(\rho, T)$

Solutions: $\rho(r)$, $T(r)$, $L(r)$, $M(r)$, $P(r)$

Mean Molecular Mass: $X = \rho_H/\rho$, $Y = \rho_{He}/\rho$, $Z = \rho_{metal}/\rho$

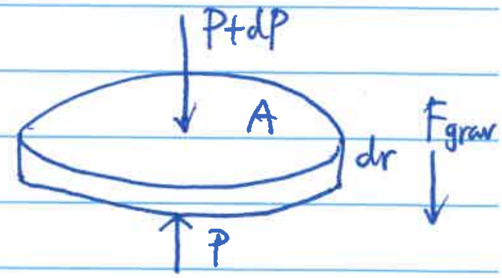
neutral gas: $n = \rho X/m_p + \rho Y/4m_p + \rho Z/Am_p$

$= \frac{\rho}{m_p}\left[X + \frac{1}{4}Y + \frac{Z}{A}\right] = \frac{\rho}{\mu m_p}$

fully ionized gas: $n = 2 \cdot \rho X/m_p + 3\rho Y/4m_p + \left(\frac{A}{2}+1\right)\frac{\rho Z}{Am_p}$

$\simeq \frac{\rho}{m_p}\left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right] = \frac{\rho}{\mu m_p}$

Equation of Hydrostatic Equilibrium $\quad \frac{dP}{dr} = -\rho g$



consider a cylinder in the atmosphere

$$F_{grav} = \rho \cdot A \cdot dr \cdot g$$

must be balanced by pressure gradient

$$F_{pres} = [P - (P + \Delta P)] \cdot A = -dP \cdot A$$

$$\Rightarrow \quad \rho A\, dr\, g = -dP\, A \quad \Rightarrow \quad \frac{dP}{dr} = -\rho g$$

Application: Pressure profile of isothermal atmosphere

mean molecular mass

ideal gas law: $P = n k T = \frac{\rho}{\mu m_p} \cdot k T$ where $\mu = \frac{\sum_i n_i A_i}{\sum_i n_i}$

$$\Rightarrow \frac{dP}{dr} = -\frac{\mu m_p}{kT} \cdot P \cdot g$$

sea level

$$\Rightarrow \frac{dP}{P} = -\frac{\mu m_p g}{kT} \cdot dr \quad \Rightarrow \quad P(r) = P_0 \cdot \exp\left(-\frac{r - R_\oplus}{H}\right)$$

where $H = \frac{kT}{\mu m_p g}$ is the scale height.

Earth's atmosphere: 78% N_2, 21% O_2, 1% Ar in # density

molecular mass numbers: 28 32 40

$$\mu = \frac{\sum_i n_i A_i}{\sum_i n_i} = 28 \cdot 78\% + 32 \cdot 21\% + 40 \cdot 1\% = 29$$

$$\Rightarrow H = \frac{kT}{\mu m_p g} = 8\ \text{km} \quad \text{for } T \sim 290\,K,\ g = 9.8\ m/s^2$$

At what elevation does pressure fall to 10% of sea level?

$$\exp\left(-\frac{r - R_\oplus}{H}\right) = 0.1 \quad \Rightarrow \quad r - R_\oplus = -H \cdot \ln(0.1) = 2.3 H$$

Polytropic Models & the Lane-Emden Equation

Goal: solve for the density profile of a star in hydrostatic equilibrium

Derivation:

assumptions: ① $P = K\rho^{\gamma}$ ② $\frac{dP}{dr} = -\rho g$ ③ $\frac{dM}{dr} = 4\pi r^2 \rho$

take the radial derivative of HSE:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM}{dr} = -G\cdot 4\pi r^2 \rho$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\cdot\frac{dP}{dr}\right) = -4\pi G\rho \quad [\text{similar to Poisson Eq: } \nabla^2\Phi = 4\pi G\rho]$$

use the equation of state to replace P

$$\frac{\gamma K}{r^2}\frac{d}{dr}\left(r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right) = -4\pi G\rho$$

define $\gamma \equiv \frac{n+1}{n} \Rightarrow \gamma - 2 = \left(\frac{1}{n} - 1\right) \Rightarrow \rho^{\gamma-2}\frac{d\rho}{dr} = n\frac{d}{dr}\left(\rho^{\frac{1}{n}}\right)$

define $D(r) \equiv \left(\frac{\rho}{\rho_c}\right)^{\frac{1}{n}}$ so that it's dimensionless.

$$\left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dD_n}{dr}\right) = -D_n^n$$

define the first term in [ ] $\equiv \lambda_n^2$

$$\Rightarrow \frac{\lambda^2}{r^2}\frac{d}{dr}\left(r^2\frac{dD_n}{dr}\right) = -D_n^n$$

define $\xi \equiv r/\lambda_n$ as the dimensionless radius

$$\Rightarrow \frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{dD_n}{d\xi}\right] = -D_n^n \quad \text{Lane-Emden equation}$$

Solutions:

$n=0$, $D_0(\xi) = 1 - \xi^2/6$ & $\xi_1 = \sqrt{6}$

$n=1$, $D_1(\xi) = \sin\xi/\xi$ & $\xi_1 = \pi$

$n=5$, $D_5(\xi) = [1 + \xi^2/3]^{-\frac{1}{2}}$ & $\xi_1 \Rightarrow \infty$

Boundary Conditions:

surface $D_n(\xi) = 0$ when $\xi = \xi_1$

center $dD_n/d\xi = 0$ when $\xi = 0$ because $\frac{dP}{dr} \to 0$ as $r \to 0$

Polytropic Model – Eddington standard model

Thermal energy density of ideal gas:

$$u_T = \frac{3}{2} kT \cdot n = \frac{3}{2} P_T \quad , \quad n = \rho / \mu m_H$$

Photon energy density

$$u_p = \frac{4 \sigma T^4}{c} = 3 \cdot P_{rad}$$

If the ratio between the two is constant β: $u_p = \beta u_T \Leftrightarrow P_T = \frac{2}{\beta} P_{rad}$

we can solve T as function of ρ:

$$\frac{4\sigma}{c} T^4 = \beta \cdot \frac{3}{2} n k T = \beta \cdot \frac{3}{2} \frac{\rho}{\mu m_H} \cdot kT$$

$$\Rightarrow \quad T \propto \rho^{\frac{1}{3}}$$

$$\Rightarrow \quad P_T = \frac{\rho}{m_H \cdot \mu} \cdot kT \propto \rho^{\frac{4}{3}} \quad \Rightarrow \text{Polytropic } n = 3$$

$$P_{rad} = \frac{\beta}{2} P_T \propto \cancel{\beta} \rho^{\frac{4}{3}}$$

Numerically Solve Lane-Emden Equation with Euler Method

$$\frac{d^2 D_n}{dx^2} + \frac{2}{x} \frac{dD_n}{dx} + D_n^n = 0$$

define $\quad f_n(x) \equiv \frac{dD_n}{dx} \quad \Rightarrow D_n^{i+1} = D_n^i + \Delta x \cdot f_n^i$

gives $\quad \frac{df_n}{dx} = -D_n^n - \frac{2}{x} f_n \quad \Rightarrow f_n^{i+1} = f_n^i + \Delta x \cdot \left[-(D_n^i)^n - \frac{2}{x_i} f_n^i \right]$

define discrete x grids: $\quad x^{i+1} = x^i + \Delta x$

~~the equation can then be solved with the boundary condition:~~

$D_n(x=0) = 1.0 \quad$ because $\quad D_n \equiv \left(\frac{\rho}{\rho_c}\right)^{1/n}$

$f_n(x=0) = 0.0 \quad$ because $\quad \frac{dP}{dr} = -g\rho \rightarrow 0$ at the center

Radiative Temperature Gradient

$$\cos\theta \, \frac{dI_\lambda}{d\tau_{\lambda,v}} = I_\lambda - S_\lambda$$ [vertical optical depth $\tau_{\lambda,v}$]

$$\Rightarrow \frac{dP_{rad}}{d\tau_v} = \frac{1}{c} F_{rad}$$ [$a \equiv \frac{4\sigma_{SB}}{c}$, radiation constant]

For Blackbody radiation, $P_{rad} = \frac{4\pi}{3c}\int_0^\infty B_\nu(T)\,d\nu = \frac{4\sigma T^4}{3c} \equiv \frac{1}{3} a T^4$

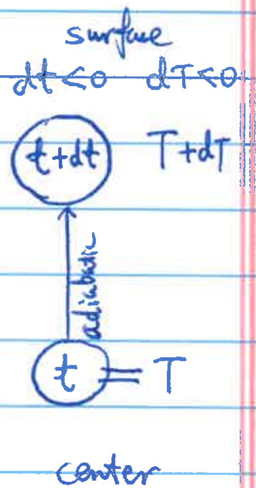
$$\Rightarrow \frac{dP_{rad}}{dr} = \frac{16\sigma}{3c} T^3 \frac{dT}{dr}$$

$$\frac{dP_{rad}}{dr} = -\frac{\kappa\rho}{c} F_{rad} = -\frac{\kappa\rho}{c} \cdot \frac{L_r}{4\pi r^2}$$

$$\Rightarrow \frac{dT}{dr} = -\frac{\kappa\rho}{c} \cdot \frac{L_r}{4\pi r^2} \cdot \frac{3c}{16\sigma} \cdot \frac{1}{T^3}$$

$$= -\frac{3}{16\sigma_{SB}} \frac{\kappa\rho}{T^3} \cdot \frac{L_r}{4\pi r^2}$$

Adiabatic Temperature Gradient



Ideal gas law: $P = \frac{1}{3}\int_0^\infty n_p \, p \, v \, dp$ [pressure integral]

$= nkT$ for Maxwell-Boltzmann velocity dist.

$= \rho kT / \mu m_H$

$$\Rightarrow \frac{dP}{dr} = -\frac{P}{\mu}\frac{d\mu}{dr} + \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

Pressure E of State: $P = K\rho^\gamma$

$$\Rightarrow \frac{dP}{dr} = \gamma \frac{P}{\rho}\frac{d\rho}{dr}$$ [$\gamma = 5/3$ for monatomic gas]

Intuitive understanding of the condition that leads to convection (Fig on the left): $P = nkT$, initially $t = T$, if $t + dt > T + dT$ then $n_{inside} < n_{outside}$ the bubble will continue to float

assuming $d\mu/dr = 0$, we have

$$\left.\frac{dT}{dr}\right|_{adiabatic} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2}$$

superadiabatic: $\left|\frac{dT}{dr}\right|_{actual} > \left|\frac{dT}{dr}\right|_{adiabatic}$ lead to convection

Stellar Energy Sources : Gravity vs. Nuclear Fusion Binding Energy

Kelvin-Helmholtz Timescale : [Gravitational Binding Energy]

gravitational potential energy of dm shell located at r :

$$dU_g = -G\frac{M_r}{r}\cdot 4\pi r^2 \rho\, dr$$

$$\Rightarrow \quad U_g = -4\pi G\int_0^R M_r \rho r\, dr$$

assuming constant density $\rho = M/\frac{4}{3}\pi R^3$, $M_r = \frac{4}{3}\pi r^3 \rho$

$$\Rightarrow \quad U_g = -4\pi G\int_0^R \frac{4}{3}\pi r^3 \cdot \frac{9M^2}{16\pi^2 R^6}\cdot r\, dr = -3GM^2\int_0^R r^4 dr \cdot \frac{1}{R^6}$$

$$= -\frac{3}{5}\frac{GM^2}{R}$$

applying virial theorem $\langle E\rangle = \frac{1}{2}\langle U\rangle = -\frac{3}{10}\frac{GM^2}{R}$, total energy

$$\text{KE timescale} = \frac{-\langle E\rangle}{L_\odot} = \frac{3}{10}\frac{GM_\odot^2}{R_\odot L_\odot} \sim 10^7\ \text{yr} = 10\ \text{Myr}$$

[nuclear ~~binding energy~~]

$$\text{Nuclear timescale} = \frac{E_{nuclear}}{L_\odot} = \frac{10\% \times 0.007 \times M_\odot c^2}{L_\odot} \simeq 10\ \text{Gyr}$$

$$\frac{E_b}{A} = \frac{AM_p + (A-Z)M_n - M_{nucleus}}{A/c^2}$$

$\sim$ ~~8~~ 9 MeV/nucleon for $^{56}_{26}Fe$

$4H \rightarrow He$ releases the binding energy of He $= 0.007 \cdot m_H \cdot c^2$

$$1u = \frac{1}{12} m(^{12}_{6}C) = 931.5\ \text{MeV}/c^2$$

$4H = 4.0313u$, $1He = 4.002603u \Rightarrow \Delta m = 0.0287u = 0.7\% / 4H$

Importance of Quantum Mechanical Tunnelling

Condition for fusion to happen is to bring two protons within the size of the proton which requires the kinetic energy to be greater than the potential energy at $r \sim 10^{-15}$ m

$$\frac{1}{2}\mu_m \bar{v}^2 = \frac{3}{2}kT \geq \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r} \Rightarrow T \geq 10^{10}\ K \gg T_c \sim 10^7 K$$

Taking into account quantum tunnelling, the kinetic energy only needs to overcome the Coulomb barrier at $\sim$ de Broglie wavelength, $\lambda = h/p$

$$\frac{1}{2}\mu_m \bar{v}^2 = \frac{3}{2}kT = \frac{p^2}{2\mu_m}, \text{ where } \mu_m = \frac{m_1 m_2}{m_1+m_2} = \frac{m_p}{2}$$

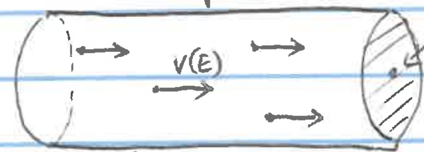
, the reduced mass of p^+ & p^+

$$\frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{\lambda} = \frac{(h/\lambda)^2}{2\mu_m}, \Rightarrow \lambda = \frac{2\pi\epsilon_0 h^2}{Z_1 Z_2 e^2 \mu_m}$$

$$\frac{3}{2}kT \geq \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{\lambda} = \frac{Z_1^2 Z_2^2 e^4 \mu_m}{8\pi^2\epsilon_0^2 h^2} \Rightarrow T \geq \frac{Z_1^2 Z_2^2 e^4 \mu_m}{12\pi^2\epsilon_0^2 h^2 k} \sim 10^7 K$$

Nuclear Reaction Rate

incident particles with $E=\frac{1}{2}\mu_m v^2$



① Maxwell - Boltzmann Distribution

$$\underbrace{n_E\, dE}_{} = \frac{2n}{\sqrt{\pi}} \frac{1}{(kT)^{3/2}} E^{1/2} e^{-E/kT} dE$$

density of all particles with energy [E, E+dE]

of incident particles that can hit target particle

$$\boxed{dN_E = \underbrace{\sigma(E)\, v(E)\, dt}_{\text{volume of cylinder}} \cdot \underbrace{n_{iE}\, dE}_{\text{number density of incident particles within } [E,\, E+dE)}}$$

② # of incident particles per target particle ~~g~~over dt period, dN_E

$$\boxed{\frac{dN_E}{dt} = \sigma(E)\, v(E)\, n_{iE}\, dE} = \sigma(E)\, v(E) \cdot \frac{n_i}{n} \cdot n_E\, dE$$

where $n_i = \int n_{iE}\, dE$, $n = \int n_E\, dE$ are the total number densities of incident particles & all particles.

$$\frac{dN_E}{dt} = \text{\# of reactions per target particle per unit time}$$

③ # of reactions per unit volume per unit time (reaction rate per volume)

$$r_{ix} = \int n_x \cdot \frac{dN_E}{dt} = \int_0^\infty n_x \cdot n_i\, \sigma(E)\, v(E) \cdot \frac{n_E}{n} dE$$

④ cross section 's dependency on energy

Barrier potential height

$$\sigma(E) = S(E) \cdot \frac{1}{E} \cdot e^{-b/\sqrt{E}} \leftarrow \text{tunnelling probability} \propto e^{-2\pi^2 U_c/E}$$

$$\sigma(E) \propto \pi\lambda^2 = \pi\left(\frac{h}{p}\right)^2 \propto \frac{1}{E} \quad \text{de Broglie wavelength}$$

$$\frac{U_c}{E} = \frac{Z_1 Z_2 e^2/4\pi\epsilon r}{\mu v^2/2}$$

$r = \lambda = h/p = \frac{h}{\mu_m v}$ ← de Broglie wavelength

$$\frac{U_c}{E} = \frac{Z_1 Z_2 e^2}{2\pi\epsilon_0 h v} \propto \frac{1}{\sqrt{E}}$$

⑤ substituting Maxwell distribution & cross section to r_{ix}

$$r_{ix} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{(\mu_m \pi)^{1/2}} \int_0^\infty S(E)\, \underline{e^{-bE^{-1/2}}\, e^{-E/kT}}\, dE$$

Gamov peak occurres because the function

$$f(E) = e^{-bE^{-1/2}} \cdot e^{-E/kT} = e^{-\frac{b}{\sqrt{E}} - \frac{E}{kT}}$$

peaks at ~~$E_0 = \frac{bkT}{2}$~~ $E_0 = \left(\frac{bkT}{2}\right)^{2/3}$ where $b = \frac{\pi \mu_m^{1/2} Z_i Z_x e^2}{\sqrt{2} \cdot \epsilon_0 h}$

for p-p collision/fusion, $E_0 = 6\text{ keV}\left(\frac{T}{10^7 K}\right)^{2/3}$

recall $1\text{ eV} \sim 10^5 K \cdot k$, $6\text{ keV} \sim 6\times10^8 K \cdot k$

⑥ Power-law approximations: $r_{ix} \simeq r_0 \cdot X_i X_x \rho^{\alpha'} T^{\beta}$

⑦ ϵ : energy generation rate per unit mass $\quad dL = \epsilon \cdot dm$

$$\rho \cdot \epsilon_{ix} = r_{ix} \cdot \mathcal{E}_0$$, where $\mathcal{E}_0$ is the energy generated per reaction

$$\Rightarrow \epsilon_{ix} = \left(\frac{\mathcal{E}_0}{\rho}\right) r_{ix} = \epsilon_0 X_i X_x \rho^{\alpha'-1} T^{\beta} \quad [W/kg]$$

Now it's interesting to think about $\boxed{\epsilon_{gravity} = \frac{1}{M} \cdot \frac{dU_g/2}{dt}}$

for homogeneous stars $\frac{1}{2}U_g = E = \frac{-3}{10}\frac{GM^2}{R}$

$$\epsilon_{gravity} = \frac{3}{10}\frac{GM}{R^2}\frac{dR}{dt}$$

Neutrinos : $m_\nu < 2.2\ eV/c^2$, $m_{e^-} = 0.5\ MeV/c^2$, $m_p = 1\ GeV/c^2$

PP chain : $4\,{}^1_1H \rightarrow {}^4_2He + 2e^+ + 2\nu_e + 2\gamma$ [How neutrino is produced? $p^+ \rightarrow n + e^+ + \nu_e$] β decay

low mass MS

$$\epsilon_{pp} \simeq 10^{-12}\ W m^3 kg^{-2} \cdot \rho \cdot \boxed{X^2 \cdot T_6^4} \quad @\ T_6 \sim 15$$

$$\epsilon_{pp} = 0.24 \rho X^2 T_6^{-2/3} e^{-33.8 T_6^{-1/3}}\ W kg^{-1}$$

CNO cycle :

high mass MS

$$\epsilon_{CNO} \simeq 8 \times 10^{-31}\ W m^3 kg^{-2} \cdot \rho \cdot \boxed{X} \cdot X_{CNO} \cdot \boxed{T_6^{19.9}} \quad @\ T_6 \sim 15$$

$$\epsilon_{CNO} = 8.67 \times 10^{20} \rho X \cdot X_{CNO} \cdot T_6^{-2/3} e^{-152.28 T_6^{-1/3}}\ W kg^{-1}$$

Triple α

Horizontal branch

$2\,{}^4_2He \Leftrightarrow {}^8_4Be$, ${}^8_4Be + {}^4_2He \rightarrow {}^{12}_6C + \gamma$

$$\epsilon_{3\alpha} \simeq \epsilon'_{0,3\alpha} \rho^2 \boxed{Y^3 T_8^{41}}$$ [three body interaction, $r_{ix} \propto (\rho Y)^3$]

$$\epsilon_{3\alpha} = 51 \rho^2 Y^3 T_8^{-3} e^{-44 T_8^{-1}}\ W kg^{-1} \quad @\ T_8 = 1$$

α capturing ${}^{12}_6C + {}^4_2He \rightarrow {}^{16}_8O + \gamma$

produces O, Ne, Na, Mg, Si, P, S

binding energy per nucleon : $\frac{E_b}{A} = \frac{\Delta mc^2}{A} = [Z m_p + (A-Z)m_n - M_{nucleus}]c^2/A$ >0

peaks at ${}^{56}_{26}Fe$ at ~ 9 MeV/nucleon

CNO cycle : ${}^{12}_6C + {}^1_1H \rightarrow {}^{13}_7N + \gamma$ $\quad$ ${}^{15}_8O \rightarrow {}^{15}_7N + e^+ + \nu_e$

${}^{13}_7N \rightarrow {}^{13}_6C + e^+ + \nu_e$ $\quad$ ${}^{15}_7N + p^+ \rightarrow {}^{12}_6C + {}^4_2He$

${}^{13}_6C + {}^1_1H \rightarrow {}^{14}_7N + \gamma$

${}^{14}_7N + p^+ \rightarrow {}^{15}_8O + \gamma$ $\quad$ total reaction $4p^+ \rightarrow {}^4_2He + 2e^+ + 2\nu_e + 3\gamma$

PP chain : $p^+ + p^+ \rightarrow {}^2_1H + e^+ + \nu_e$, ${}^2_1H + p^+ \rightarrow {}^3_2He + \gamma$, $2\,{}^3_2He \rightarrow {}^4_2He + 2p^+$

Faint Young Sun Paradox

Stars become more luminous as they evolve along/within the main sequence (H-burning)

Start with the Lane-Emden equation for polytropic models

$$\frac{\gamma \cdot K}{r^2}\frac{d}{dr}\left[r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right] = -4\pi G\rho$$

the solution is $\rho(r, \gamma, K)$, where $K = P\cdot\rho^{-\gamma}$ is a constant

during the MS evolution, γ & K remain constant, so the density structure of the star remains essentially the same, i.e., the core density ρ_c is constant

Given $P = \frac{\rho}{\mu m_p}\cdot kT$ & $P = K\cdot\rho^{\gamma}$

we have $kT = \mu m_p \cdot K\cdot\rho^{\gamma-1}$ or $T \propto \mu\cdot\rho^{\gamma-1}$

so the core temperature should increase as $\mu \nearrow$ because $4\,{}^1_1H \rightarrow {}^2_4He + 2e^+ + 2\nu_e$

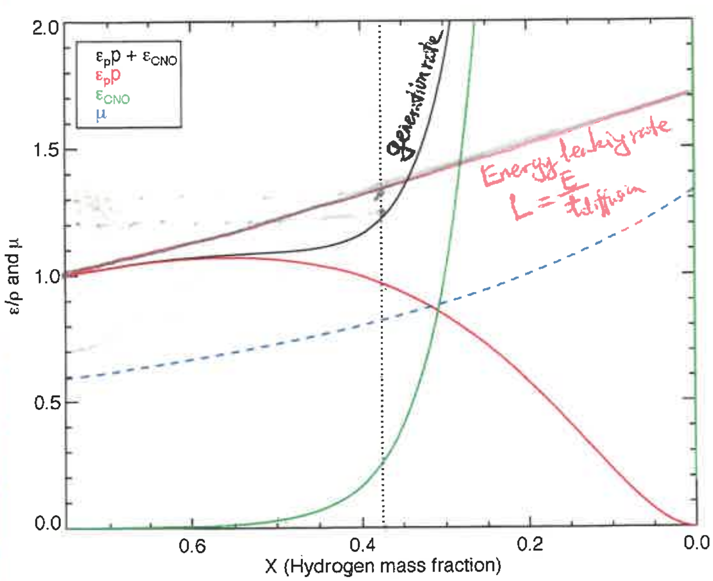
For pp chain, $\epsilon_{pp} \propto \cancel{\rho X^2 T_6^4}\ \rho X^2 \cdot T_6^{-2/3}\, e^{-33.8\,T_6^{-1/3}}$ $W\,kg^{-1}$

$$T_6 = 15\,(\mu/0.8) = T/10^6 K$$ because currently $\mu_\odot = 0.8$ at the core

$$\mu \simeq (2X + \tfrac{3}{4}Y)^{-1} = \frac{4}{5X+3}$$

Plug the above two to ϵ_{pp}, we can calculate ϵ_{pp} as a function of X

CNO cycle becomes important as $X \searrow$ and $T \nearrow$, so we compute ϵ_{CNO} as well



see ms_evolution.pro

since $\frac{dL}{dM} = \epsilon$, if the core mass does not change, the solar luminosity has increased by 20% since it became a MS star 4.5 Gyrs ago

More sophisticated models predicts a change of 30%

Another way to explain the MS evolution is to assume the core T does not evolve, but the star becomes less opaque due to loss of e^- density

Ken Gayley: from raditive energy transport: $\frac{T_c}{R} = \frac{3\rho\kappa L}{64\pi\sigma_{SB}T_c^3R^2} \Rightarrow L \propto \frac{T_c^4\cdot R}{\kappa\rho}$, $\kappa \propto n_e = \frac{\rho}{m_p}(X + \frac{1}{2}Y) = \frac{\rho}{2m_p}(1+X)$

If the Sun was 30% less luminous 4 Ga, what was the surface temperature of the Earth?

Solar input energy: $P_{in} = \pi R_\oplus^2 \cdot S(1-\text{albedo})$, $S = L_\odot / 4\pi d^2$

Earth's radiative energy: $P_{out} = 4\pi R_\oplus^2 \cdot \epsilon\, \sigma_{SB} T_\oplus^4$, ϵ is surface emissivity

Equalibrium condition $P_{in} = P_{out}$

$$\Rightarrow \quad T_\oplus^4 = \frac{S(1-\text{albedo})}{4\epsilon \cdot \sigma_{SB}} = \frac{L_\odot(1-\text{albedo})}{16\pi\epsilon\,\sigma_{SB}\cdot d^2}$$

Green house gas increases the surface temperature by $\left(1+\frac{3\tau}{4}\right)$, τ the optical depth

$$\Rightarrow \quad T_\oplus^4 = \frac{L_\odot(1-\text{albedo})}{16\pi\epsilon\,\sigma_{SB}\cdot d^2}\cdot\left(1+\frac{3\tau}{4}\right)$$

$T_\oplus = 288\ K$ ($15\,^\circ C$) at the present $L_\odot$

$\Rightarrow T_\oplus = 263\ K$ when the Sun had 70% of $L_\odot$ today, which is below freezing

Geological evidence for liquid water 4 Gyrs ago.

(1) oxygen isotopes in Jack Hills zircons (dated 4.4 - 4.3 Ga), suggests parent rock interactions with liquid water

(2) 3.8 Ga old sedimentary rocks from West Greenland suggests large & deep ocean.

Kelvin-Helmholtz timescale of the Earth $\rightarrow$ 20 Myr too short to maintain an ocean

$$\tau = \frac{GM^2}{R\cdot\sigma 4\pi R^2\cdot T^4} \propto \frac{M^2}{R^3 T^4}, \quad \tau_\odot = 10^7\,yr$$

$M_\odot = 2\times10^{30}\,kg$, $R_\odot = 7\times10^8\,m$, $T_\odot = 5800\,K \Rightarrow \tau_\odot = 10^7\,yr$

$M_\oplus = 6\times10^{24}\,kg$ $R_\oplus = 6.4\times10^6\,m$, $T_\oplus = 288\,K \Rightarrow \tau_\oplus = 2\times10^7\,yr$

Ocean freezing timescale $\rightarrow$ 100 years, really short compared to other timescales.

$V = 4\pi R_\oplus^2\cdot h$, $A = 4\pi R_\oplus^2$, $\tau \propto h$ the depth of the ocean

for a bottle of water with diameter of 5 cm, it takes 0.5 hr to freeze:

$V = \frac{\pi D^2}{4}\cdot H$, $A = \pi D\cdot H \Rightarrow \tau \propto \text{Diameter} \simeq 1\ hr/10\,cm$

suppose the ocean is 10 km deep. $\tau = \frac{10\,km}{10\,cm}\,hr \simeq 10^5\,hr \simeq 10\,yrs.$

Virial theorem & L-M relation of stars (Ken Gayley)

virial theorem $K = -\frac{1}{2}U \Rightarrow \frac{M}{\mu m_p} \cdot kT = \frac{GM^2}{R} \Rightarrow kT = \frac{\mu m_p \cdot GM}{R}$

luminosity of a star is determined by radiative diffusion rate

$$L = \frac{E}{t_{diffusion}} = \frac{aT^4 \cdot R^3}{R/(c/\tau)} \quad , \quad \tau = \kappa\rho R = \kappa \frac{M}{R^3} \cdot R$$

$$\Rightarrow L \propto \cancel{T^4 R^4 / M\kappa} \propto \frac{\mu^4 \cdot M^4}{R^4} \cdot \frac{R^4}{M \cdot \kappa} \propto \frac{\mu^4}{\kappa} M^3$$

$$\Rightarrow L \propto M^3 \text{ with no } R \text{ dependency \& } T \text{ dependency}$$

Proving $t_{diffusion} = R/[c/\tau]$, effective speed of light $= c/\tau$

① random walk with mean free path l

$$D = l \cdot \sqrt{N} \quad \Rightarrow \quad N = \left(\frac{R}{l}\right)^2 \text{ is the number of scatting required}$$

② definition of optical depth & mean free path $(\kappa\rho l = 1)$

$$\tau = \kappa\rho \cdot R = \frac{R}{l}$$

$$\Rightarrow t_{diffusion} = \frac{Nl}{c} = \frac{\tau^2 \cdot l}{c} = \tau^2 \cdot \frac{R}{\tau} \cdot \frac{1}{c} = R/[c/\tau] = \frac{\tau \cdot R}{c}$$

$$\Rightarrow L = E/t_{diffusion} \propto \frac{1}{\tau} \propto \frac{1}{\kappa}$$

Alternatively, start from $\frac{dP_{rad}}{d\tau_v} = \frac{1}{c} F_{rad}$, $P_{rad} = \frac{1}{3}u = \frac{4\sigma T^4}{3c}$

$$\Rightarrow P_{rad} = \frac{1}{c}\int F_{rad}\, d\tau_v$$

$$\Rightarrow \frac{dP_{rad}}{dr} = -\frac{\kappa\rho}{c} F_{rad}$$

using the T gradient equation

Lane-Emden Equation Homework

$$\frac{1}{x^2}\frac{1}{dx}\left[x^2\frac{dD_n}{dx}\right] = -D_n^n$$

$$\gamma = \frac{n+1}{n}, \quad D_n = \left(\frac{\rho}{\rho_c}\right)^{\frac{1}{n}}, \quad x = \frac{r}{\lambda_n}, \quad \lambda_n = \left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]^{1/2}$$

$$M = 4\pi\int_0^R r^2\rho\, dr = 4\pi\lambda_n^3\rho_c\int_0^{x_1} x^2 D_n^n\, dx$$

For $n=3$, $\lambda_n \propto \left(\rho_c^{\frac{1-3}{3}}\right)^{1/2} = \rho_c^{-\frac{1}{3}}$ $\Rightarrow$ M is independent of ρ_c

Eddington Standard model

$$P_{gas} = \frac{\rho k T}{\mu m_H} = \beta P \qquad P_{rad} = \frac{1}{3}aT^4 = (1-\beta)P$$

$$P = K\rho^{4/3} \quad \& \quad K = \left[\frac{3(1-\beta)}{a}\right]^{1/3}\left(\frac{k}{\beta\mu m_H}\right)^{4/3}$$

For $R = R_\odot$, $\beta = 0.01$, $\mu = 0.8$, $n=3$

$\Rightarrow$ $\rho_c = 2\times10^{10}\ kg/m^3$, $T_c = 5\times10^{10}\ K$ $\quad M = 7.3\times10^{35}\ kg$

For the Sun, $\beta = 0.99935$ [below from Table 11.1]

$\rho_c = 1.5\times10^5\ kg/m^3$, $T_c = 1.57\times10^7\ K$, $M = 2\times10^{30}\ kg$

Chap 12 : ISM & Star Formation

Closed box system

$$ISM \xrightarrow{\text{Star formation}} Stars$$

stellar evolution

Interstellar Extinction :

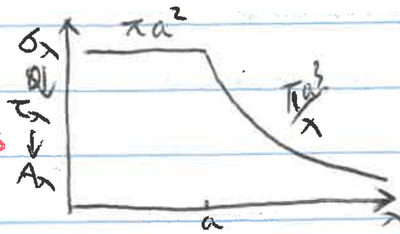
$$m_\lambda = M_\lambda + 5\log d - 5 + A_\lambda \iff m_{\lambda,obs} = m_{\lambda,int} + A_\lambda$$

radiative transfer for pure absorption $\quad F_{\lambda,obs} = F_{\lambda,int}\, e^{-\tau_\lambda}$

given the definition of magnitude $\quad m_\lambda = -2.5\log F_\lambda / F_{\lambda,ref}$

$$A_\lambda = m_{\lambda,obs} - m_{\lambda,int} = -2.5\log F_{\lambda,obs}/F_{\lambda,int} = 2.5\cdot\tau_\lambda\cdot\log e = 1.086\,\tau_\lambda$$

$$\tau_\lambda = \int_0^S n_d(s')\,\sigma_\lambda\, ds' \simeq \underset{\text{Cross section}}{\sigma_\lambda}\ \underset{\text{column density}}{N_d}$$



The Mie ~~Theory~~ Scattering [EM scattering by homogeneous sphere

$\sigma_g = \pi a^2 \rightarrow$ geometric cross section

define extinction coeff $\quad Q_\lambda \equiv \sigma_\lambda/\sigma_g \quad \begin{cases} \text{when } \lambda \gtrsim a, & Q_\lambda \sim a/\lambda \\ \text{when } \lambda \ll a, & Q_\lambda \sim \text{const} \end{cases}$

$\frac{N_H}{A_V} = 1.8\times10^{21}$ atoms/cm^2/mag

$\rho_{silicate} \sim 2\,g/cm^3$

$\Rightarrow$ dust-to-gas ratio in mass

$$\Rightarrow \boxed{\sigma_\lambda \propto a^3/\lambda \quad (\lambda \gtrsim a) \ /\ a^2 \quad (\lambda \ll a)} \Rightarrow A_\lambda \ \&\ \tau_\lambda \propto \frac{1}{\lambda} \text{ when } \lambda \gtrsim a$$

Application:

Estimate the amount of extinction (A_λ) @ ~~0.5~~ 1.0 μm from interstellar dust with $\bar a \sim 0.2\,\mu m$, $\bar n \sim 10^{-13}\,cm^{-3}$, and a column/distance of 1 kpc.

$$\boxed{A_\lambda = 1.086\,\tau_\lambda} = 1.086\cdot\pi a^3/\lambda\cdot\bar n\cdot d = 0.17 \quad @\,0.5\,\mu m$$

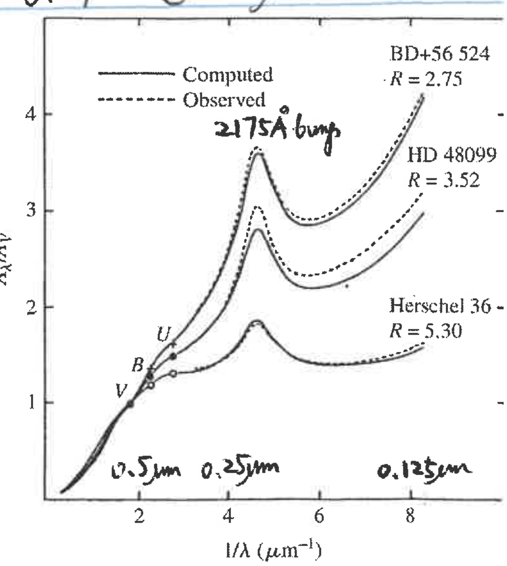
Observed extinction curves:

$E(B-V) = (B-V)_{obs} - (B-V)_{int}$

color excess $\quad E(B-V) = -(B-V)_{intrinsic} + (B-V)_{obs}$

Page 402, wrong definition $\quad = -A_V + A_B = A_B - A_V$

2175Å bump $\rightarrow$ graphite or PAHs



① Thermal reemission

Polarization of scattered light

polarization ⊥ to B

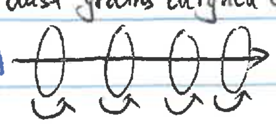
② Scattered light polarization // to B

a few percent polarization & level of Polarization depends on λ, this has two implications.

① non-spherical dust grains

② dust grains aligned along a unique direction by B

see Andersson 15 ARA&A



long axis perpendicular to B direction. ~~polarization angle aligned with B~~ leading to polarized light with E direction parallel to B

HI 21 cm line in emission:

The brightness of an $\boxed{\text{optically thin radio emission line}}$ is proportional to the column density but can be independent of the gas temperature.

$\nu_{10} = \frac{8}{3} g_I \left(\frac{m_e}{m_p}\right) \alpha^2 R_H c \simeq 1.4\,\text{GHz}$

$g_I = 5.6$ nuclear g factor for proton

$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$ fine structure constant.

$A_{10} \simeq 2.85 \times 10^{-15}\,\text{Hz}$

$\Rightarrow \tau = 1/A_{10} \simeq 11\,\text{Myr}$

$n_{crit} \ll 1\,\text{cm}^{-3}$

Start from RTE: $\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$

$$\int_{I_\nu(s=0)}^{I_\nu(s=s)} \frac{dI_\nu}{I_\nu - S_\nu} = \int_{s=0}^{s=s} d\tau_\nu = -\tau_\nu$$

if S_ν is constant of s, we have

$$I_\nu(s=s) = I_\nu(s=0)\, e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

for pure emission-line cloud in LTE: $I_\nu(s=0) = 0$ & $S_\nu = B_\nu(T_b)$ (brightness Temper.)

for optically thin cloud: $1 - e^{-\tau} \simeq [1 - (1 - \tau)] = \tau$

$$\boxed{I_\nu(s) = B_\nu(T) \cdot \tau_\nu}, \quad B_\nu(T_b) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT_b} - 1}$$

Gaussian line profile at line center

$$\phi(\nu_0) = \sqrt{\frac{\ln 2}{\pi}} \frac{2}{\Delta\nu} = \sqrt{\frac{\ln 2}{\pi}} \frac{2c}{\nu_0 \Delta u}$$

$\Delta\nu$ is FWHM in Hz

Δu is FWHM in km/s

$\frac{1}{2}\sqrt{\frac{\pi}{\ln 2}} = 1.064$

Express the opacity in Einstein coefficients:

$$\kappa_\nu \rho = \frac{h\nu_0}{4\pi}(n_l B_{lu} - n_u B_{ul})\phi_\nu, \text{ where } B_{lu} = \frac{g_u}{g_l} B_{ul}, \quad A_{ul} = \frac{2h\nu^3}{c^2} B_{ul}$$

$$= \frac{h\nu_0}{4\pi} n_l B_{lu}\left[1 - \frac{n_u}{n_l}\frac{g_l}{g_u}\right] \cdot \phi_\nu \quad \text{where } \frac{n_u}{n_l} = \frac{g_u}{g_l} \exp\left(-\frac{h\nu}{kT_s}\right)$$

excitation (spin) temper.

$$= \frac{c^2}{8\pi\nu_0^2} \frac{g_u}{g_l} n_l \cdot A_{ul} \cdot \left[1 - \exp\left(-\frac{h\nu}{kT_s}\right)\right] \cdot \phi_\nu$$

For absorption line:

$EW = \int (1 - e^{-\tau_\lambda}) d\lambda \simeq \int \tau_\lambda\, d\lambda$

$\tau_{\lambda_0} = \tau_{\nu_0} \propto \frac{N_l}{kT_s} \frac{1}{\Delta\nu}$

[Eq. 12.7 C&O]

at line center for Gaussian Profile

when $h\nu \ll kT$

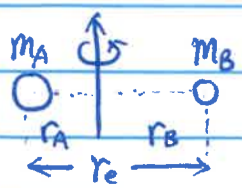
$$B_\nu(T) \simeq \frac{2kT_b\nu^2}{c^2} \qquad \tau_\nu = \kappa_\nu \rho \cdot s \simeq \frac{c^2}{8\pi\nu^2} \cdot \frac{g_u}{g_l} N_l \cdot A_{ul} \cdot \frac{h\nu}{kT} \propto \frac{N_l}{kT_s}$$

$$\Rightarrow I_\nu(s=s) = B_\nu(T_b) \cdot \tau_\nu(T_s)$$

$$= \frac{2\cancel{kT_b}\cancel{\nu^2}}{\cancel{c^2}} \cdot \frac{\cancel{c^2}}{8\pi\cancel{\nu^2}} \frac{g_u}{g_l} N_l \cdot A_{ul} \frac{h\nu}{\cancel{kT_s}}$$

$$= \frac{h\nu}{4\pi} \cdot \frac{g_u}{g_l} A_{ul} \cdot N_l \propto N_l \quad \text{no } T \text{ dependency}$$

Molecular Lines [§7.7 Condon & Ransom]



polar molecule : non-zero permanent electric dipole moment.

$$\vec{p} \equiv \int \vec{x}\, \rho(\vec{x})\, d^3\vec{x} \quad \text{[electric dipole moment]}$$

moment of inertia $\quad I = m_A r_A^2 + m_B r_B^2 = \dfrac{m_A m_B}{m_A + m_B} r_e^2 \equiv \mu r_e^2 m_H$

angular momentum $\quad L = I \cdot \omega = \mu r_e^2 \omega m_H$

rotational kinetic energy: $\quad E = \frac{1}{2} I \omega^2 = L^2/2I$

Quantization of L : $L = \sqrt{J(J+1)} \cdot \hbar \qquad , \; J = 0, 1, 2, \dots$

$$\Rightarrow \qquad E = \left(\frac{\hbar^2}{2I}\right)(J(J+1))$$

$$\Delta E (J \to J-1) = \hbar^2 J / I \quad \Rightarrow \nu(J \to J-1) = \frac{\Delta E}{h} = \frac{\hbar J}{2\pi I} = \frac{hJ/m}{4\pi^2 \mu r_e^2}$$

$$\mu(CO) = \frac{12 \times 16}{12 + 16} = 6.86 \quad \text{reduced mass of CO}$$

① CO ladder $\nu_J \propto J$, $\quad \nu(2 \to 1) = 230.538 \text{ GHz} < 2 \cdot \nu(1 \to 0)$

$\qquad 2 \cdot \nu(1 \to 0) = 230.542 \text{ GHz}$

② Calculate freq. of ^{13}CO

$$\frac{\nu(^{13}CO)}{\nu(^{12}CO)} = \frac{\mu(^{12}CO)}{\mu(^{13}CO)} = \frac{6.86}{7.17} = \frac{1}{1.046}$$

$$\Rightarrow \nu_{1 \to 0}(^{13}CO) = 110.2 \text{ GHz}$$

③ Infer r_e, the equilibrium separation between C & O.

$$r_e = \frac{1}{2\pi} \left(\frac{hJ}{\mu m_H \cdot \nu_J} \right)^{1/2} = 1.13 \times 10^{-8} \text{ cm}$$

for $J = 1$, $\nu_J = 115.27 \text{ GHz}$.

ASTR: 6782 Fall 2017

Chap 3. Star Formation

Jeans Mass: For cores, $T \sim 10K$, $M = 10 M_\odot$, $D \sim 0.1pc$, $n \sim 10^{10} m^{-3}$

$$\rho \sim 2\, m_p \cdot n \sim 3.4 \times 10^{-17}\, kg/m^3$$

Jeans Mass, $M_J = 8\, M_\odot \,(T/10K)^{3/2} \,(10^{10} m^{-3}/n)^{1/2}$

Jeans length, $R_J = 20{,}000\,AU\,(T/10K)^{1/2}\,(10^{10} m^{-3}/n)^{1/2} \sim 0.1\,pc$

Free fall timescale $t_{ff} = \sqrt{\frac{3\pi}{32 G \rho_0}} = 4 \times 10^5\, yr \left(\frac{10^{10} m^{-3}}{n}\right)^{1/2}$

Kelvin-Helmholtz timescale

$$t_{KH} = \frac{GM^2/R}{4\pi R^2 \cdot \sigma_{SB} T^4} \sim 10^7 yr \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-3} \left(\frac{T}{T_\odot}\right)^{-4}$$

Jeans Mass & length Derivation

freefall timescale t_{ff} from K3 law $P^2 = \frac{4\pi^2}{GM} a^3$

$$P = 2\, t_{ff}, \quad a = \frac{1}{2} r_0$$

$$t_{ff} = \sqrt{\frac{3\pi}{32 G \rho_0}} = 4 \times 10^5 yr \sqrt{\frac{10^{10} m^{-3}}{n_0}}$$

Jeans instability criterion: $t_{ff} < t_{sound} = r_0/c_s$ or $2K + U < 0$

$$\left(\frac{3\pi}{32 G \rho_0}\right)^{1/2} < \frac{r_0}{c_s} = \frac{r_0}{(\gamma k T/\mu m_p)^{1/2}} \quad \text{where} \begin{cases} \gamma = 5/3 & \text{for monatomic g.} \\ \gamma = 7/5 & \text{for diatomic ga.} \end{cases}$$

$\mu \equiv \frac{\rho}{n m_p}$

$$\Rightarrow r_0 > r_J \equiv \left(\frac{3\pi \gamma k T}{32 G \rho_0 \mu m_p}\right)^{1/2} = 0.1\,pc \left(\frac{T}{10K}\right)^{1/2} \left(\frac{10^{10} m^{-3}}{n_0}\right)^{1/2} \text{ for } \mu = 2$$

$$\Rightarrow M > M_J \equiv \frac{4}{3}\pi r_J^3 \rho_0 = 8\, M_\odot \left(\frac{T}{10K}\right)^{\frac{3}{2}} \left(\frac{10^{10} m^{-3}}{n_0}\right)^{1/2} \cdot \left(\frac{2}{\mu}\right)^{\frac{3}{2}}$$

Cloud Fragmentation

$$M_J \propto T^{3/2} \rho_0^{-\frac{1}{2}} \propto \begin{cases} \rho_0^{-\frac{1}{2}} & \text{for isothermal condition} \\ \rho^{(3\gamma-4)/2} & \text{for adiabatic condition, since } T \propto \rho^{\gamma-1} \end{cases}$$

$2K = -U$

$\Rightarrow 3 \frac{M_J}{\mu m_p} kT = \frac{3}{5} \frac{G M_J^2}{R_J}$

$\Rightarrow R_J = \frac{G M_J}{5 k T} \cdot \mu m_p$

so there is a minimum Jeans mass at the transition when $P \propto \rho^\gamma$

$$L_{ff} = L_{rad} \quad \text{where } L_{ff} = \frac{\frac{3}{10} GM^2/R}{t_{ff}}, \quad L_{rad} = 4\pi R^2 e \sigma T^4$$

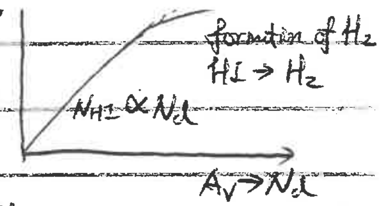
this is equivalent to $t_{ff} = t_{KH} = \frac{\frac{3}{10} GM^2/R}{4\pi R^2 e \sigma T^4}$, along with $2K = -U$

we have $M_{J,min} = 0.5 M_\odot (T/1000K)^{1/4} (0.1/e)^{\frac{1}{2}} (1/\mu)^{9/4}$ $\quad t_{ff} = \sqrt{\frac{3\pi}{32 G \rho_0}}$

Molecular clouds → traced by CO   SF sites

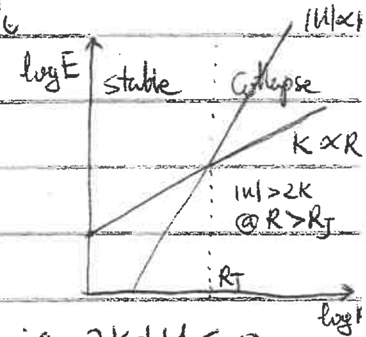
	GMC	Complexes	Clumps	Cores	Hot Cores
T	15K	10K	10K	10K	100–300K
M	$10^5 M_\odot$	$10^4 M_\odot$	$30 M_\odot$	$10 M_\odot$	10–$3000 M_\odot$
d	50pc	10pc	1pc	0.1pc	0.1pc
n	$10^8\,m^{-3}$	5×10^8	10^9	~~10~~ $10^{9}\text{-}10^{10}$	10^{13}–10^{15}

$\rho = 2m_p n$, $m_p = 1.7\times10^{-27}$ kg 3×10^{-18} kg/m^3 ⇒ $M_J \simeq 24 M_\odot$



Formation of Protostars

in comparison, $\rho_{air} = 1.2$ kg/m^3



Jeans mass : condition for collapse $2K < |U| = -U$, i.e. $2K + U < 0$

$$U = -\frac{3}{5}\frac{GM^2}{R}, \quad K = \frac{3}{2}NkT = \frac{3}{2}\frac{M}{\mu m_p}\cdot kT$$

$$2K < -U \Rightarrow \frac{3MkT}{\mu m_p} < \frac{3}{5}\frac{GM^2}{R}, \quad R = \left(\frac{3M}{4\pi\rho_0}\right)^{1/3} \Leftrightarrow M = \frac{4}{3}\pi\rho_0 R^3$$

$$\Rightarrow M > \left(\frac{5kT}{G\mu m_p}\right)^{3/2}\left(\frac{3}{4\pi\rho_0}\right)^{1/2} \equiv M_J \quad \text{or} \quad R > \left(\frac{15kT}{4\pi G\mu m_p\rho_0}\right)^{1/2} \equiv R_J$$

Minimum Jeans mass

$$M_{J,min} = 0.03 M_\odot (T/10K)^{1/4}(0.1/e)^{1/2}\cdot(2/\mu)^{9/4}$$

$$R_{J,min} = \frac{GM_{J,min}}{5kT}\mu m_p = 824\,AU\,(T/10K)^{-3/4}(0.1/e)^{1/2}\cdot(2/\mu)^{5/4}$$

[Neptune at 30AU]

Jeans Mass $M_J = \left(\frac{5kT}{G\mu m_p}\right)^{3/2}\left(\frac{3}{4\pi\rho_0}\right)^{1/2} = \boxed{8\,M_\odot\left(\frac{T/10K}{\mu/2}\right)^{3/2}\cdot\left(\frac{3\times10^{-17}kg/m^3}{\rho_0}\right)^{1/2}}$

$$= \frac{5^{3/2}}{G^{3/2}}\cdot\left(\frac{kT}{\mu m_p}\right)^{4/2}\left(\frac{\mu m_p}{kT}\,\frac{1}{\rho_0}\right)^{1/2}\cdot\left(\frac{3}{4\pi}\right)^{1/2}$$

$$= \frac{5.46\,V_T^4}{G^{3/2}\cdot\rho_0^{1/2}} \quad \text{where } V_T = \sqrt{\frac{kT}{\mu m_p}} \text{ is the isothermal sound speed}$$

Sound speed $V_s = \sqrt{\gamma P/\rho}$. for isothermal gas $P = \frac{\rho}{\mu m_p}kT \propto \rho^1$ so $\gamma = 1$

$= \sqrt{\gamma kT/\mu m_H}$

~~Homogeneous~~ Homologous collapse: (in the absence of pressure supported by rotation, turbulence & B)

free fall motion: $\frac{d^2r}{dt^2} = -G\frac{M_r}{r^2} = -\left(\frac{4\pi}{3}G\rho_0 r_0^3\right)\frac{1}{r^2}$

$$\Rightarrow \quad \frac{dr}{dt}\cdot\frac{d^2r}{dt^2} = -\left(\frac{4\pi}{3}G\rho_0 r_0^3\right)\frac{1}{r^2}\frac{dr}{dt}$$

integrate over dt on both side , $\int\frac{dr}{dt}\frac{d^2r}{dt^2}dt = \frac{1}{2}\int\frac{d(\frac{dr}{dt})^2}{dt}\cdot dt = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + C$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \left(\frac{4\pi}{3}G\rho_0 r_0^3\right)\frac{1}{r} + C_1 ,$$

evaluate @ initial condition $dr/dt = 0$ when $r = r_0$

$$\frac{dr}{dt} = -\left[\frac{8\pi}{3}G\rho_0 r_0^2\left(\frac{r_0}{r} - 1\right)\right]^{1/2}$$

substitute $\theta = r/r_0$ & $\chi = \sqrt{\frac{8\pi}{3}G\rho_0}$ into the above, ($\chi \propto \frac{1}{t}$)

$$\frac{d\theta}{dt} = -\chi\left(\frac{1}{\theta} - 1\right)^{1/2}$$

substitute $\theta = \cos^2\beta = r/r_0 \le 1$ (we can do this because $\theta \le 1$)

$$\frac{d\cos^2\beta}{dt} = -\chi\left(\frac{1-\cos^2\beta}{\cos^2\beta}\right)^{1/2} \Rightarrow 2\cos\beta\sin\beta\frac{d\beta}{dt} = \chi\cdot\frac{\sin\beta}{\cos\beta}$$

Trigonometric identity

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos 2A = 2\cos^2 A - 1$

$\Rightarrow \quad \cos^2\beta\frac{d\beta}{dt} = \frac{\chi}{2}$, now integrate over dt, replace $\cos^2\beta = \frac{1}{2}[1+\cos 2\beta]$

$\frac{\beta}{2} + \frac{1}{4}\sin 2\beta = \frac{\chi}{2}t$. ; note that when $t=0$, $\beta=0$, $\theta=1$, $r=r_0$

when the free fall completes, $r=0$, $\beta = \pi/2$, $t = t_{ff}$

$$\Rightarrow \quad \frac{\pi}{4} = \frac{\chi}{2}t_{ff} \quad \Rightarrow \quad t_{ff} = \frac{\pi}{2\chi} = \left(\frac{3\pi}{32}\,\frac{1}{G\rho_0}\right)^{1/2}$$

$$\boxed{t_{ff} = 3.8\times10^5\,yr\left(\frac{3\times10^{-17}kg/m^3}{\rho_0}\right)^{1/2}}$$

Free Fall timescale & Jeans Length (alternative derivation, RP17.1)

Kepler's 3rd law: $P^2 = \frac{4\pi^2}{GM} a^3$

a free fall particle @ r_0 from the center must follow this law even though $e \simeq 1$

so $t_{ff} = \frac{P}{2}$, $a = \frac{r_0}{2}$, we also know $M = \frac{4}{3}\pi r_0^3 \cdot \rho_0$

$$\Rightarrow 4 t_{ff}^2 = \frac{4\pi^2}{G} \cdot \frac{3}{4\pi r_0^3 \cdot \rho_0} \cdot \frac{r_0^3}{8} \Rightarrow t_{ff} = \sqrt{\frac{3\pi}{32 G\rho_0}} = 4\times10^4\,\mathrm{yr}\left(\frac{\rho_0}{3e\text{-}15\,\mathrm{kg}\ldots}\right.$$

$\rho_0 = 3\times10^{-15}\,\mathrm{kg/m^3} \Leftrightarrow n_{H_2} = \frac{\rho}{2m_p} = 10^{12}\,\mathrm{m^{-3}}$ [dense cores]

Jeans criterion for a cloud that is $\boxed{\text{stable}}$ against perturbations:

$\boxed{t_{ff} > t_{sound}}$ free-fall time longer than sound travel time

$$\left(\frac{3\pi}{32 G\rho_0}\right)^{1/2} > \frac{r_0}{c_s} = \frac{r_0}{(\gamma kT/\mu m_p)^{1/2}}, \quad \gamma \text{ is the adiabatic index}$$

$$\Rightarrow r_0 < r_J = \left(\frac{3\pi\gamma kT}{32 G\rho_0 \mu m_p}\right)^{1/2} = 2000\,\mathrm{AU}\left(\frac{T}{10\,\mathrm{K}}\right)^{1/2}\left(\frac{\rho_0}{3\times10^{-15}\,\mathrm{kg\,m^{-3}}}\right)^{-1/2}$$

Jeans mass is simply the mass enclosed within r_J

$$\Rightarrow M < M_J = \frac{4}{3}\pi r_J^3 \cdot \rho_0 = 0.2\,M_\odot\left(\frac{T}{10\,\mathrm{K}}\right)^{3/2}\left(\frac{\rho_0}{3\times10^{-15}\,\mathrm{kg\,m^{-3}}}\right)^{-1/2}$$

After cloud fragmentation (see next page), talk about the formation of disks

Collapse stopped by rotation $\longrightarrow$ Protoplanetary Disk

Conservation of angular momentum $v_0 r_0 = v_f r_f$

the cloud would stop falling when it forms a rotationally supported disk

the disk would have a radius of r_f & velocity v_f, & gravity balanced by rotatio[n]

$$\frac{GM}{r_f^2} = \frac{v_f^2}{r_f} \Rightarrow r_f = \frac{v_f^2 r_f^2}{GM} = \frac{v_0^2 r_0^2}{GM}$$

$$\Rightarrow r_f \simeq 200\,\mathrm{AU}\left(\frac{v_0}{0.1\,\mathrm{km/s}}\right)^2\left(\frac{r_0}{4000\,\mathrm{AU}}\right)^2\left(\frac{M}{1\,M_\odot}\right)^{-1}$$

How to reduce angular momentum?

① viscous torques from outer disk ② magnetized protostellar wind

Cloud fragmentation. $M_J \propto T^{3/2} \rho_0^{-\frac{1}{2}}$ so M_J decreases as ρ_0 increases if T is constant

This causes a cascading collapse ~~of~~ leading to formation of large # of smaller objects if T remains constant, i.e., $\boxed{\text{isothermal}}$ collapse.

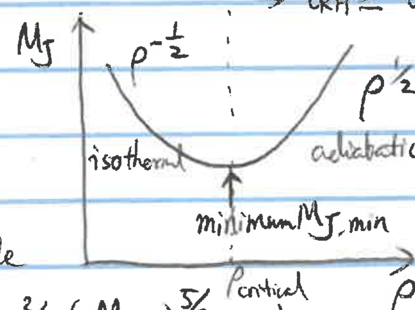
But if the collapse is $\boxed{\text{adiabatic}}$ $T = K' \cdot \rho^{\gamma-1}$, then $M_J \propto \rho^{(3\gamma-4)/2}$

γ: heat capacity ratio $\gamma = C_p/C_v = 1 + \frac{2}{\text{dof}}$

for monatomic gas, $\gamma = 5/3$, $M_J \propto \rho^{1/2}$

for diatomic gas, $\gamma = 7/5$, $M_J \propto \rho^{0.1}$

$\rightarrow t_{KH} \geq t_{ff}$



$\rho_{critical} \propto T^{\frac{8}{3}}$

Now estimate the minimum Jeans mass

to remain isotherml, the energy generated due to gravity must be released within the ff time scale

$M_J = \frac{4}{3}\pi R_J^3 \rho_0$

$$L_{ff} = \frac{\Delta(U_g + K)}{t_{ff}} = \frac{\frac{3}{10} G \frac{M_J^2}{R_J}}{\left(\frac{3\pi}{32} \frac{1}{G\rho_0}\right)^{1/2}} \sim G^{3/2} \left(\frac{M_J}{R_J}\right)^{5/2}$$

but the proto star can only radiate away energy through its surface

$$L_{rad} = 4\pi R_J^2 e \sigma T^4 \quad \text{, where} \quad 0 < e < 1 \quad \text{(radiative efficiency)}$$

Heating $\geq$ Cooling

the transition from isotherml to adiabatic collapse occures when

$\boxed{t_{ff} \leq t_{KH}} \Longleftrightarrow \boxed{L_{rad} \leq L_{ff}}$, radiation becomes insufficient to remove gravitational energy

or $M_J^{5/2} \geq \frac{4\pi}{G^{3/2}} R_J^{9/2} e \sigma T^4$ → now we need to elemnate R_J without introducing ρ

Since $\boxed{2 \cdot \frac{3M_J}{2\mu m_p} \cdot kT = \frac{3}{5} \frac{GM_J^2}{R_J}}$ $[2K = -U]$ virial theorem

we have $R_J = \frac{GM_J}{5kT} \mu m_p$, plug this into the above, solve for M_J

$$M_J \geq 0.03 M_\odot \left(\frac{T^{1/4}}{e^{1/2} \mu^{9/4}}\right) = M_{J,min}$$

e.g. $M_{J,min} = \boxed{0.5 M_\odot} (T/1000K)^{1/4} (0.1/e)^{1/2} \cdot (1/\mu)^{9/4}$

$R_{J,min} = \frac{\mu m_p}{5kT} \cdot G M_{J,min} = \boxed{10\ AU} (T/1000K)^{-3/4} \cdot (0.1/e)^{1/2} (1/\mu)^{5/4}$

Estimating MW's SFR. For a core with mass above M_J, the SFR is simply M_J/t_{ff}

Jeans SFR: $SFR_{Jeans} = \frac{8M_\odot}{4\times10^5 yr} \left(\frac{T/10K}{\mu/2}\right)^{3/2} = 2\times10^{-5} M_\odot/yr$ for a single core with $8M_\odot$ mass

Heyer & Dame 15 → total mass in H_2 is $(1 \pm 0.3)\times10^9 M_\odot$, if all within cores, $SFR = \frac{10^9}{8} \times 2\times10^{-5} = 2\times10^3 M_\odot/yr$

Current $SFR_{observed} \sim 1\ M_\odot/yr \ll SFR_{estimated}$

How to resolve this apparent contradiction?

Explaining MW & all local galaxies' low SF efficiency

$$SFR_{MW} = \frac{\epsilon \cdot M_{H_2}}{M_{Jeans}} \cdot \frac{M_{Jeans}}{t_{SF}} = \frac{\epsilon\, M_{H_2}}{t_{SF}}$$

① the freefall timescale $t_{ff} = 4 \times 10^5\, yr \cdot \sqrt{\frac{10^{10}\, m^{-3}}{n}}$ is too short to describe the SF process, instead, SF proceeds at almost the KH timescale, $t_{KH} \sim 10^7 yr$

② the fraction of H_2 that is actively forming stars is $\epsilon \sim 10^{-2} = 1\%$

99% of H_2 mass are in structures that are stable againsts gravitational collapse

This doesn't make sense when the Jeans mass is only $140\, M_\odot$ for a GMC with $10^5\, M_\odot$

③ requires turbulence support to increase Jeans mass

Kinetic energy $\frac{1}{2} \mu m_p v^2 = \frac{3}{2} k T_{kinetic} \Rightarrow T_K = 10\,K \cdot \left(\frac{v}{0.3\,km/s}\right)^2$

sound speed $c_s = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma k T}{\mu m_p}} = 540\, m/s \left(\frac{\gamma}{7/5}\right)^{1/2} \left(\frac{2}{\mu}\right)^{1/2} \left(\frac{T}{50K}\right)^{1/2}$

supersonic turbulence can increase T significantly, causing the Jeans mass increase

$$M_J = 8\, M_\odot \left(\frac{T}{10K}\right)^{3/2} \cdot \left(\frac{10^{10} m^{-3}}{n}\right)^{1/2}$$

$$= 10^4 M_\odot \left(\frac{\sigma_T}{3\,km/s}\right)^3 \cdot \left(\frac{10^{10} m^{-3}}{n}\right)^{1/2}$$ for a turbulence v dispersion of 3km/s

④ turbulence increases in larger clouds (Larson's Law [1981])

$\sigma = 3\, km/s \left(\frac{R}{20pc}\right)^{0.4}$, $\sigma \sim c_s(50K)\, 0.5\, km/s$ @ 0.23 pc

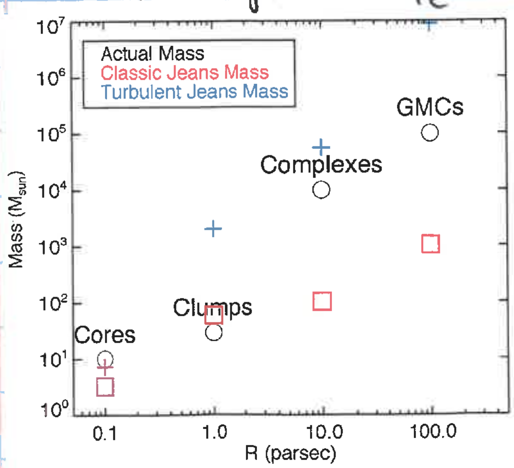
$\sigma \sim c_s(10K) \sim 0.24\, km/s$ @ 0.04 pc

comparable to core size ~ 0.1 pc

kinetic temperature $T_k = \frac{\mu m_p \sigma^2}{3k}$

excitation temperature $T_e = \frac{\mu m_p}{\gamma k} c_s^2$

} so when $\sigma \sim c_s$, $T_k \sim T_e \sim T$ and Jeans mass' original definition becomes valid



Kelvin-Helmholtz timescale

$$t_{KH} \sim 10^7 yr \cdot \left(\frac{M}{M_\odot}\right)^2 \cdot \left(\frac{R}{R_\odot}\right)^{-3} \cdot \left(\frac{T}{T_\odot}\right)^{-4}$$

$R_\odot = \frac{1}{215} AU$, $T_\odot = 5800K$

End of Semester Review Session

Understanding of Solar System

① Celestial Sphere ② Celestial Mechanics.

Understanding Stellar ~~Spectra~~ Atmosphere via spectroscopy

① Light ~~has~~ as a tool / Nature of light ② Interaction of Light & Matter ③ Stellar Spectra ④ Rad. Transfer

Understanding Stellar Structures & MS evolution

① Stellar Interior. ② MS evolution (Chap 13 part 1)

Understanding the origin of stars

① ISM (focus on cold) ② Gravitational collapse.

Key Equations.

$LST = H(M) = RA(\text{meridian})$, $r = \dfrac{a(1-e^2)}{1+e\cos\varphi}$, $\mu = \dfrac{m_1 m_2}{m_1+m_2}$, $M = m_1+m_2$

$\mathcal{L}(r,\dot{r},\varphi,\dot{\varphi}) = \frac{1}{2}[\dot{r}^2+(r\dot{\varphi})^2] - \Phi(r)$, energy per unit mass

$P^2 = \dfrac{4\pi^2}{GM}a^3$, $v^2 = GM\left(\dfrac{2}{r}-\dfrac{1}{a}\right) = \dfrac{GM}{a}\left(\dfrac{2a}{r}-1\right)$

$\vec{g} = -\vec{\nabla}\Phi$ $\quad 2\langle K\rangle + \langle U\rangle = 0$

$m = -2.5\log(F_x/F_{x,\text{ref}})$ $\quad m - M = 5\log(d/pc) - 5$

$I_\nu = \dfrac{dE}{dt\,dA\cos\theta\,d\Omega\,d\nu}$ $\quad \langle I_\nu\rangle = \dfrac{1}{4\pi}\int I_\nu\,d\Omega$, $F_\nu = \int I_\nu\cos\theta\,d\Omega$

$P_{rad,\nu} = \dfrac{1}{c}\int I_\nu\cos^2\theta\,d\Omega$, $u_\nu = \dfrac{1}{c}\int I_\nu\,d\Omega$

$P \sim \dfrac{dp/dt}{dA} = \dfrac{dE}{c\cdot dA\,dt}$ $\quad u \sim \dfrac{dE}{dA\,dL} = \dfrac{dE}{dA\cdot c\,dt}$

$\dfrac{n_2}{n_1} = \dfrac{g_2}{g_1}e^{-\frac{E_2-E_1}{kT_x}} \longrightarrow B_\nu(T) = \dfrac{2h\nu^3}{c^2}\dfrac{1}{e^{h\nu/kT}-1}$

Thermodynamical Equilibrium

$E_n = -13.6\,\mathrm{eV} \cdot \frac{1}{n^2}$, Kirchhoff's Laws (1860)

$q_{ij} = \int_0^\infty u\,\sigma_{ij} f(u)\,du$, $f(u) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi u^2 \exp\left(-\frac{mu^2}{2kT}\right)$

$n_{crit} \equiv \frac{A_{21}}{q_{21}}$ when $n < n_{crit}$; $\frac{n_2}{n_1} \ll \frac{g_2}{g_1} e^{-\Delta E/kT}$

collision-dominated regime.

when $n > n_{crit}$ $\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-\Delta E/kT}$.

Natural: $\Delta t\,\Delta E \gtrsim \hbar$, $\Delta t \sim A^{-1}$, $\gamma_n = \sum_{n'<n} A_{nn'} \sim 10^8\,\mathrm{Hz}$ for permitted lines

Thermal $\frac{\nu-\nu_0}{\nu_0} \simeq -\frac{u}{c}$, $\frac{\lambda-\lambda_0}{\lambda_0} \simeq \frac{u}{c}$, thermal broadening $\sigma_z = \sqrt{\frac{kT}{m}}$

Pressure ~~Re~~ $\Gamma = \gamma_n + 2N_{col}$

Zeeman splitting $\nu = \nu_0 + [-1, 0, 1]\frac{eB}{4\pi\mu}$, $\Delta\nu = 2.8\,\mathrm{Hz}\,\frac{B}{0.2\,nT}$

$B = 5e{-5}\,T$

n, l, m_l, m_s

L, S, J

Saha Eq. $\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{Z_i}\frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT}$

$Z_i = \sum_{j=1}^{\infty} g_i^j \exp\left(-\frac{E_i^j - E_i^1}{kT}\right) \simeq g_i^1$

Rad. Transfer: $dI = -\kappa\rho I\,ds + j\rho\,ds$

$\frac{dI}{d\tau} = I - S$, $\tau = \int_0^s \kappa\rho\,ds$, $d\tau = -\kappa\rho\,ds$

B-B transition: $\kappa\rho = \frac{h\nu}{4\pi}(n_l B_{lu} - n_u B_{ul})$. LTE: $S_\nu = B_\nu(T_B)$

$j\rho = \frac{h\nu}{4\pi} A_{ul}\, n_u$ Vertical τ: $\tau = \tau_V \cdot \sec\theta$

Eddington Approximations plane-parallel

$\langle I\rangle = \frac{1}{2}(I_{out} + I_{in})$, $F = \pi(I_{out} - I_{in})$

$P_{rad} = \frac{2\pi}{3c}(I_{out} + I_{in}) = \frac{4\pi}{3c}\langle I\rangle$

$$\sigma_{12} = \frac{e^2}{4\varepsilon_0 m_e c} f_{12}\,\phi_\nu \quad , \int_0^\infty \phi_\nu\, d\nu = 1.0$$

$$B_{12} = \frac{4\pi}{h\nu}\sigma_{12} \quad , \quad \tau_\nu = \sigma_\nu \cdot N \propto f N/\nu_0 (\phi_\nu \nu_0)$$

$$EW = \int \frac{F_{\lambda cont} - F_\lambda}{F_{\lambda cont}}\, d\lambda = \int (1 - e^{-\tau_\lambda})\, d\lambda$$

Curve of growth. $W \propto N,\ \sqrt{\ln N},\ \sqrt{N}$

What is thermodynamical equilibrium?

$$T_x = T_b = T_k.$$

Radiation Pressure

intrinsic $P_{rad}^{int} = \frac{1}{c}\int I_\nu \cos^2\theta\, d\Omega \overset{\text{isotropic BB}}{=} \frac{4\pi}{3c}\int B_\nu(T)\, d\nu = \frac{4\sigma_{SB}T^4}{3c}$

external $P_{rad}^{ex} = \frac{1}{c}\int\left(\int F_{rad,\nu}\, d\tau_\nu\right) d\nu$

in both cases $\frac{dP_{rad}}{d\tau_\nu} = \frac{F_{rad}}{c}$

existence of gas $\rightarrow \frac{dP_{rad}}{d\tau} \rightarrow F_{rad} \Rightarrow P_{rad}^{ext}$

HSE. $\frac{dP}{dr} = -\frac{GM}{r^2}\rho = -\rho g$

$$P = K\rho^\gamma \quad , \quad \frac{dP}{dr} = -\rho g \quad , \quad \frac{dM}{dr} = 4\pi r^2 \rho$$

$\Rightarrow$ Lane-Emden Eq.

condition for convection to occur $\left|\frac{dT}{dr}\right|_{act} > \left|\frac{dT}{dr}\right|_{adiabatic}$

Binding energy per nucleon $\frac{E_b}{A} = \frac{\Delta m c^2}{A} = \frac{Z m_p + (A-Z) m_n - m_{nucleus}}{A} \cdot c^2$

Radiative T gradient

$$\frac{dT}{dr} = -\frac{3}{16\sigma_{SB}} \cdot \frac{\kappa\rho}{T^3}\, \frac{L(r)}{4\pi r^2} \Rightarrow \frac{T_c}{R} = \frac{3\kappa\rho L}{64\pi\sigma T_c^3 \cdot R^2} \Rightarrow L \propto \frac{T_c^4 R}{\kappa}$$

$$\kappa \propto n_e = \frac{\rho}{m_p}\left(X + \frac{1}{2}Y\right) = \frac{\rho}{2m_p}(1+X) \quad , \quad X + Y = 1$$

$A_\lambda = 2.5\,\tau_\lambda \log e = 1.086\,\tau_\lambda$

Mie scattering $\quad \sigma_\lambda \approx a^3/\lambda \quad$ for $\lambda \gtrsim a$

$\tau_\lambda = \sigma_\lambda \cdot N$

HI 21 cm line, optically thin emission

$$I_\nu(s=s) = I_\nu(s=0)\cdot e^{-\tau_\lambda} + S_\nu(1-e^{-\tau_\nu}) \qquad h\nu \ll kT$$

$$\simeq 0 + S_\nu \cdot \tau_\nu = B_\nu(T)\,\tau_\nu \;\cancel{\propto N}$$

absorption

$$\propto \frac{2\cancel{kT}\nu^2}{c^2}\cdot\frac{N}{\cancel{kT}} \propto N$$

$$EW = \int(1-e^{-\tau})\,d\lambda \simeq \int \tau_\lambda\, d\lambda$$

$$\tau_{\lambda_0} = \tau_{\nu_0} \propto \frac{N}{kT}\cdot\frac{1}{\Delta\nu}$$

~~eq~~ Jeans mass

$$M_J \propto T^{3/2}\, n^{-\frac{1}{2}} \quad , t_{ff} \propto n^{-\frac{1}{2}} \sim 4\times10^5\,\text{yr}$$

$$\tau_{KH} = \frac{\frac{3}{10}GM^2/R}{4\pi R^2 \sigma_{SB} T^4} \propto M^2 R^{-3} T^{-4} \quad \sim 10^7\,\text{yr}$$

$$M_{J,min} \propto T^{1/4}\, e^{-\frac{1}{2}}\, \mu^{-9/4}$$

$$SFR_{MW} = \frac{\epsilon \cdot M_{H_2}}{M_J}\cdot\frac{M_J}{t_{SF}} = \frac{\epsilon\, M_{H_2}}{\tau_{KH}}$$