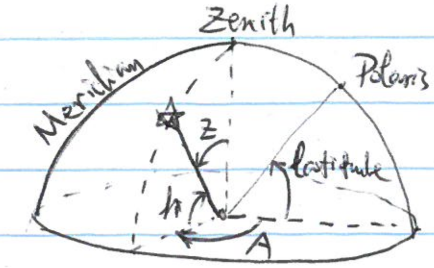
Chap 1. Celestial Sphere

Readings: CO §1.3 Homework: Python exercise & Visibility Plot

Alt-Az System:
Altitude (h)
Zenith distance (z)
Azimuth (A)



Equatorial System
Declination (δ) $\rightarrow$ degrees N/S of celestial equator
Right Ascension (α) $\rightarrow$ degrees from vernal equinox (Υ) upsilon to the hour circle

Time
Solar vs Sidereal
Local Sidereal Time (LST) $\equiv$ hour angle of Υ (H)
Hour Angle (H): angle to the observer's meridian, similar to α
Solar Day: average interval between meridian crossing of the Sun
Sidereal Day: interval between consecutive meridian crossing of a star
$LST = H(\Upsilon) = RA(\text{meridian})$

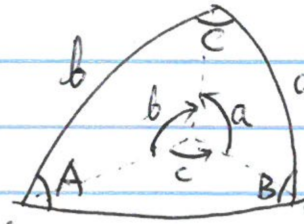
Precession
Period = 25,770 years
J2000 = noon on Jan 1, 2000 in Greenwich, England (UT)
$\Delta\alpha = M + N \sin\alpha \tan\delta$
$\Delta\delta = N \cos\alpha$
M & N are 3rd order polynomials of
$T = (t - 2000.0)/100$
$= (JD - 2451545)/36525$

Julian Date
zero time for JD: Jan 1, 4713 BC, noon UT
J2000.0 = JD 2451545.0 , 1 year = 365.25 solar days
MJD = JD − 2400000.5 so that it begins at midnight.

Proper Motion
$\mu \equiv \frac{d\theta}{dt} = \frac{v_\theta}{r}$. transverse velocity divided by distance

Spherical Trigonometry

Law of sines $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$

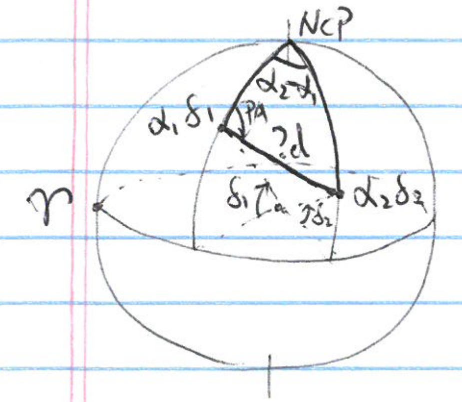


Law of cosines

$$\cos a = \cos b \cos c + \sin b \sin c \cdot \cos A$$

$$\cos A = -\cos B \cdot \cos C + \sin B \cdot \sin C \cdot \cos a$$

Application – distance ~~batween~~ between two coordinates & PA



d is the angular distance

$$\cos d = \cos(90^\circ-\delta_1)\cdot\cos(90^\circ-\delta_2) + \sin(90^\circ-\delta_1)\sin(90^\circ-\delta_2)\cdot\cos(\alpha_2-\alpha_1)$$

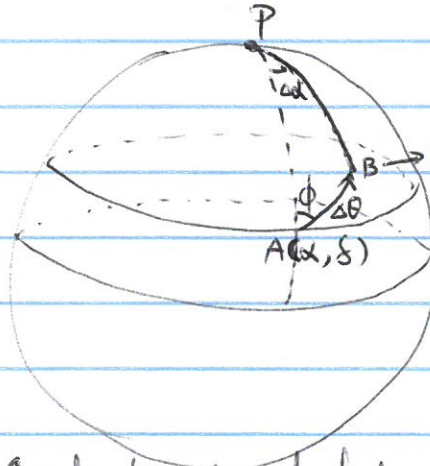
Position angle of (α_2,δ_2) relative to (α_1,δ_1), $PA=\phi$

$$\frac{\sin(90^\circ-\delta_2)}{\sin PA} = \frac{\sin d}{\sin(\alpha_2-\alpha_1)}$$

Application to proper motion → small angle approximations

$$\Delta\alpha = \Delta\theta\cdot\frac{\sin\phi}{\cos\delta}, \quad \Delta\delta = \Delta\theta\cos\phi$$

$$\Rightarrow (\Delta\theta)^2 = (\Delta\alpha\cdot\cos\delta)^2 + (\Delta\delta)^2$$

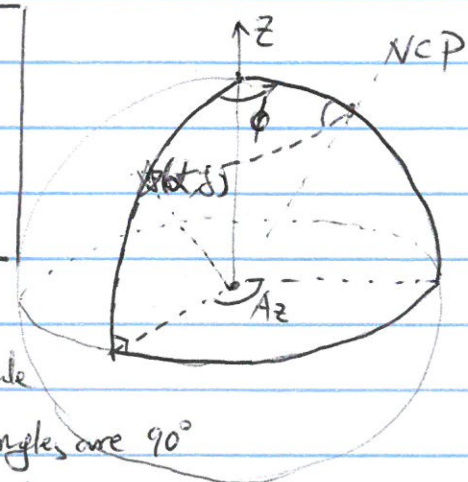


$B \to (\alpha+\Delta\alpha, \delta+\Delta\delta)$

Problem: from $\Delta\theta, \phi, \alpha, \delta$, solve for $\Delta\alpha$ and $\Delta\delta$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$



Application to calculate Alt & Azimuth

coordinates of Zenith $\alpha = LST$, δ = latitude

$Az = PA = \phi$ because the other two angles are 90°

so this problem is the same as the first application

Chap 2 Celestial Mechanics

Tycho Brahe 1546-1601, positional accuracy $\lesssim 4'$

SN1572

Kepler 1571-1630 analyzed Tycho's data of planets

1609: 1st & 2nd laws, elliptical orbits & equal area law

1629: 3rd law $P^2 = a^3$

Geometry of Elliptical orbits

$\boxed{\text{Worksheet Problems}}$

$b^2 = a^2(1-e^2)$

Ellipse: $e<1$

$r = \dfrac{a(1-e^2)}{1+e\cos\theta}$ $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

parabola $(e=1)$

$r = \dfrac{2p}{1+\cos\theta}$

hyperbola $e>1$

$r = \dfrac{a(e^2-1)}{1+e\cos\theta}$

Newtonian Mechanics

Astronomy started to become a physical science

Galileo (1564-1642)

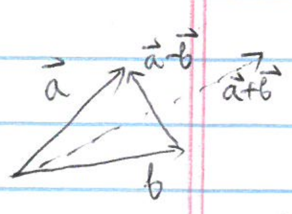
① father of modern observational astronomy — telescope

② varying phases of Venus, moons of Jupiter — supporting Copernican

③ g is constant for bodies of different weights

④ under house arrest by the Roman church from 1632 to 1642.

⑤ 1992, Pope John Paul II officially appologized



Newton (1642-1727)

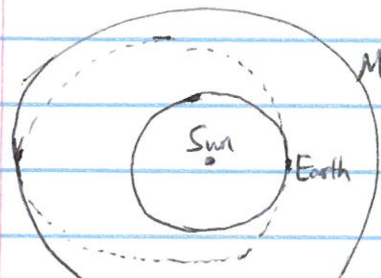
Law 1: $\vec{p} = m\vec{v} = \text{const}$. Law 2: $\vec{F} = \dfrac{d\vec{p}}{dt}$

Law 3: $\vec{F}_{12} = -\vec{F}_{21}$ Gravity: $\vec{F} = -G\dfrac{Mm}{r^2}\vec{e}_r$

Law of Gravity derived from Kepler's 3rd law and Newton's 3 laws of motion. Key assuption: $F_{centripetal} = F_G$

Transfer Orbit

The Hohmann transfer orbit is an ellipse whose perihelion is at the orbit of the inner circular orbit and whose aphelion is at the outer circular orb



$$a_{transfer} = (a_{low} + a_{high})/2$$

$$t_{transfer} = \frac{1}{2} P(a_{transfer}) = \frac{1}{2} a_{transfer}^{3/2}$$

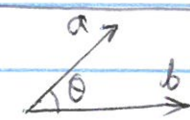
$$V_{peri} = \frac{2\pi a_{transfer}}{P_{transfer}} \left(\frac{2 a_{transfer}}{a_{low}} - 1 \right)^{1/2}$$

$$V_{ap} = \frac{2\pi a_{transfer}}{P_{transfer}} \left(\frac{2 a_{transfer}}{a_{high}} - 1 \right)^{1/2}$$

The last two equations used the vis viva equation

$$\frac{1}{2} v^2 - \frac{GM}{r} = \frac{GM}{2a} \Rightarrow v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

and Keplers 3rd law: $P^2 = \frac{4\pi^2}{GM} a^3$



$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cdot \cos\theta$$
$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \cdot \sin\theta$$

Work & Energy

$$U_f - U_i = \Delta U = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

for gravitational field for a Mass at the origin,

$$\Delta U = -\int \frac{GMm}{r^2}(-\hat{e}_r) \cdot d\vec{r} = \int \frac{GMm}{r^2} dr$$

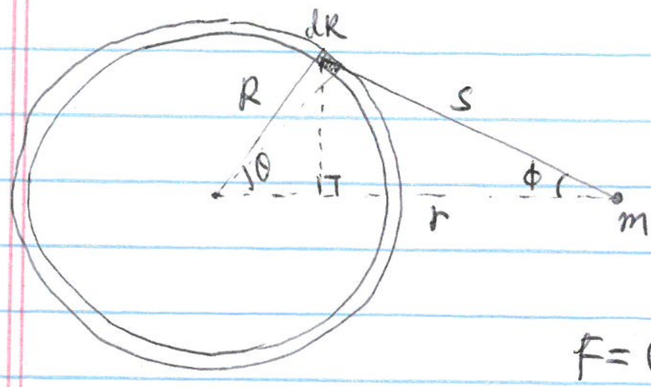
$$U_f - U_i = -GMm\left(\frac{1}{r_f} - \frac{1}{r_i}\right) \Rightarrow U = -G\frac{Mm}{r} \rightarrow 0 \text{ @ infinity}$$

also, $\vec{F}_g = -\nabla U$

$K = \frac{1}{2} m v^2$ Kinetic Energy

Gravity from a spherically symmetric object

Textbook derivation – direct integration



$$dV = R\cdot d\theta\cdot dR\cdot 2\pi R\cdot\sin\theta$$

$$dM = \rho\cdot dV = 2\pi R^2\sin\theta\,\rho(R)\,dR\,d\theta$$

$$s^2 = R^2 + r^2 - 2r\cdot R\cdot\cos\theta$$

$$\cos\phi = \frac{r - R\cdot\cos\theta}{s}$$

$$F = Gm\int_0^{R_0}\int_0^{\pi}\frac{2\pi R^2\sin\theta\,\rho(R)\,dR\,d\theta}{s^2}\cdot\cos\phi$$

trick is to replace variables

$$u \equiv s^2 = r^2 + R^2 - 2rR\cos\theta \;\Rightarrow\; \cos\theta = \frac{r^2+R^2-u}{2rR}$$

$$-d\cos\theta = \sin\theta\,d\theta = \frac{du}{2rR}$$ since both r & R are independent of θ

$$F = 2\pi Gm\int_0^{R_0} R^2\rho(R)\,dR\,\boxed{\int_0^{\pi}\frac{(r-R\cdot\cos\theta)\cdot\sin\theta\,d\theta}{s^3}}$$. box must equal to $\frac{2}{r^2}$

$$\text{box} = \int\frac{r-\frac{r^2+R^2-u}{2r}}{u^{3/2}}\cdot\frac{du}{2rR} = \int\frac{r^2-R^2+u}{4r^2R\,u^{3/2}}\,du$$

$$= \frac{1}{4r^2R}\left[\int(r^2-R^2)\cdot u^{-\frac{3}{2}}\,du + \int u^{-\frac{1}{2}}\,du\right]_{u_0}^{u_1}$$

$$= \frac{r^2-R^2}{4r^2R}\left[-2u^{-\frac{1}{2}}\Big|_{u_0}^{u_1}\right] + \frac{1}{4r^2R}\,2\cdot u^{\frac{1}{2}}\Big|_{u_0}^{u_1}$$

$$u_0 = r^2+R^2-2rR \quad , \quad u_1 = r^2+R^2+2rR = (r+R)^2$$

$$\text{box} = \frac{1}{4r^2R}\left[(r^2-R^2)(-2)\cdot\left(\frac{1}{r+R}-\frac{1}{r-R}\right)\right] + \frac{1}{4r^2R}\cdot 2\left[r+R-(r-R)\right]$$

$$= \frac{1}{2Rr^2}\Big\{\left[r+R-(r-R)\right] + \left[r+R-(r-R)\right]\Big\} = \frac{4R}{2Rr^2} = \frac{2}{r^2}$$

Gravity from a spherically symmetric object with a density profile $\rho(R)$

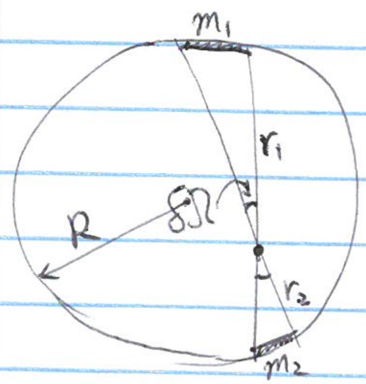
$$F = \frac{Gm}{r^2}\int_0^{R_0} 4\pi R^2 \rho(R)\, dR = G\frac{Mm}{r^2}$$

Proof of this, Newton's second theorem is easy with Potential Theory

See Sec 2.2 of Binney & Tremaine

Theorem 1: object inside a spherical shell experiences no net gravity

Theorem 2: gravity from a spherical shell to an object outside of the shell is the same as if the shell's matter were in its center.

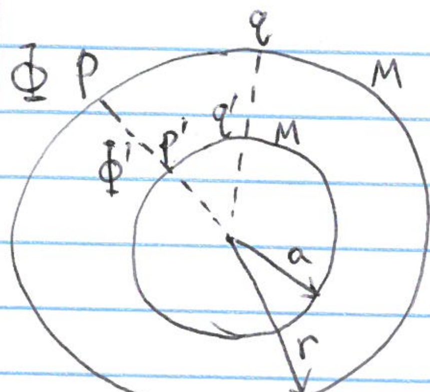


$$m_1 = r_1^2\,\delta\Omega\,\rho \quad , \quad m_2 = r_2^2\,\delta\Omega\,\rho$$

$$\left.\begin{aligned} F_1 &= G\frac{m_1\,\delta m}{r_1^2} = G\,\delta\Omega\,\rho\,\delta m \\ F_2 &= G\frac{m_2\,\delta m}{r_2^2} = G\,\delta\Omega\,\rho\,\delta m \end{aligned}\right\} \text{cancels out}$$

$$\Rightarrow \Phi = -GM/R = \text{const inside shell}$$

Here we consider two cases, one outside of a shell, the other inside a shell.



$$\Phi(p) = \int -\frac{GM}{|\vec{p}-\vec{q}'|}\frac{\delta\Omega}{4\pi}$$

1st theorem

$$\Phi(p') = \int -\frac{GM}{|\vec{p}'-\vec{q}|}\frac{\delta\Omega}{4\pi} = -\frac{GM}{r}$$

$\Phi(p) = \Phi(p')$ because $|\vec{p}-\vec{q}'| = |\vec{p}'-\vec{q}|$

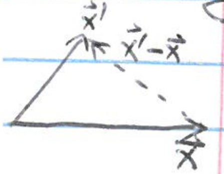
therefore $\Phi(p) = -\frac{GM}{r}$

$$\vec{F}(r) = -\nabla\Phi = -\frac{GM}{r^2}\vec{e}_r$$

$$\vec{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\vec{F}(r) = -\vec{e}_r\cdot\frac{d}{dr}\left(-\frac{GM}{r}\right) = -\vec{e}_r\,\frac{GM}{r^2}$$

Potential Theory



$$\left.\begin{aligned} \vec{g}(\vec{x}) &= G\int d^3\vec{x}'\,\frac{\vec{x}'-\vec{x}}{|\vec{x}'-\vec{x}|^3}\,\rho(\vec{x}') \\ \Phi(\vec{x}) &\equiv -G\int d^3\vec{x}'\,\frac{\rho(\vec{x}')}{|\vec{x}'-\vec{x}|} \end{aligned}\right\} \Rightarrow \vec{g}(\vec{x}) = -\nabla\Phi$$

Virial Theorem : $2\langle K\rangle + \langle U\rangle = 0$

Proof: define $Q = \sum_{i=1}^{N} \vec{p}_i \cdot \vec{r}_i$ (recall that $\vec{L} = \cancel{\vec{p}\times}\ \vec{r}\times\vec{p}$)

$$\frac{dQ}{dt} = \sum_{i=1}^{N}\left(m_i \frac{d\vec{r}_i}{dt}\cdot\frac{d\vec{r}_i}{dt} + m_i \frac{d^2\vec{r}_i}{dt}\cdot\vec{r}_i\right)$$

$$= 2K + \underbrace{\sum_{i=1}^{N} \vec{F}_i\cdot\vec{r}_i}_{\boxed{\text{virial of Clausius}}}$$

$$\frac{dQ}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt}\cdot\vec{r}_i\right) = \frac{d}{dt}\left(\sum_{i=1}^{N}\frac{1}{2}\frac{d}{dt}(m_i\,\vec{r}_i\cdot\vec{r}_i)\right) = \frac{1}{2}\frac{d^2 I}{dt^2}$$

where $I = \sum_{i=1}^{N} m_i r_i^2$ is the $\boxed{\text{moment of inertia}}$

For gravitational systems $\vec{F}_i = \sum_{j\neq i} \frac{G m_i m_j (\vec{r}_j - \vec{r}_i)}{|\vec{r}_j - \vec{r}_i|^3} = \sum_{j\neq i} F_{ij}$

$$\text{Virial} = \sum_i \vec{F}_i\,\vec{r}_i = \sum_i\left(\sum_{j\neq i}\vec{F}_{ij}\right)\cdot\frac{1}{2}\left[(\vec{r}_i+\vec{r}_j)+(\vec{r}_i-\vec{r}_j)\right]$$

$$= \frac{1}{2}\sum_i\left(\sum_{j\neq i}\vec{F}_{ij}\right)\cdot\vec{r}_i + \frac{1}{2}\sum_i\left(\sum_{j\neq i}\vec{F}_{ij}\right)\cdot\vec{r}_j + \frac{1}{2}\sum_i\sum_{j\neq i}\vec{F}_{ij}\cdot(\vec{r}_i-\vec{r}_j)$$

$$= \frac{1}{2}\sum_i\left[\sum_{j\neq i}\vec{F}_{ij}\cdot\vec{r}_i - \sum_{j\neq i}\vec{F}_{ji}\cdot\vec{r}_j\right] + \frac{1}{2}\sum_i\sum_{j\neq i}\left(-\frac{G m_i m_j}{|\vec{r}_j - \vec{r}_i|}\right)$$

$$= 0 + \frac{1}{2}\sum_i\sum_{j\neq i} U_{ij}$$

$$= U$$

$$\sum_{i=1}^{3}\sum_{j\neq i} F_{ij}(\vec{r}_i+\vec{r}_j)$$
$$= F_{12}(r_1+r_2) + F_{13}(r_1+r_3)$$
$$+ F_{21}(r_2+r_1) + F_{23}(r_2+r_3)$$
$$+ F_{31}(r_3+r_1) + F_{32}(r_3+r_2)$$

$$\Rightarrow \quad \frac{dQ}{dt} = 2K + U$$

do time integral over a period of τ. $\langle U\rangle = \frac{1}{\tau}\int_0^{\tau} U\,dt$

$$\frac{Q(\tau)-Q(0)}{\tau} = \langle 2K\rangle + \langle U\rangle = 0 \quad \text{when } \tau\to\infty$$

because $Q = \sum \vec{p}_i\cdot\vec{r}_i$ is bounded for a system that reached an equilibrium or steady-state configurations

Application: Virial mass $\sigma^2 = \frac{G M_{vir}}{R_{vir}} \Rightarrow M_{vir} = \frac{\sigma^2 R_{vir}}{G}$

Faber-Jackson relation: $R \propto \frac{L}{\sigma^2}$, $L = 4\pi R^2 B \Rightarrow L \propto L^2/\sigma^4 \cdot B$

Fundamental Plane $R \propto \sigma^A I^B$

$$\frac{GM}{R} = k \cdot \frac{\sigma^2}{2} \quad , \quad M = L \cdot (M/L) \ , \ L = I R^2$$

$$\Rightarrow \frac{L\,(M/L)}{R} \propto \sigma^2 \Rightarrow I \cdot R\,(M/L) \propto \sigma^2$$

$$\Rightarrow R \propto \sigma^2 I^{-1} (M/L)^{-1}$$

$$L \propto I \cdot R^2 \propto \sigma^4 I^{-1} (M/L)^{-2}$$

Chap 3 Light

Parallax $d = \frac{1}{p''}$ pc $\min(p'') = 0.77''$ for Proxima Centauri

in 1836, Bessel measured $p = 0.316''$ for 61 Cygn

① distance to the moon measured by Hipparchus

solar eclipse on Mar 14 189 BC

total eclipse @ Nicaea (40°N) 4/5 eclipse @ Alexandria (31°N

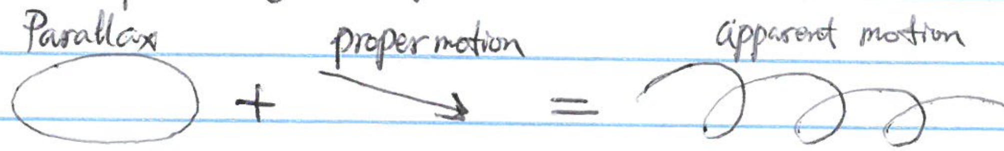
$\Rightarrow$ Dis to moon $= 71-81\ R_\oplus$ vs $60.27\ R_\oplus$ from better observ

② Hipparchus satellite, ESA 1989 - 1993

$\sigma(p) = 1$ mas for 10^5 stars $\Rightarrow \max(d) = 1$ kpc

③ Gaia, ESA

$\sigma(p) = 10\ \mu$as for 10^9 stars $\Rightarrow \max(d) = 100$ kpc



Magnitude $m = -2.5 \log_{10}(F_\lambda / F_{\lambda,\text{vega}})$ Vega magnitude system, A0V star

$m_{AB} = -2.5 \log_{10}(F_\nu / 3631\ \text{Jy})$ AB system

$1\ \text{Jy} = 10^{-23}\ \text{erg}\ s^{-1}\ \text{Hz}^{-1}\ \text{cm}^{-2}$

DM: $m - M = 5\log(d) - 5 = 5 \log \frac{d}{10\text{pc}}$

Abs Mag: M = apparent magnitude at $d = 10$pc

derivation:

$$m - M = -2.5 \log(F/F_{10}) = -2.5 \log \left[\frac{L}{4\pi d^2} \Big/ \frac{L}{4\pi (10\text{pc})^2} \right]$$
$$= 5 \log\left(\frac{d}{10\text{pc}}\right)$$

ΔM gives luminosity ratio: $L_2 / L_1 = 10^{\frac{M_1 - M_2}{2.5}}$

Bolometric luminosity of the Sun: $L_\odot = \int_0^\infty L_{\odot,\lambda}\, d\lambda = 3.839 \times 10^{33}$ erg/s

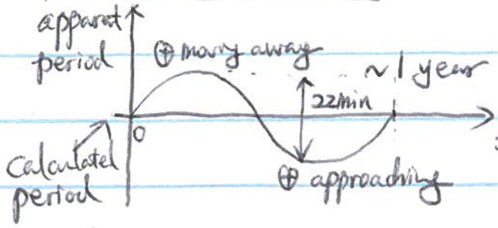
$M_\odot = +4.74$ magnitude, $m_\odot = -26.83$, $(m-M)_\odot = -31.57$

EM Wave

second is defined by the duration of N periods of transition of Cs-55.

$c = 3 \times 10^5$ km/s , meter is defined by c in vacuum.

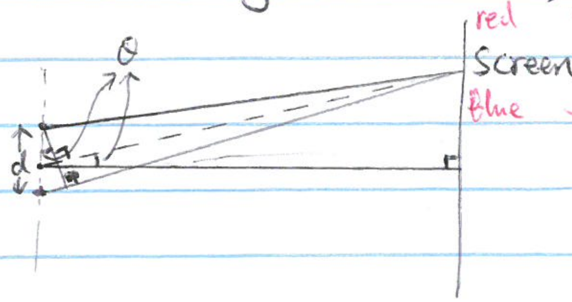
$\boxed{1675, \text{ Roemer}}$ timed the eclipse of Jupiter's moons, esp. Io (P=42hr



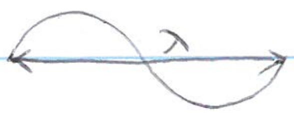
$c \cdot 22 \text{ min} = 2 \text{ AU}$

$\Rightarrow c = 2.2 \times 10^5$ km/s

Thomas Young (1773-1829) double slit experiment (after Newton died



$$d \sin\theta = \begin{cases} n\lambda & \text{constructive} \\ (n - \frac{1}{2})\lambda & \text{destructive} \end{cases}$$



$c = \lambda \nu$

Maxwell (1831-1879)

discovered EM waves from his equations and noticed the similar propert with light. He then inferes that light is EM wave. (Polarization, speed reflection, refraction)

Heinrich Hertz in 1889 produced radio EM waves in a lab

Poynting vector (describes the energy transfer rate of EM wave)

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$ unit: W/m^2, μ_0, vacuum permeability

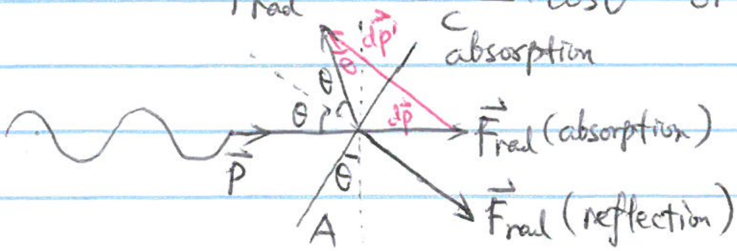
E: N/Coulomb, B: Tesla N/m/A

$$\langle S \rangle = \frac{1}{2\mu_0} E_0 B_0$$ $\rightarrow E_0$ & B_0 are the amplitudes & in vacuum $E_0 = c \cdot B_0$

Radiation Pressure (because EM wave also carries momentum) $\vec{F} = \frac{d\vec{p}}{dt}$

$$F_{rad} = \frac{\langle S \rangle A}{c} \cos\theta \quad \text{or} \quad \frac{2 \langle S \rangle A}{c} \cos^2\theta = 2 \cdot \cos\theta \, F_{abs}$$

absorption reflection



$|d\vec{p}'| = 2 \cdot \cos\theta \, |d\vec{p}|$

$E^2 = (mc^2)^2 + p^2c^2$, $E = h\nu$, m=

$\Rightarrow p = h\nu/c$

Thomas Wedgewood : English Porcelain maker, Color $\sim T$

Blackbody — an ideal emitter, absorbs all light, refects none

Wien's displacement law $\lambda_{max} \cdot T \simeq 3\ mm \cdot K$

CMB, $T = 3K$, $\lambda_{max} = 1mm$

Stefan-Boltzmann Eq. $L = A \cdot \sigma \cdot T^4$ (W)

$F_{surf} = \sigma T^4$ (W/m^2)

$$F = \int_{\nu=0}^{\infty} d\nu \int_0^{2\pi} d\phi \int_0^{\pi/2} B_\nu \cdot \cos\theta \sin\theta\, d\theta = \frac{2\pi^5}{15} \frac{k^4}{c^2 h^3} T^4 = \sigma_{SB} T^4$$

Energy Quantization and the derivation of Planck function

See Blackbody worksheet

luminosity of a star $L_\lambda = 4\pi R^2 B_\lambda$, $F_\lambda = \frac{L_\lambda}{4\pi r^2}$

$L_\nu = 4\pi R^2 B_\nu$

Color Index — Observing through a filter, sensitivity function $S(\lambda)$

$$F_{\lambda, obs} = F_{\lambda, intrinsic} \cdot S(\lambda)$$

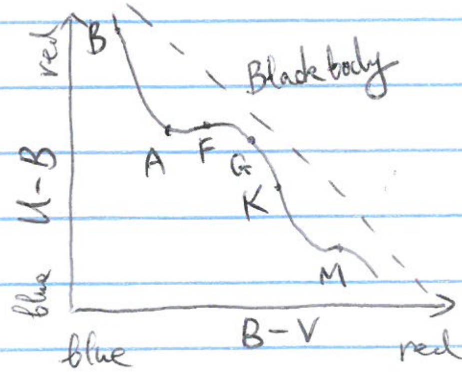
$$m = -2.5 \log_{10} \left(\frac{\int_0^\infty F_\lambda S(\lambda)\, d\lambda}{\int_0^\infty F_{\lambda, vega} \cdot S(\lambda)\, d\lambda} \right)$$

Vega magnitude (think about AB mag)

Vega is a A0V star, you find another A0V star with $B = 3.5$, what is its V magnitude? Also 3.5 mag.

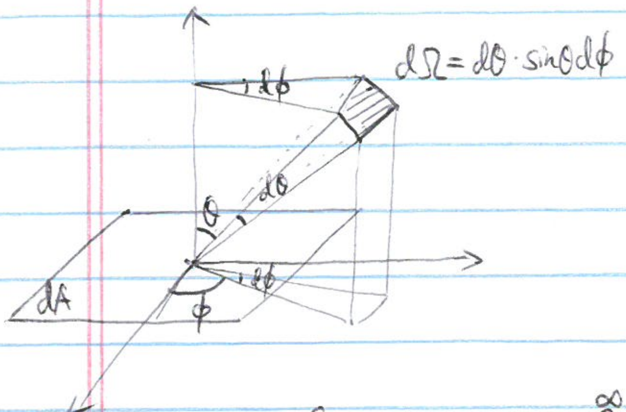
$$U-B \text{ color index} = U - B = m_U - m_B = -2.5 \log_{10} \left(\frac{\int F_\lambda S_U d\lambda}{\int F_\lambda S_B d\lambda} \right) + C_{U-B}$$

Color-Color Diagram



$$I_\nu \equiv \frac{\Delta E}{\Delta t \Delta A \Delta \nu \Delta \Omega} \cdot \frac{1}{\cos\theta}$$

Stefan-Boltzmann Equation (derivation)



specific flux ↓ specific intensity ↓

$$F_\nu = \frac{\Delta E}{\Delta t \Delta A \Delta \nu} = \int B_\nu \cos\theta \cdot d\Omega$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} B_\nu \cos\theta \sin\theta \, d\theta \, d\phi$$

$$= 2\pi B_\nu \int_0^{\pi/2} \sin\theta \, d(\sin\theta) = \pi B_\nu$$

$$F = \int F_\nu \, d\nu = \pi \int_0^\infty B_\nu \, d\nu = \frac{2\pi h}{c^2} \int \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}$$

replace $x = h\nu/kT$

$$F = \frac{2\pi h}{c^2} \cdot \left(\frac{kT}{h}\right)^4 \cdot \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

the integral $= \dfrac{\pi^4}{15}$

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

$$\Rightarrow F = \frac{2\pi^5}{15} \cdot \frac{k^4}{c^2 h^3} \cdot T^4 = \sigma_{SB} T^4$$

$$L = 4\pi R^2 \sigma T^4 = \int F \cdot dA$$

Doing the integral with expansion & integrate by parts.

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}} = \sum_{n=1}^\infty e^{-nx}$$

$$\int \frac{x^3}{e^x - 1} dx = \int x^3 \cdot \frac{e^{-x}}{1 - e^{-x}} dx = \sum_{n=1}^\infty \int_0^\infty x^3 e^{-nx} dx$$

$$= \sum_{n=1}^\infty \frac{6}{n^4} = 6 \cdot \zeta(4) = 6 \cdot \frac{\pi^4}{90} = \pi^4/15$$

↑ Reimann zeta function

$d(uv) = v\,du + u\,dv$ let $u = x^3$, $dv = e^{-x} dx$

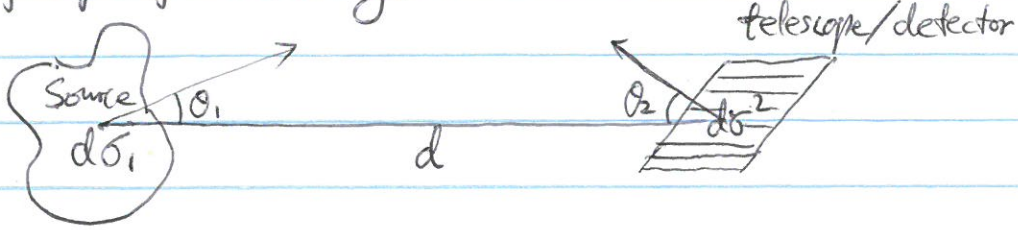
$\Rightarrow du = 3x^2 dx$, $v = -e^{-x}$

$$\int x^3 e^{-x} dx = \int u \, dv = \int d(uv) - \int v \, du = -x^3 e^{-x} + 3 \int x^2 e^{-x} dx$$

then let $u = x^2$, $dv = e^{-x} dx$ $\Rightarrow \int x^2 e^{-x} dx = -x^2 e^{-x} + 2 \int x e^{-x} dx$

dE/dt = power

Conservation of specific intensity : $dW = I_\nu \cos\theta \, d\sigma \, d\Omega \, d\nu$



$$dW_1 = (I_\nu)_1 \cos\theta_1 \frac{\cos\theta_2 \, d\sigma_2}{d^2} d\sigma_1 \, d\nu$$

$$dW_2 = (I_\nu)_2 \cos\theta_2 \frac{\cos\theta_1 \, d\sigma_1}{d^2} d\sigma_2 \, d\nu$$

energy conservation

$$dW_1 = dW_2 \quad \Rightarrow \quad (I_\nu)_1 = (I_\nu)_2$$

Chap 5 Interaction of light & matter

Spectral Lines [Empirical understanding]

1814 Fraunhofer cataloged 475 dark lines in Solar spectrum
Identified sodium in Solar atmosphere

1860 Bunsen's burner & Kirchhoff's laws
every element produces its own pattern of spec lines
- ☆ hot, dense gas or solid object produces continuum spectrum
- ☆ hot, diffuse gas produces emission lines
- ☆ cool, diffuse gas in front of a continuous spectrum produces absorption lines

Kirchhoff identified 70 Fe absorption lines in Solar spectrum

1868 ^4_2He was discovered spectroscopically on the Sun
helium was not found on Earth until 1895

Doppler Shift

$$z = \frac{\lambda_{obs} - \lambda_{rest}}{\lambda_{rest}} = \frac{\Delta\lambda}{\lambda_{rest}}$$

$$\lambda_{obs} = \lambda_{rest}\sqrt{\frac{1+V_r/c}{1-V_r/c}}$$ relativistic doppler shift

$$\lambda_{obs} = \lambda_{rest}(1+V_r/c)\Big/\sqrt{1-V_r^2/c^2} \simeq \lambda_{rest}(1+V_r/c)$$

$$\Rightarrow \frac{V_r}{c} = \frac{\lambda_{obs}-\lambda_{rest}}{\lambda_{rest}} = z$$ low-speed approximation ($V_r \ll c$)

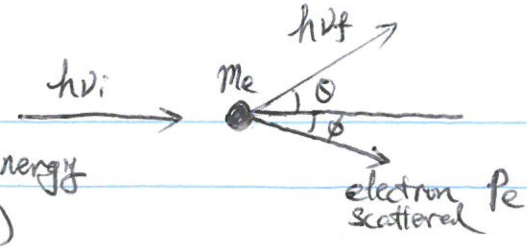
Resolving Power of Grating $\boxed{R = \frac{\lambda}{\Delta\lambda} = n \cdot N}$ ← in practice, R is determined by slit width.
n is the order number. N is the total number of lines illuminated.

Photoelectric Effect

Maximum kinetic energy of e^-, $K_{max} = h\nu - \phi$
Cutoff frequency, $\nu_c = hc/\phi$, where ϕ is the minimum binding energy.
Einstein's introduction of massless particles, photons

$$E_{photon} = h\nu = \frac{hc}{\lambda}, \quad hc = 1240\ eV\cdot nm$$

Compton Effect



$$E_{photon} = h\nu = \frac{hc}{\lambda} = pc$$

$$\boxed{E^2 = (mc^2)^2 + p^2c^2}$$ relativistic energy

$$\Delta\lambda = \lambda_f - \lambda_i = \frac{h}{m_e c}(1-\cos\theta)$$

derivation requires $\boxed{\text{conservation of momentum \& Energy}}$

$$\left.\begin{aligned} p_i &= p_f \cdot \cos\theta + p_e \cdot \cos\phi \\ 0 &= p_f \cdot \sin\theta - p_e \cdot \sin\phi \end{aligned}\right\} \Rightarrow p_i^2 - 2p_i p_f \cos\theta + p_f^2 = p_e^2$$

multiply $\times c^2$, and add $m_e^2c^4$ to both side

$$m_e^2c^4 + E_i^2 - 2E_iE_f\cos\theta + E_f^2 = p_e^2c^2 + m_e^2c^4 = E_e^2$$

Energy conservation

$$E_i + m_ec^2 = E_f + E_e$$

$\Rightarrow$

$$\begin{aligned} m_e^2c^4 + E_i^2 - 2E_iE_j\cos\theta + E_f^2 &= (E_i + m_ec^2 - E_f)^2 \\ &= E_i^2 + m_e^2c^4 + E_f^2 \\ &\quad + 2E_im_ec^2 - 2E_iE_f - 2m_ec^2E_f \end{aligned}$$

$$\Rightarrow \quad -\cancel{2}E_iE_f\cos\theta = \cancel{2}E_im_ec^2 - \cancel{2}E_iE_f - \cancel{2}m_ec^2E_f$$

divide by E_iE_f on both side, and take $-E_iE_f$ to the left side

$$1 - \cos\theta = m_ec^2\left(\frac{1}{E_f} - \frac{1}{E_i}\right)$$

$$\Rightarrow \quad \frac{1}{E_f} - \frac{1}{E_i} = \frac{1}{m_ec^2}(1-\cos\theta), \quad E_f = \frac{hc}{\lambda_f}, \; E_i = \frac{hc}{\lambda_i}$$

$$\Rightarrow \quad \cancel{\frac{1}{\lambda_f} - \frac{1}{\lambda_i}} \quad \lambda_f - \lambda_i = \frac{h}{m_ec}(1-\cos\theta)$$

Bohr Model

b 1911, Rutherford discovered that an atom's positive charge was concentrated in tiny, massive nuclei (10^4 times smaller than atoms

$\boxed{\text{Experiment}}$: high speed He nuclei bombardment of metal foils

a 1890s Thomson discovered electrons (1906 Nobel Prize)

$\boxed{\text{Experiment}}$: electric discharges in low pressure gas tubes

Thomson measured $e/m = 1.8\times10^{-11}$ ~~Cal~~ Coulombs/kg which is the charge and mass ratio of the particle.

ⓒ 1885 Balmer found an empirical formula to reproduce H lines

$$\frac{1}{\lambda} = R_H\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \quad m < n, \quad \boxed{R_H: \text{Rydberg constant}}$$

Balmer series: $m=2$, $n \geq 3$

Paschen series: $m=3$, $n \geq 4$

Lyman series: $m=1$, $n \geq 2$

Brackett series: $m=4$, $n \geq 5$

The empirical works ⓐ, ⓑ, ⓒ led to the theoretical understanding of Niels Bohr in 1913 for the H atom

$\boxed{\text{Key assumptions}}$: ① e^- doesnot radiate EM wave if their $L = nh/2\pi$

② e^- angular momentum is quantized.

③ $e^- - p^+$ system is controled by Coulomb's law

④ circular motion

Force balance: $\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = \frac{m_e m_p}{m_e+m_p}\frac{v^2}{r} = \mu\frac{v^2}{r}$

permittivity $\rightarrow \epsilon_0 = \frac{1}{\mu_0 c^2}$ $\leftarrow$ permeability

$$\Rightarrow \quad K = \frac{1}{2}\mu v^2 = \frac{1}{8\pi\epsilon_0}\frac{e^2}{r}$$

Virial theorem: $E = K + U = K - 2K = -K = -\frac{1}{8\pi\epsilon_0}\frac{e^2}{r}$

$(U + 2K = 0)$

Quantization of L: $L = \mu v r = nh/2\pi$, $\Rightarrow v = \frac{n\hbar}{\mu r}$

use this to replace v in the force balance equation, and solve for r

$$r_n = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}n^2 = a_0 n^2 \quad a_0: \text{Bohr radius}$$

$$E_n = -\frac{\mu e^4}{32\pi^2\epsilon_0^2\hbar^2}\frac{1}{n^2} = -13.6\ \text{eV}\cdot\frac{1}{n^2}$$

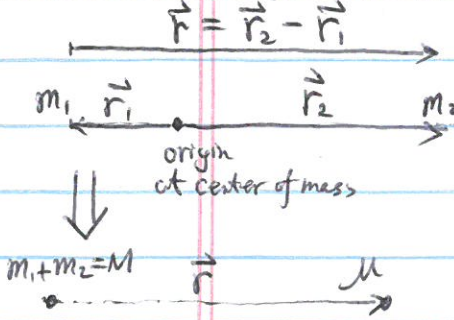
$\boxed{1\,eV = 1.6\times10^{-19}C\cdot 1V = 1.6\times10^{-19}\ \text{Joules}}$

n is called the principle quantum number.

$E_{photon} = \cancel{E_m}\ E_n - E_m$, $E_{photon} = \frac{hc}{\lambda} = \frac{2\pi\hbar c}{\lambda}$

$$\Rightarrow \quad R_H = \frac{\mu e^4}{64\pi^3\epsilon_0^2\hbar^3 c}$$

Reduced mass & center-of-mass coordinate system for two body problem



definition of CoM: $\dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = 0$

$$\Rightarrow \vec{r}_1 = -\frac{m_2}{m_1+m_2}\vec{r} \quad , \quad \vec{r}_2 = \frac{m_1}{m_1+m_2}\vec{r}$$

$\vec{r}_2 - \vec{r}_1 = \vec{r}$ the displacement vector

define reduced mass $\mu = \dfrac{m_1 m_2}{m_1+m_2}$ we have $\vec{r}_1 = -\dfrac{\mu}{m_1}\vec{r}$

$$\vec{r}_2 = \frac{\mu}{m_2}\vec{r}$$

this definition greatly simplifies the energy & angular momentum expression

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|} \quad , \quad \vec{v}_1 = \frac{d\vec{r}_1}{dt}$$

$$= \underline{\frac{1}{2}\mu v^2} - G\frac{M\mu}{r} \quad , \quad M = m_1 + m_2 ,$$

$$m_1 v_1^2 + m_2 v_2^2 = m_1 \frac{\mu^2}{m_1^2}\left(\frac{dr}{dt}\right)^2 + m_2 \cdot \frac{\mu^2}{m_2^2}\left(\frac{dr}{dt}\right)^2$$

$$= \mu^2 v^2 \left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \mu v^2$$

$$\vec{L} = m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2 = m_1 \cdot \left(-\frac{\mu}{m_1}\vec{r}\right) \times \left(-\frac{\mu}{m_1}\frac{d\vec{r}}{dt}\right)$$

$$+ m_2 \left(\frac{\mu}{m_2}\vec{r}\right) \times \left(\frac{\mu}{m_2}\frac{d\vec{r}}{dt}\right)$$

$$= \mu^2 \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \cdot \vec{r} \times \vec{v} = \mu \vec{r} \times \vec{v}$$

for Coulomb forces: $U = -\dfrac{1}{4\pi\epsilon_0}\dfrac{q_1 q_2}{r}$

$$K = \frac{1}{2}\mu v^2$$

$$\vec{L} = \mu \vec{r} \times \vec{v} \quad \Rightarrow L = \mu r v$$

Wave-Particle Duality

1927 de Broglie wavelength & frequency

$$\lambda = \frac{h}{p}, \quad \nu = \frac{E}{h}$$

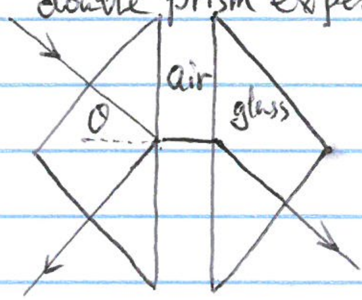
Heisenberg's uncertainty principle:

$$\Delta x \, \Delta p \simeq \hbar \iff \lambda = \frac{h}{p}, \quad \lambda = \Delta x$$

$$\Delta E \, \Delta t \simeq \hbar \iff \nu = \frac{E}{h}, \quad \nu = \frac{1}{\Delta t}$$

Quantum Mechanical Tunneling

double prism experiment



gap < 30λ

total reflection when $\sin\theta > \frac{n_{air}}{n_{glass}}$
but tunneled light is detected
on the right side when gap $\lesssim 30\lambda$

Schrödinger's Equation

Quantum Numbers: (n, l, m_l, s, m_s) for electrons

n: principle quantum number → energy: $E = -13.6\,\text{eV} \cdot \frac{1}{n^2}$

l: angular momentum quantum number → $L = \sqrt{l(l+1)}\,\hbar$

$l = 0, 1, 2, \ldots, n-1$, $\boxed{s\,p\,d\,f\,g\,h}$

magnetic quantum number m_l: z-component of L, direction of L → $L_z = m_l \hbar$

$m_l = -l, \ldots, 0, \ldots, l$ (Zeeman Effect breaks degeneracy)

$$\nu = \nu_0 \;\&\; \nu_0 \pm \frac{eB}{4\pi\mu}, \quad \mu = \frac{m_e m_n}{m_e + m_n}, \text{ where } m_n \text{ is nuclei mass}$$

Note that Zeeman effect can produce at most 3 lines because of selection rules

s: spin angular momentum $S = \frac{\sqrt{3}}{2}\hbar = \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)}\,\hbar$

spin quantum number → m_s $\quad S_z = m_s \hbar, \quad m_s = \pm\frac{1}{2}$

Pauli Exclusive Principle: No two electrons can share the same quantum state

Dirac Equation: Fermions have ½ spins, $S = \frac{2n+1}{2}\hbar$ (e.g. e^-, p^+, ν)
Bosons have integral spins: $S = 0, 1, 2, 3\,\hbar$ (e.g. photons)
Bosons do not follow Pauli exclusion principle
⇒ Prediction of anti-particles
opposite electric charges & magnetic moments

Selection Rules: allowed transitions: $\Delta l = \pm 1$, $\Delta m_l = 0$ or ± 1 ($\Delta S = 0$) (Zeeman effec
forbidden transitions: $\Delta l \neq \pm 1$, $\underline{m_l = 0 \ \& \ \Delta m_l = 0}$, $\Delta S \neq 0$

Spectroscopic Notation:
individual electrons: n, l, m_l, ~~m_s~~ s, m_s — $(n \geq 1)$ $(0, n-1)$ $(-l, l)$ $(\frac{1}{2})$ $\pm\frac{1}{2}$ — $m_s = -s, -s+1, \cdots s-1, s$; for e^-, $s = \frac{1}{2}$
multiple electrons: $L = \sum_i l_i$, $S = \sum_i s_i$, $L_z = M_L = \sum_i m_l$
$J = \sum_i (l+s) = L + S$ $\quad S_z = M_S = \sum_i m_s$

For example, the electron configuration of He^0 in ground state
$n_1 l_1 n_2 l_2 = 1s\,1s = 1s^2$ ← two e^- at $n=1$, $l=0$ state
if written for the total system, $L = \sum_i l_i = 0 + 0 = 0 = S$
often we add a superscript $2S+1$, $S = \sum_i s_i = -\frac{1}{2} + \frac{1}{2} = 0$
so the ground state is ${}^1S = {}^{2S+1}L_J$, $J = L + S = 0$
⇒ ground term of He^0 is $1s^2\ {}^1S_0$

$(n_i l_i)^{2S+1} L_J$ is the complete notation for a given electron configuration

---

n: 1, 2, 3, 4, ... KLMOPQ
l: 0, 1, 2, ..., $n-1$ spdfg
m_l: $-l, -l+1, \ldots, l-1, l$
m_s: $-\frac{1}{2}$, $\frac{1}{2}$

A filled shell contains $2n^2\ e^-$
# of states $= g = 2\sum_{l=0}^{n-1}(2l+1) = 2n^2$
$L = 0$ & $S = 0$ for filled shells

L: $\sum l$, e.g., $2p^2$ has $L = 1+1, 1+1-1, 1-1 = 2, 1, 0 = D, P, S$
S: $\sum_i m_s$, e.g., 2 electrons, $S = \frac{1}{2} - \frac{1}{2} = 0^{(2p^2)}, \frac{1}{2} + \frac{1}{2} = 1$ (2s2p)
J: $L+S, L+S-1, \cdots |L-S|$ e.g. ${}^3P$ has 3 levels, $L=1$, $S=1$, $J = 2, 1, 0$

Configuration (n, l) e.g. Boron $1s^2 2s^2 2p$ Neon $1s^2 2s^2 2p^6$

Carbon $1s^2 2s^2 2p^2$

Terms (S, L) e.g. Boron's outer most e^- has $l = 1$, $s = \frac{1}{2}$

so the term is ^2P, where $2 = 2S+1$

Levels (S, L, J) e.g. $J = L+S, L+S-1, \cdots, \mathrm{abs}(L-S)$ for $L > S$

(fine structure) doublet $\rightarrow$ $J = \frac{3}{2}, \frac{1}{2}$, multiplicity $= 2S+1$

so ^2P term has 2 levels $(2S+1)$: $^2P_{3/2}$, $^2P_{1/2}$

Fine Structure line: relativistic corrections (spin-orbit, etc) split degenerate levels

e.g. Na I doublet @ 5890, 5896 Å. $3^2P_{1/2} \rightarrow 3^2S_{1/2}$ & $3^2P_{3/2} \rightarrow 3^2S$

Hyperfine structure: coupling between J and the spin of nucleus I cause energy differences, $F = I + J = J-I, J-I+1, \cdots, J+I-1, J+$

e.g. $1s\,^2S_{1/2}$ level of the Hydrogen ground state has $J = \frac{1}{2}$

$I = \frac{1}{2}$ for the spin of proton

we have $F = 0, 1$

the transition between $F=1$ to $F=0$ produces 21 cm photons

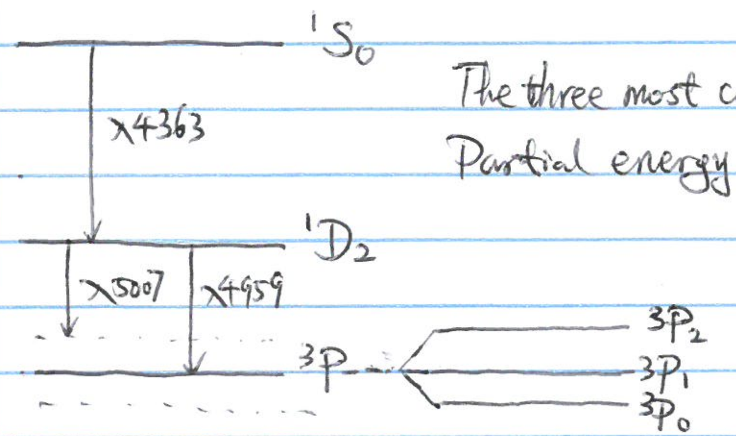
[O III], 6 electrons, configuration $1s^2 2s^2 2p^2$ C-like

For the outer two electrons @ $2p^2$ $L = 1+1, 1+1-1, 1-1 = 2, 1, 0$

we have ~~three~~ six terms $S = \frac{1}{2} - \frac{1}{2}, \frac{1}{2} + \frac{1}{2} = 0, 1$

^1S, ^1P, ^1D, ^3S, ^3P, ^3D

$J =$ 0 1 2 ~~012~~ 1 012 123



The three most commonly observed optical lines of [O III]

Partial energy level diagram

selection rules
- $\Delta J = 0, \pm 1$, but no $J = 0 \rightarrow$
- $\Delta L = 0, \pm 1$, but no $L = 0 \rightarrow$
- $\Delta S = 0$

Chap 8 Stellar Spectra

Harvard classification scheme: O B A F G K M ⋮ L T

a temperature sequence — He I (B), H I (A), Ca II (K) — Brown Dwarfs $T < 2500\,K$

Annie Cannon: 1901, employed by Pickering

HD catalogue, named after Henry Draper.

Equivalent Widths: $W = \int \frac{F_c - F_\lambda}{F_c} d\lambda$, F_c is the continuum flux density

Maxwell-Boltzmann Velocity distribution:

The most likely populated electron levels/velocity at certain temperature

$$n_v\, dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, 4\pi v^2\, dv$$

$$v_{mp} = \sqrt{\frac{2kT}{m}}$$, i.e. $\frac{1}{2} m v_{mp}^2 = kT$, $mp \rightarrow$ most probable speed

$$v_{rms} \equiv \sqrt{\overline{v^2}} = \sqrt{\frac{3kT}{m}}$$ root-mean-square ~~velocity~~ speed

Boltzmann Equation:

$$\frac{P(E_b)}{P(E_a)} = \frac{g_b}{g_a} \cdot \overbrace{e^{-(E_b - E_a)/kT}}^{\text{Boltzmann factor}}$$

$1\,eV = 1.6\times10^{-19}$ Joules

$k = 8.62\times10^{-5}\, eV\cdot K^{-1}$

$kT = \frac{1}{40}\, eV$ @ 300 K

g_a, g_b ← statistical weights of energy levels

e.g. ground state of H I, $E_a = -13.6\,eV$, $S_a = \{n=1, l=0, m_l=0, m_s=\frac{1}{2}\}$

$g(n) = 2n^2$ for H I, $E(n) = -13.6\,eV/n^2$ ⇒ $= 1s\ {}^2S_{\frac{1}{2}}$

If $E_b > E_a$, ① when $T \rightarrow 0$, $P(E_b)/P(E_a) = 0$

② when $T \rightarrow \infty$, $P(E_b)/P(E_a) = g_b/g_a$ (all levels equally accessible

For large number of particles

$$N(E_b)/N(E_a) = P(E_b)/P(E_a) = \frac{g_b}{g_a} \cdot e^{-(E_b - E_a)/kT}$$

> Example 8.1.3 At what temperature would there be equal #s of n=1 & n=2 H I?
>
> $E_b = E(n=2) = -13.6\,eV/n^2$, $E_a = E(n=1) = -13.6\,eV$
>
> $g_b = 2n^2 = 8$, $g_a = 2$, plug in BE ⇒ $\frac{10.2\,eV}{kT} = \ln 4$, $T = 85{,}400\,K$

Saha Equation: the relative number of atoms in different stages of ionization

Named after Meghnad Saha, derived in 1920

Partition function:

$$Z = \sum_{j=1}^{\infty} g_j \, e^{-(E_j - E_1)/kT}$$

Z_i & Z_{i+1} are the PF for atoms in its initial & final ionization stages

$$\frac{N_{i+1}}{N_i} = \frac{2 Z_{i+1}}{n_e Z_i} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_i / kT}$$

where χ_i is the ionization ~~sta~~ energy to remove an electron in the ground state

Combine Saha & Boltzmann equations

Example 8.1.4 Pure H ~~atomosphere~~ with $5000\,K < T < 25{,}000\,K$

① calculate partition functions Z_I & Z_{II}.

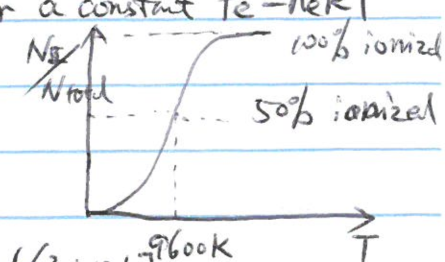
$Z_{II} = 1$ because H II is a single proton

$Z_I = g_1 = 2 \cdot n^2 = 2$ because almost all e^- are in ground state

$kT < 2.15$ eV

② use Saha equation to evaluate N_{II}/N_I for a constant $P_e = n_e kT$

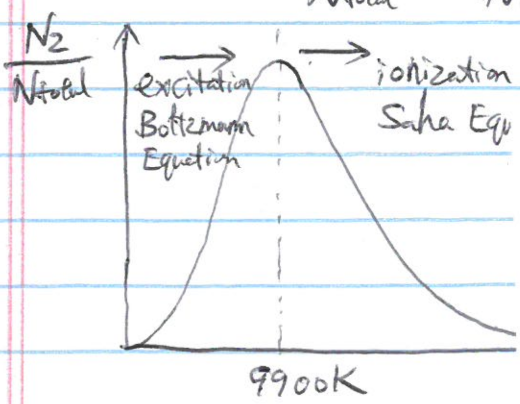
③ evaluate $\dfrac{N_{II}}{N_{total}} = \dfrac{N_{II}}{N_I + N_{II}} = \dfrac{N_{II}/N_I}{1 + N_{II}/N_I}$



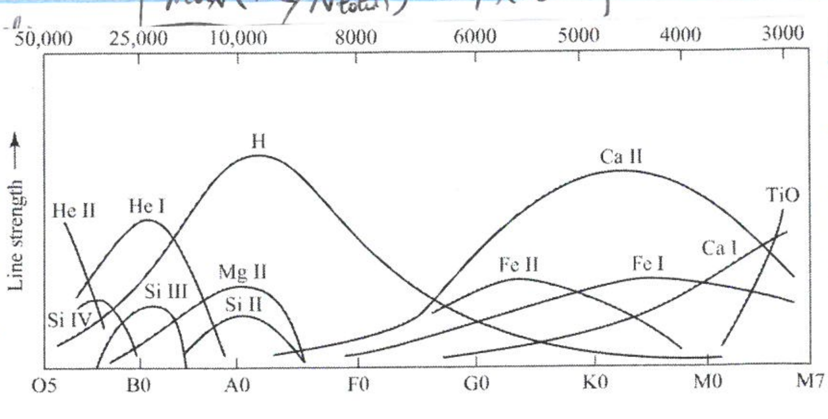
④ use Boltzmann equation to evaluate N_2/N_1

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/kT} = \frac{2(2)^2}{2(1)^2} \exp\left[- \frac{-13.6/2^2 + 13.6}{kT} \right]$$

⑤ calculate $\dfrac{N_2}{N_{total}} \simeq \left(\dfrac{N_2}{N_1 + N_2} \right) \left(\dfrac{N_I}{N_I + N_{II}} \right) = \dfrac{N_2/N_1}{1 + N_2/N_1} \cdot \dfrac{1}{1 + N_{II}/N_I}$



$\boxed{\max(N_2/N_{total}) = 9 \times 10^{-6}}$



abundance: $\boxed{12 + \log(Ca/H) = 12 - 5.65 = 6.35}$

Mixed atmosphere : $N(He) : N(H) = 1 : 10$, $N(Ca) : N(H) = 1 : 500{,}000$

again we assume a constant electron pressure $P_e = n_e kT = 1.5\ N\,m^{-2}$ @ $T = 5800$ K

$^{40}_{20}Ca$ Balmer lines : H I $n=2$ to higher levels , $\chi_I = 13.6$ eV

e^- configuration Ca H & K : Ca II $n=1$ ground state , $\chi_I = 6.11$ eV , $Z_I = 1.32$

[Ar] $4s^2$ $T = 5800$K $\Rightarrow kT = 0.5$ eV $Z_{II} = 2.30$

$1s^2 2s^2 2p^6 3s^2 3p^6 4s^2$ For the excited state of Ca II $E_2 - E_1 = 3.12$ eV , $g_1 = 2$, $g_2 = 4$

Ca II $\lambda = hc/\Delta E = 398.8$ nm

$3s^2 3p^6 4s^1$ $\Rightarrow$ for H , $(N_2/N_{total})_{HI} = 5.06 \times 10^{-9}$

① $N_{II}/N_I = 1/13{,}000$ ② $\dfrac{N_2}{N_1} = \dfrac{g_2}{g_1} e^{-\Delta E/kT} \simeq \dfrac{1}{198{,}000{,}000}$

$\Rightarrow$ for Ca , $(N_1/N_{total})_{CaII} = 0.995$

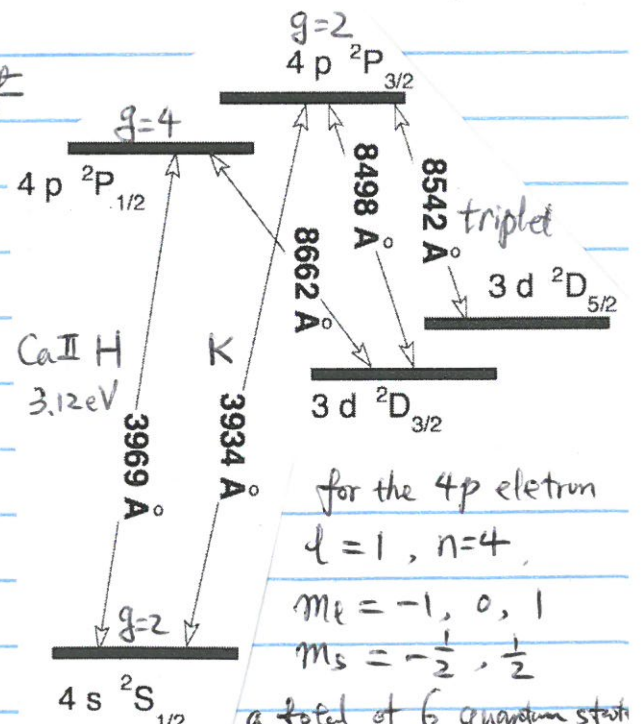
① $N_{II}/N_I = 918$ ② $N_2/N_1 = 1/264$

multiply the abundance ratio between H & Ca to $(N_2/N_{total})_{HI}$

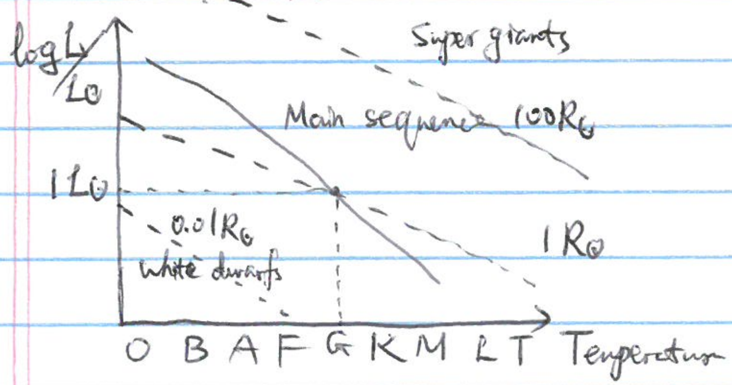
we have $500{,}000 \times 5.06 \times 10^{-9} = 1/395 \ll (N_1/N_{total})_{CaII} \simeq 1$

HR diagram : $L = 4\pi R^2 \cdot \sigma \cdot T^4$, $L = 4\pi d^2 \cdot F$

$\Rightarrow R = \dfrac{1}{T^2}\sqrt{\dfrac{L}{4\pi\sigma}}$



L vs T diagram first introduced by Russell 1914.



for the 4p eletron

$l = 1$, $n = 4$

$m_l = -1, 0, 1$

$m_s = -\frac{1}{2}, \frac{1}{2}$

a total of 6 quantum states

4 give $J = \frac{1}{2}$, 2 $\Rightarrow J = \frac{3}{2}$

Morgan-Keenan Luminosity classes

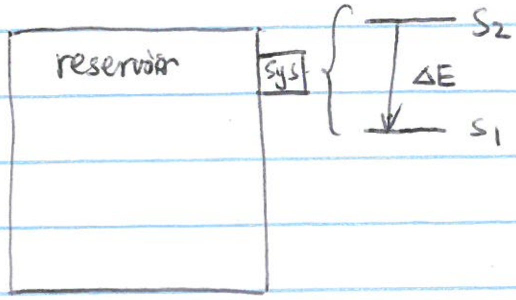
Ia-0, Ia, Ib	II	III	IV	V	D
supergiants	bright giants	Normal Giants	Subgiants	MS	white Ds

Discussion on Boltzmann Equation & Saha Equation

Boltzmann Equation: $\dfrac{N_b}{N_a} = \dfrac{P_b}{P_a} = \dfrac{g_b}{g_a} \cdot e^{-(E_b - E_a)/kT}$

assumptions:
① all microstates have equal probability of being occupied
② kT is the energy required to increase Ω by a factor of e
③ when a reservior absorbs energy, Ω increases
④ no energy transfer between reseriors at the same temperature (0th law

Derivation:



$\Omega_{total} = \Omega_R \cdot \Omega_{sys} = \Omega_R$ because $\Omega_{sys} = 1$

$\Omega(S_1) > \Omega(S_2)$ because reservior absorbs ΔE after the system makes the transition

(A) derivation of the Boltzmann factor:

① suppose the res has Ω number of microstates before it receives ΔE, which made Ω to increase to $\Omega' = \Omega + f(\Delta E)$ or $\Omega' = \Omega \cdot f(\Delta E)$

② divide the res into two identical parts so that $\Omega = \Omega_1 \cdot \Omega_2 = \Omega_1^2$ and each part receives $\Delta E/2$, so that $\Omega_1' = \Omega_2' = \Omega_1 + f(\frac{\Delta E}{2})$ or $\Omega_1 f(\frac{\Delta E}{2})$

③ because the division is artificial, we have $\Omega' = \Omega_1' \cdot \Omega_2'$

so that $\Omega' = \Omega + f(\Delta E) = [\Omega_1 + f(\frac{\Delta E}{2})]^2 = \Omega_1^2 + f^2(\frac{\Delta E}{2}) + 2\Omega_1 f(\frac{\Delta E}{2})$

or $\Omega' = \Omega \cdot f(\Delta E) = \Omega_1^2 \cdot f^2(\frac{\Delta E}{2}) \Rightarrow f(\Delta E) = f^2(\frac{\Delta E}{2})$

④ only the latter works, so $f(\Delta E) = e^{\Delta E/b}$, where b is a constant

(B) derivation of the kT factor:

① from the ~~first theorem~~ zeroth law of ~~thermal dynamics~~, thermodynamics, systems in thermal equilibrium have the same temperature (T).

② imagine two different reserviors with the same T are in contact to each other they allow heat transfer but no heat flows between them.

③ each system takes or loses ΔE but the total Ω is conserved.

$\Omega_0 = \Omega_a \cdot \Omega_b$, a loses ΔE to b

$\Omega_a' = \Omega_a \cdot e^{-\Delta E / f(a)}$, $\Omega_b' = \Omega_b \cdot e^{\Delta E / f(b)}$

$\Rightarrow \Omega_a' \cdot \Omega_b' = \Omega_a \cdot \Omega_b \cdot e^{-\Delta E\left(\frac{1}{f(a)} - \frac{1}{f(b)}\right)} = \Omega_0$

$\Rightarrow f(a) = f(b)$. now let's recall what is the same between the two

their temperature, so that we can propose $\boxed{f = kT}$

Saha Equation: $$\frac{N_{i+1}}{N_i} = \frac{2 Z_{i+1}}{Z_i} \left(\frac{2\pi m_e kT}{h^2 \cdot n_e^{2/3}}\right)^{3/2} e^{-\chi_i / kT}$$

a useful formula, $$\frac{h^2 n_e^{2/3}}{2\pi m_e} = 4.8 \times 10^{-19}\, eV \left(\frac{n_e}{1\, m^{-3}}\right)^{2/3}$$
$$= 10^{-5}\, eV \left(\frac{n_e}{10^{20}\, m^{-3}}\right)^{2/3}$$

assumptions:

① thermal equilibrium among e^-, atoms, and ions (same T)

② Boltzmann equation

③ Z_i is almost independent of T in the regime of interest

properties:

① it doesn't care about the detailed physical process that established the thermal equilibrium

② the atomic weight of the ions are irrelevant, only the ionization energy χ_i matters, e.g. $\chi_i = -13.6\, eV \frac{Z^2}{1^2} = -13.6 Z^2\, eV$ for Z proton, $1e^-$ sys

③ it requires other equations to be solvable.
e.g. for pure H, $N_{II} = n_e$, $n_I + n_{II} = n_{total} \simeq 10^{20}\, m^{-3}$

④ H reaches full ionization @ $T \sim 15{,}000\, K \sim 1.5\, eV \ll \chi_i = 13.6\, eV$
why? Because recombination is equally difficult/unlikely as ionization

Planck Function for Blackbody emission. $B_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$.

$x = h\nu / kT$, $x_{max} = 2.821 \Rightarrow h\nu_{max} = 2.821 \cdot kT = 4.5\, eV \ll \chi_i$

Maxwell-Boltzmann velocity distribution:

$n(v) = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, 4\pi v^2 dv \Rightarrow \frac{1}{2} v_{mp}^2 \cdot m = kT$ (most probable KE)

Derivation of Maxwell-Boltzmann Velocity distribution

$$n_v \, dv = n \cdot \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}} \cdot 4\pi v^2 \cdot dv$$

① from Boltzmann Eq. $\cancel{n_{p_e} dp_e = g_e}$

$$n_v dv = n_p dp = g(p) e^{-E/kT} dp$$

$$= \frac{8\pi m^2 v^2}{nh^3} \cdot e^{-\frac{mv^2}{2kT}} \, d\,mv$$

$$= \frac{2m^3}{nh^3} \cdot 4\pi v^2 \cdot e^{-\frac{mv^2}{2kT}} \, dv$$

② normalization

$$n = \int_0^\infty n_p \, dp = \frac{2}{nh^3} (2\pi mkT)^{3/2} \qquad \cancel{\Rightarrow \frac{nh^3}{2} = n \cdot (2\pi mkT)^{-3/2}}$$

$$\Rightarrow \quad \frac{2}{nh^3} = \frac{n}{(2\pi mkT)^{3/2}}$$

① + ②, eleminate $\frac{2}{nh^3}$, we have the M-B distribution

$$n_v dv = n \cdot m^3 \cdot \left(\frac{1}{2\pi mkT}\right)^{3/2} e^{-\frac{mv^2}{2kT}} \cdot 4\pi v^2 \, dv$$

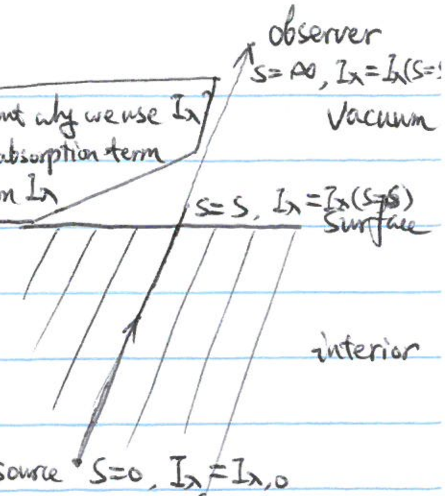
Properties: ① $V_{mp} = \sqrt{\frac{2kT}{m}}$, i.e., $kT = \frac{1}{2} m V_{mp}^2$

② $V_{rms} = \sqrt{\overline{v^2}} = \sqrt{\int_0^\infty v^2 P(v) \, dv} = \sqrt{\frac{3kT}{m}}$

③ $\sigma_z = \sqrt{\frac{kT}{m}}$, $FWHM = 2\sqrt{2\ln 2} \cdot \sigma_z$

Chap 9 : Stellar Atmosphere

Think about why we use I_λ? Why the absorption term depends on I_λ



Transfer Equation: $$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds + j_\lambda \rho\, ds$$

Solution for absorption only:
$$I_\lambda = I_{\lambda,0}\, e^{-\int_0^S \kappa_\lambda \rho\, ds}$$

define optical depth : $d\tau_\lambda = -\kappa_\lambda \rho\, ds$
$$\int_{\tau(s=0)}^{\tau(s=S)} d\tau_\lambda = -\int_0^S \kappa_\lambda \rho\, ds \Rightarrow \tau_\lambda(s=S) - \tau_\lambda(s=0) = -\int_0^S \kappa_\lambda \rho\, ds$$

define $\tau(s=S)=0$, optical depth is zero @ surface, we have
$$\tau_\lambda(s=0) = \int_0^S \kappa_\lambda \rho\, ds$$
$$\Rightarrow I_\lambda(s=S) = I_\lambda(s=0)\cdot e^{-\tau_\lambda(s=0)}$$

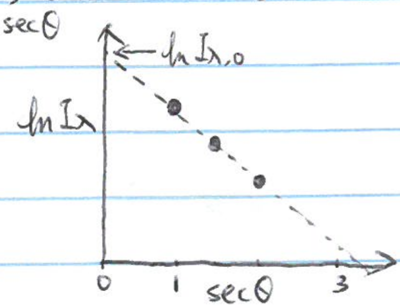
for constant ρ & κ_λ, $\tau_\lambda = \kappa_\lambda \rho \cdot s = \frac{s}{l}$ [# of mean free path of photons]

Example #1 : estimate the optical depth of Saturn's ring
$$\tau = \kappa \rho s = n\cdot\sigma\cdot d$$
$$A_{eff} = n\cdot V\cdot \sigma = n\cdot A_{geometric}\cdot d\cdot\sigma = \tau\cdot A_{geometric}$$
$$\Sigma_{obs} = \frac{\Sigma_{intrinsic}\cdot A_{eff}}{A_{geometric}}$$, observed surface brightness
$$\Rightarrow \tau = \frac{A_{eff}}{A_{geo}} = \frac{\Sigma_{obs}}{\Sigma_{intrinsic}}$$

Example #2 : atmosphere extinction (use plane-parallel geometry)

define vertical optical depth $\tau_{\lambda,v} = \int_0^h \kappa_\lambda \rho\, dz$

if the incident ray is at an angle from zenith, $ds = z\cdot\sec\theta$
$$\tau_\lambda = \tau_{\lambda,v}\sec\theta \Rightarrow I_\lambda = I_{\lambda,0}\, e^{-\tau_{\lambda,v}\sec\theta}$$
$$\Rightarrow \ln I_\lambda = \ln I_{\lambda,0} - \tau_{\lambda,v}\cdot\sec\theta$$

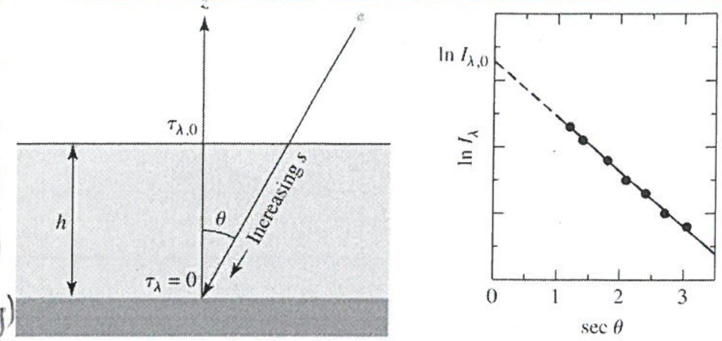


optical depth of Saturn's ring : $\tau = n\sigma \overset{\text{thickness}}{d} = \dfrac{A_{eff}}{A_{geometric}} = \dfrac{\Sigma_{obs}}{\Sigma_{intrinsic}}$, where Σ is surface brightness

airmass : $1/\cos\theta$ where θ is the zenith angle

$$\tau_\lambda = \int_0^s \kappa_\lambda \rho\, ds = -\int_h^0 \kappa_\lambda \rho \frac{dz}{\cos\theta} = \left[\int_0^h \kappa_\lambda \rho\, dz\right]\cdot \sec\theta = \tau_\lambda(\theta=0)\sec\theta$$

$$I_\lambda = I_{\lambda,0}\cdot e^{-\tau_\lambda(\theta=0)\cdot\sec\theta} \Rightarrow \ln I_\lambda = \ln I_{\lambda,0} - \tau_\lambda(\theta=0)\cdot\sec\theta$$



Sources of Opacity → stellar atmosphere

① Bound-bound transition (excitation)

② Bound-free absorption (photoionization)

③ Free-free absorption (Bremsstrahlung)

④ Electron scattering (Thomson scattering) ⑤ H^- (binding $E = 0.75$ eV) $\lambda \le 1.6\ \mu m$ for BF

{ FF absorption is a three-body interaction ($h\nu$, e^-, ion)
{ E scattering is a two-body interaction ($h\nu$, e^-)

important for stars later than F0 type, including the Sun

$$\sigma_T = \frac{1}{6\pi\epsilon_0^2}\left(\frac{e^2}{m_e c^2}\right)^2 = 6.65\times10^{-29}\ m^2 \text{ at all wavelengths}$$

$\sigma_T \ll \sigma_{bf}$ → the cross-section for H photoionization

$h\nu \ll mc^2$ Thomson scattering : ~~$h\nu$ + free electron (w/ KE), $E_e = K + \phi > 0$~~ → elastic scattering & elastic collision, low energy limit of Compton scattering

$h\nu \gg mc^2$ Compton scattering : ~~$h\nu$ + an electron loosely bound to an nucleus $KE_e \simeq 0$~~ → inelastic scattering but elastic collision

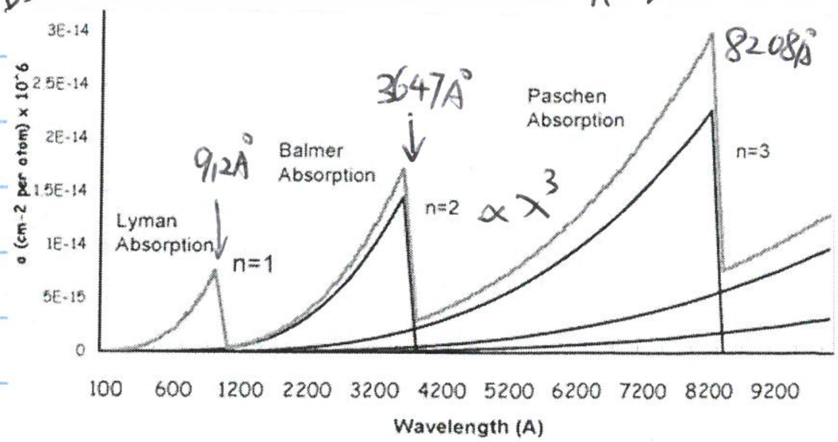
polarized → Rayleigh scattering : ~~same as Compton scattering but when $\lambda \gg r_{atom}$~~

~~note that $r = a_0 = 0.05$ nm for n=1, Hydrogen atom~~

$\sigma_{Rayleigh} \propto 1/\lambda^4$, $\sigma_{Thomson}$ independent of wavelength

$$\sigma_{bf} = \frac{64\pi^4 m_e e^{10}}{3\sqrt{3}\, c h^6}\frac{Z^4}{n^5\nu^3}\cdot g(\nu, n, l, Z) = 2.8\times10^{29}\ cm^2\cdot\frac{g Z^4}{n^5\nu^3}$$

$$\lambda_{cutoff} = \frac{hc}{\chi} = \frac{hc}{13.6\,eV/n^2} = 912\,\text{Å}\cdot n^2$$



Rosseland mean opacity :

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T} d\nu}$$

$$\left.\begin{aligned} \overline{\kappa_{bf}} &= 4.34\times10^{21}\ m^2\,kg^{-1}\ \frac{g_{bf}}{t}\ Z(1+X)\frac{\rho}{T^{3.5}} \\ \overline{\kappa_{ff}} &= 3.68\times10^{18}\ m^2\,kg^{-1}\ g_{ff}(1-Z)(1+X)\cdot\frac{\rho}{T^{3.5}} \end{aligned}\right\} \propto \kappa_0 \rho / T^{3.5}$$

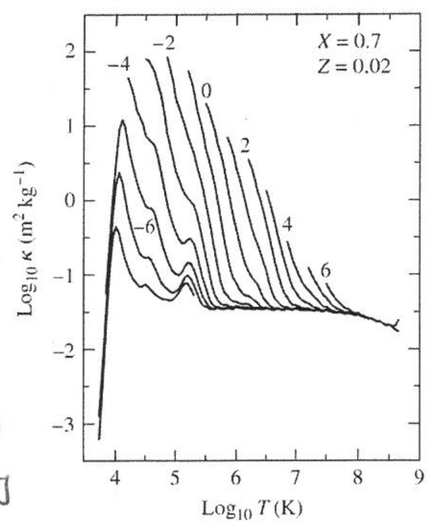
Kramers opacity law

where ρ in $kg\ m^{-3}$, T in Kelvin, $Z = \frac{m(\text{metal})}{m_{total}}$ (metalicity), $X = \frac{m(H)}{m_{total}}$

g_{bf} and g_{ff} are Gaunt factors ($\simeq 1$), and t is guillotine factor (1-100,

$$\overline{\kappa_{es}} = 0.02(1+X)\ m^2\,kg^{-1}$$

$$\overline{\kappa_{H^-}} = 7.9\times10^{-34}\ m^2\,kg^{-1}\,(Z/0.02)\,\rho^{1/2}\,T^9$$



emission : adding photons to a beam of light < absoption, scatterring

opacity : removing photons from a beam of light < emission

emission bound-bound, free-bound, free-free, e^- scattering

opacity bound-bound, bound-free, free-free, e^- scattering

random walk of photons due to the absoption, scattering, & emission processes

$$\vec{d} = \sum_{i=1}^{N} \vec{l}_i \Rightarrow \vec{d}\cdot\vec{d} = \sum_{i=1}^{N}\sum_{j=1}^{N} \vec{l}_i \vec{l}_j = N l^2 + l^2 \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \cos\theta_{ij}$$

$$\Rightarrow d^2 = N l^2 \Rightarrow d = \sqrt{N}\cdot l \quad ①$$

Now think about optical depth, $\tau_\lambda = \int_0^s \kappa_\lambda \rho\, ds \overset{\text{uniformity}}{=} \kappa_\lambda \rho \cdot s \overset{\text{definition of } l}{=} \frac{s}{l}$

if $s = d$, we have $d = \tau_\lambda \cdot l$ ②

combining ① & ②, we have $N = \tau_\lambda^2$, consistent with the definition of $\tau_\lambda = 1$, which defines the surface where a photon is free to escape.

Conclusion: Looking into a star at any angle or wavelength, we always look back to $\tau_\lambda = 2/3 \simeq 1$

Quantitative Explanations

Applications: ① absorption line formation

② limb darkening

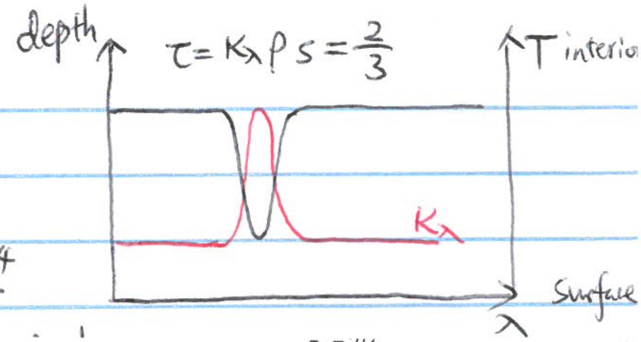
③ radiation pressure gradient

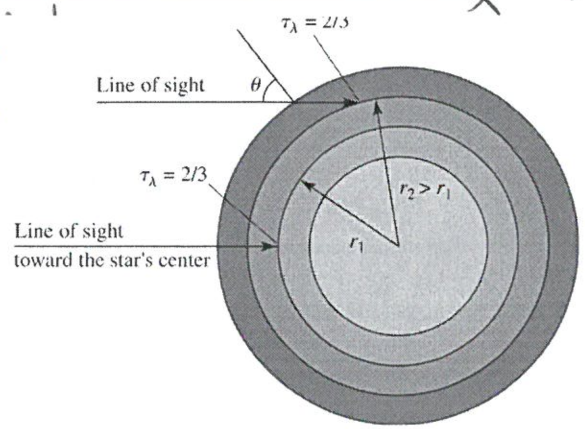
$$P_{rad} = \frac{4\pi}{3c}\int_0^\infty B_\lambda(T)\,dT = \frac{4\sigma T^4}{3c}$$

so P_{rad} decreases as $T \searrow$

this causes the radiative flux, F_{rad}

$$\frac{dP_{rad}}{dr} = -\frac{\bar{\kappa}\rho}{c}F_{rad}$$





Transfer equation: $dI_\lambda = -\kappa_\lambda \rho I_\lambda ds + j_\lambda \rho ds$

$$\Rightarrow -\frac{1}{\kappa_\lambda \rho}\frac{dI_\lambda}{ds} = I_\lambda - \frac{j_\lambda}{\kappa_\lambda}$$

definition of source function

$$\Rightarrow \boxed{-\frac{1}{\kappa_\lambda \rho}\frac{dI_\lambda}{ds} = I_\lambda - S_\lambda}$$

if $I_\lambda = S_\lambda$, then $dI_\lambda/ds = 0$, I_λ remains constant

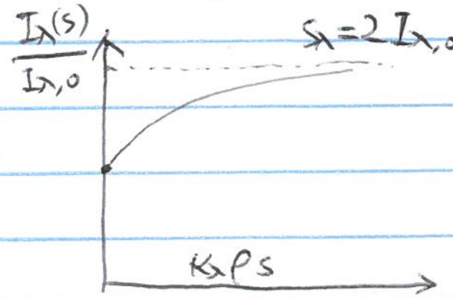
if $I_\lambda < S_\lambda$, then $dI_\lambda/ds > 0$, $I_\lambda \nearrow$ with distance

if $I_\lambda > S_\lambda$, then $dI_\lambda/ds < 0$, $I_\lambda \searrow$ as the beam travels

~~Thermodynamic equilibrium: $I_\lambda = B_\lambda$ and $dI_\lambda/ds = 0 \Rightarrow I_\lambda = \boxed{S_\lambda = B_\lambda}$~~ see notes later

~~LTE assumption is $S_\lambda = B_\lambda$, which does not necessarily give $I_\lambda = B_\lambda$~~:

Solution to the Transfer equation.



Assuming ρ, κ_λ, S_λ are all constant

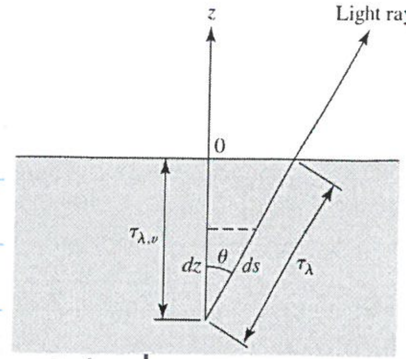
$$\int_{I_\lambda(0)}^{I_\lambda(s)} \frac{dI_\lambda}{I_\lambda - S_\lambda} = -\kappa_\lambda \rho \int_0^s ds$$

$$\ln\frac{I_\lambda(s) - S_\lambda}{I_\lambda(0) - S_\lambda} = -\kappa_\lambda \rho s \Rightarrow I_\lambda(s) = I_\lambda(0)\cdot e^{-\kappa_\lambda \rho s} + S_\lambda(1 - e^{-\kappa_\lambda \rho s})$$

$I_\lambda(s) \to S_\lambda$ as $\kappa_\lambda \rho s \to \infty$

@ $s = \frac{1}{\kappa_\lambda \rho}$ (one mean free path), $I_\lambda = \frac{I_{\lambda,0}}{e} + S_\lambda - \frac{S_\lambda}{e} = S_\lambda + \frac{1}{e}(I_{\lambda,0} - S_\lambda)$

Explanation of Limb Darkening



$$I_\lambda(\theta)/I_\lambda(0) = 0.4 + 0.6\cdot\cos\theta$$

Begin by writing down the transfer equation

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda \qquad (1) \quad [d\tau_\lambda = -\kappa_\lambda \rho\, ds]$$

multiply $e^{-\tau_\lambda}$ to both sides, we have, $\frac{d}{d\tau_\lambda}(e^{-\tau_\lambda} I_\lambda) = -S_\lambda e^{-\tau_\lambda}$

so the general solution for the transfer equation is

$$e^{-\tau_\lambda} I_\lambda \Big|_{\tau_{\lambda,0}}^{\tau_\lambda \text{ @ surface}} = -\int_{\tau_{\lambda,0}}^{\tau_\lambda \text{ @ surface}} S_\lambda e^{-\tau_\lambda}\, d\tau_\lambda$$

because $\tau_\lambda = 0$ @ surface & $I_\lambda = I_\lambda(0)$ @ surface

$\tau_\lambda = \tau_{\lambda,0}$ @ bottom & $I_\lambda = I_{\lambda,0}$ @ bottom → initial position of ray

$$\Rightarrow I_\lambda(0) = I_{\lambda,0} e^{-\tau_{\lambda,0}} - \int_{\tau_{\lambda,0}}^{0} S_\lambda e^{-\tau_\lambda}\, d\tau_\lambda \quad \cdots (2)$$

$$= I_{\lambda,0} e^{-\tau_{\lambda,0}} + S_\lambda(1 - e^{-\tau_{\lambda,0}}) \text{ when } S_\lambda \text{ is constant}$$

Assumption #1. Plane-Parallel atmosphere (separate out angle dependence)

define vertical optical depth $\tau_{\lambda,v}(z) = \int_z^0 \kappa_\lambda \rho\, dz$

we have $\tau_{\lambda,v} = \tau_\lambda \cdot \cos\theta \iff \tau_\lambda = \sec\theta \cdot \tau_{\lambda,v}$

$$d\tau_\lambda = \sec\theta\, d\tau_{\lambda,v}$$

Assumption #2. $\tau_{\lambda,0} = \infty$ optically thick atmosphere, Eq (2) can be simplified

$$I_\lambda(0) = \int_0^\infty S_\lambda \sec\theta \cdot e^{-\tau_{\lambda,v}\sec\theta}\, d\tau_{\lambda,v} \quad \cdots (3)$$

This equation has a general solution if $S_\lambda = a_\lambda + b_\lambda \tau_{\lambda,v}$

$$I_\lambda(0) = a_\lambda + b_\lambda \cdot \cos\theta \quad \cdots (4)$$

To evaluate a_λ & b_λ, we make further assumptions.

Assumption #3. Local Thermodynamic Equilibrium (LTE), $\cancel{dI_\lambda/ds = 0}$

$\Rightarrow \cancel{I_\lambda =} S_\lambda = B_\lambda(T)$, the Planck function $\cdots (5)$

$$S_\lambda = B_\lambda(T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1} \text{ is a function of } T \,\&\, \lambda$$

To express S_λ as a function of $\tau_{\lambda,v}$, we need to solve the vertical structure of the atmosphere.

Assumption #4: Grey atmosphere, $\tau_\lambda = \bar{\tau} = \text{const.}$ independent of λ

Integrate over λ on both side of the transfer equation, we have

$$\cos\theta \cdot \frac{dI}{d\tau_v} = I - S \quad \text{where } I = \int_0^\infty I_\lambda \, d\lambda, \quad S = \int_0^\infty S_\lambda \, d\lambda$$

Result ①; integrate over $d\Omega$ on both side

isotropic source function

$$\frac{d}{d\tau_v}\int I\cos\theta \, d\Omega = \int I \, d\Omega - S\int d\Omega$$

$$\boxed{\frac{dF_{rad}}{d\tau_v} = 4\pi\left(\overset{\text{mean intensity}}{\langle I\rangle} - S\right)} \quad \cdots (6)$$

Result ②. multiply $\cos\theta$ to both side then integrate over $d\Omega$

$$\frac{d}{d\tau_v}\int I\cos^2\theta \, d\Omega = \underline{\int I\cdot\cos\theta \, d\Omega} - S\underbrace{\int \cos\theta \, d\Omega}_{\text{zero}}$$

$$(7) \quad \boxed{\frac{dP_{rad}}{d\tau_v} = \frac{1}{c}F_{rad}}$$ ← radiative flux driven by radiation pressure gradient

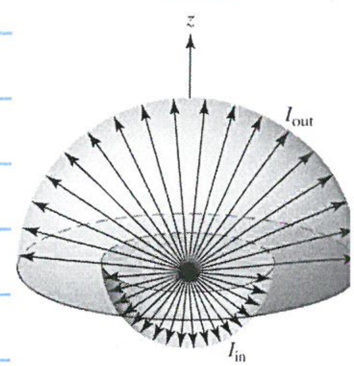
Assumption #5: Eddington Approximation

$$\langle I\rangle = \frac{1}{2}(I_{out} + I_{in}), \quad F_{rad} = \pi(I_{out} - I_{in}), \quad \boxed{P_{rad} = \frac{4\pi}{3c}\langle I\rangle}$$

at top of atmosphere, $I_{in} = 0$

$$\langle I\rangle = \frac{1}{2}I_{out}, \quad F_{rad} = \pi I_{out} = 2\pi\langle I\rangle$$

$$P_{rad} = \frac{2\pi}{3c}I_{out} = \frac{2}{3c}F_{rad}$$



Assumption #6: equilibrium

steady state atmosphere, energy conservation

$$F_{rad} = \text{constant} = F_{rad}(\text{surface}) = \sigma T_e^4$$

At top of the atm

$$P_{rad,\lambda} = \frac{1}{c}\int B_\lambda \cos^2\theta \, d\Omega \; [\text{for half hemisphere}]$$

$$= \frac{1}{c}B_\lambda \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta$$

$$= \frac{2\pi}{3c}B_\lambda$$

$$\Rightarrow P_{rad} = \int \frac{2\pi}{3c}B_\lambda \, d\lambda = \frac{2}{3c}\sigma T_e^4$$

so we can integrate Eq (7) over $d\tau_v$

$$P_{rad} = \frac{1}{c}F_{rad}\tau_v + A$$

This doesn't require Eddington approximation ~~isotropic blackbody radiation~~ ~~this is needed~~

at top of the atmosphere $A = P_{rad} = \cancel{\frac{2}{3c}F_{rad}} = \frac{2}{3c}\sigma T_e^4$

[This is simply a result of isotropic Blackbody radiation]

$$\Rightarrow \boxed{P_{rad} = \frac{1}{c}F_{rad}\left(\tau_v + \frac{2}{3}\right)} \quad \cdots (8)$$

because $F_{rad} = \text{const}$, Eq.(6) tells us that $\langle I\rangle = S = \int B_\lambda \, d\lambda = \frac{\sigma T^4}{\pi}$

based on Eddington approximation ($P_{rad} = \frac{4\pi}{3c}\langle I\rangle$), we can rewrite Eq.(8) as

$$S = \langle I\rangle = \frac{\sigma T^4}{\pi} = \frac{3\sigma}{4\pi}T_e^4\left(\tau_v + \frac{2}{3}\right) \Rightarrow T^4 = \frac{3}{4}T_e^4\left(\tau_v + \frac{2}{3}\right)$$

Solution: $S = a + b\tau_v = \frac{3\sigma}{4\pi} T_e^4 \cdot \tau_v + \frac{\sigma}{2\pi} T_e^4$

$$\Rightarrow a = \frac{\sigma T_e^4}{2\pi}, \quad b = \frac{3\sigma}{4\pi} T_e^4$$

Eq (4) becomes $I_\lambda(\theta) = \frac{\sigma T_e^4}{2\pi} + \frac{3\sigma}{4\pi} T_e^4 \cdot \cos\theta$

$$\frac{I_\lambda(\theta)}{I_\lambda(\theta=0)} = \frac{a + b\cdot\cos\theta}{a+b} = \frac{1 + \frac{3}{2}\cos\theta}{1 + \frac{3}{2}} = \frac{2}{5} + \frac{3}{5}\cos\theta$$

Eddington Approximation formula derived:

$$\langle I_\lambda \rangle = \frac{1}{4\pi}\int I_\lambda \, d\Omega = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi}\left[\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} I_{out} \cdot \sin\theta \, d\theta \, d\phi + \int_{\phi=0}^{2\pi}\int_{\theta=\pi/2}^{\pi} I_{in} \sin\theta \, d\theta \, d\phi\right]$$

$$= \frac{1}{4\pi}\left[2\pi \cdot I_{out} + 2\pi I_{in}\right] = \frac{1}{2}(I_{out} + I_{in})$$

$$F_{rad,\lambda} = \int I_\lambda \cdot \cos\theta \, d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda \sin\theta\cos\theta \, d\theta \, d\phi$$

$$= 2\pi\left(\int_{\theta=0}^{\pi/2} I_{out} \cdot \sin\theta\cos\theta \, d\theta + \int_{\theta=\pi/2}^{\pi} I_{in} \cdot \sin\theta\cos\theta \, d\theta\right)$$

$$= 2\pi \cdot \left(I_{out} \cdot \frac{\sin^2\theta}{2}\Big|_0^{\pi/2} + I_{in} \cdot \frac{\sin^2\theta}{2}\Big|_{\pi/2}^{\pi}\right)$$

$$= \pi(I_{out} - I_{in})$$

$$P_{rad,\lambda} = \frac{1}{c}\int I_\lambda \cos^2\theta \, d\Omega = \frac{1}{c}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda \cos^2\theta \sin\theta \, d\theta \, d\phi$$

$$= \frac{2\pi}{3c}(I_{out} + I_{in}) = \frac{4\pi}{3c}\langle I_\lambda \rangle \quad \left[\int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta \, d\theta = -\frac{1}{3}\cos^3\theta\Big|_0^{\pi/2}\right.$$

Thermodynamic Equilibrium: $S_\lambda = B_\lambda$

Imagine an ideal reflective box of hot gas & photons at equilibrium, so there is no net flow of energy between matter & radiation, they share the same T.

In this case, $dI_\lambda/ds = 0$, intensity is constant throughout the box

Using the Transfer equation, we have $I_\lambda = S_\lambda = B_\lambda(T)$

Since $S_\lambda = j_\lambda/\kappa_\lambda$ is a property of the matter, we can generalize this $S_\lambda = B_\lambda(T)$ to LTE

Is the solar photosphere in LTE?

① the scale height $H_T = \frac{T}{|dT/dr|} = 677$ km

② mean free path of H atoms (Bohr's model)

$$l = \frac{1}{n\sigma} \simeq 10^{-4}\,m, \quad \text{for } n = 10^{23}\,m^{-3},\ \sigma = \pi(2a_0)^2,\ a_0 = 0.0529\,nm$$

③ mean free path of photons

$$l = \frac{1}{\kappa_\lambda \rho} \simeq 160\,km \quad \text{for } \rho = 2\times10^{-4}\,kg\,m^{-3},\ \kappa_\lambda = 0.03\,m^2/kg \text{ @ } 500\,nm$$

Spectral line Profiles:

$$EW = \int \frac{F_{cont} - F_\lambda}{F_{cont}}\, d\lambda, \quad FWHM = (\Delta\lambda)_{1/2}$$

① natural broadening $\Delta E \Delta t = \hbar$, $E = hc/\lambda$ $\Delta\lambda \sim 10^{-5}\,nm$

$$\Delta\lambda = \frac{\lambda^2}{2\pi c}\left(\frac{1}{\Delta t_i} + \frac{1}{\Delta t_f}\right) \leftarrow \text{formal derivation}$$

$$(\Delta\lambda)_{1/2} = \frac{\lambda^2}{\pi c}\cdot\frac{1}{\Delta t_0} \leftarrow \text{using Lorentz profile}$$

② Doppler broadening $\Delta\lambda \sim 10^{-2}\,nm$

$$\sigma = \frac{\lambda}{c}\sqrt{\frac{kT}{m}}, \quad FWHM = 2\sqrt{2\ln 2}\,\sigma = \frac{2\lambda}{c}\sqrt{\frac{2kT\ln 2}{m}}$$

Gaussian Profile: $\phi(\nu) = \frac{1}{\sqrt{2\sigma^2\pi}} \exp\left[-\frac{(\nu-\nu_0)^2}{2\sigma^2}\right]$

③ Turbulence $\Delta\lambda \sim 10^{-2}\,nm$

$$FWHM = \frac{2\lambda}{c}\sqrt{v_{turb}^2 \ln 2}, \quad \sigma = \frac{\lambda}{c}\sqrt{\frac{v_{turb}^2}{2}}$$

④ Pressure broadening $\Delta\lambda \sim 10^{-5}\,nm$

damping / Lorentz profile: $\phi(\nu) = \frac{\gamma_n/4\pi^2}{(\nu-\nu_0)^2 + (\gamma_n/4\pi)^2}$

$$\Delta t_0 \simeq \frac{l}{v_{mp}} = \frac{1}{n\sigma\sqrt{2kT/m}} \leftarrow \text{average time between collisions}$$

$$\Rightarrow \Delta\lambda = \frac{\lambda^2}{\cancel{2}\pi c}\cdot\frac{1}{\Delta t} \simeq \frac{\lambda^2}{c}\,\frac{n\sigma}{\pi}\sqrt{\frac{2kT}{m}}, \quad \Delta\lambda = 2\gamma,\ \gamma = \frac{\gamma_n}{4\pi}$$

Curve of Growth: determine N_a by comparing calculated & observed line profiles

oscillator strength: the relative probability of an e^- making a transition from the same initial orbital to different orbitals.

$f = 0.637$ for $H\alpha$, $f = 0.119$ for $H\beta$. fN_a: effective Column density

Absorption Clouds

$$W = \frac{1}{f_0}\int [f_0 - f(\lambda)]\, d\lambda = \int_0^\infty [1 - e^{-\tau(\lambda)}]\, d\lambda$$

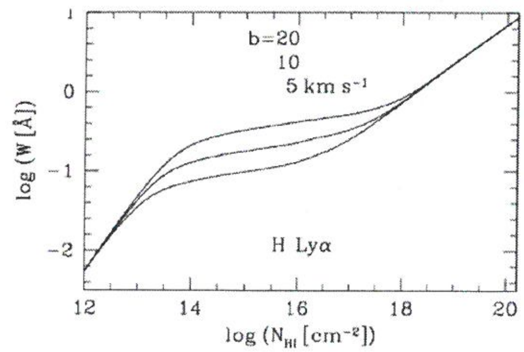
$$\tau(\lambda) = N\sigma(\nu) = N \cdot \frac{1}{\sqrt{\pi}}\frac{c}{b}\frac{a_{ij}}{\nu_{ij}} \cdot \text{Voigt}\Big(\underbrace{\frac{c}{b}\frac{\gamma}{\nu}}_{A_\nu},\ \underbrace{\frac{c}{b}\frac{\nu - \nu_{12}}{\nu}}_{B_\nu}\Big)$$

b is the Doppler parameter for thermal + turbulent motion : $b = \sqrt{2}\,\sigma$

$$b^2 = \frac{2kT}{m} + b_{turb}^2$$

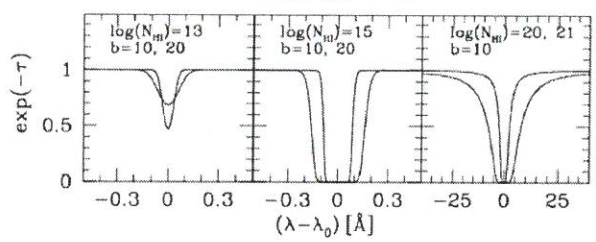
a_{ij} is another way of writing the oscillator strength f_{ij}

$$a_{ij} = \frac{\pi q_e^2}{m_e c} f_{ij}$$

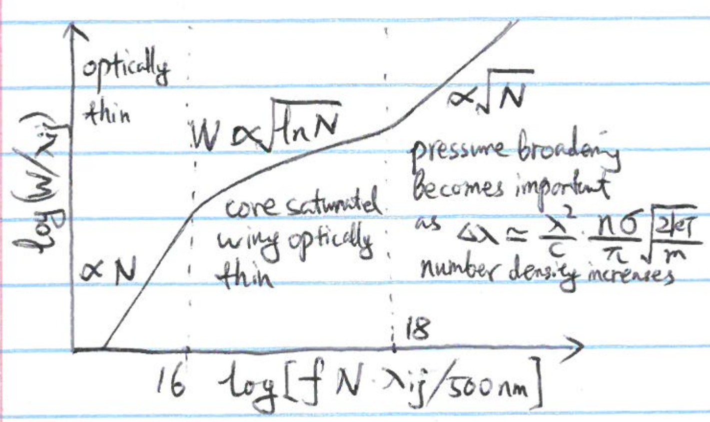


Clearly $\tau(\lambda)$ depends on ① $f_{ij} \cdot N \cdot \lambda_{ij}$
and ② the b parameter

Also W scales w/ wavelength because the absorption line show the same velocity profile at different wavelengths : $\frac{\Delta\lambda}{\lambda} \propto b$
so for the same velocity profile $W \propto \lambda_{ij}$



So we make a $\boxed{\text{general curve of growth}}$
by plotting W/λ_{ij} vs. $f N \cdot \lambda_{ij}$ (Fig 9.22)



Voigt Profile: $\phi(\nu)$

$$\mathcal{L}(\nu) = \frac{1}{\pi}\frac{\gamma}{(\nu - \nu_{12})^2 + \gamma^2}$$

$$\text{FWHM}(\mathcal{L}) = 2\cdot\gamma \Rightarrow \gamma = \frac{\lambda^2}{2\pi c}\frac{1}{\Delta t_0}$$

$$\mathcal{D}(\nu) = \frac{1}{\sqrt{\pi}\, b}\exp\left(-\frac{\nu^2}{b^2}\right)$$

$$b = \sqrt{2}\,\sigma = \sqrt{\frac{2kT}{m} + b_{turb}^2}$$

$$\phi(\nu) = \mathcal{L}(\nu) \otimes \mathcal{D}(\nu) = \frac{1}{\sqrt{\pi}}\frac{c}{b}\frac{V(A,B)}{\nu}$$

where $V(A,B) = \frac{A}{\pi}\int_{-\infty}^{\infty}\frac{\exp(-y^2)\,dy}{(B-y)^2 + A^2} \simeq \exp(-B^2) + \frac{1}{\sqrt{\pi}}\frac{A}{A^2 + B^2}$

Voigt function

$$A = \frac{c}{b}\frac{\gamma}{\nu}, \quad B = \frac{c}{b}\frac{\nu - \nu_{12}}{\nu}$$

Chap 10 : Stellar Interior

Equations of Stellar Structure

① $\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$ (Hydrostatic Equilibrium)

e.g. $P_c \simeq \frac{G(M_\odot/2)\rho_\odot}{(R_\odot/2)^2} = \frac{8\pi}{3}G\rho_\odot^2 R_\odot^2 \propto GM\rho/R$, $\rho_\odot \simeq 10^3\ kg\,m^{-3}$

$\simeq 5\times10^9$ atm

② $dM = 4\pi r^2\rho(r)\,dr$ (mass continuity)

③ $P(r) = \frac{\rho(r)kT(r)}{\mu_\odot m_p}$ (gas pressure, ideal) $\mu_\odot \simeq 0.6$

$P_{rad}(r) = \frac{4\sigma}{3c}T^4$ (radiation pressure)

e.g. $T_c = P_c \cdot \frac{\mu m_p}{\rho_\odot k} = \frac{2GM_\odot \mu_\odot m_p}{R_\odot k} \simeq 3\times10^7$ K

$T_c \propto \frac{M\mu}{R}$

④ Equation of radiative/convective energy transport. ⑤ Equation of Energy generation

Mean molecular mass.

$\rho = \rho_H + \rho_{He} + \rho_{metal} \Rightarrow 1 = X + Y + Z$ ← mass fractions

$P = nkT = \frac{\rho kT}{\mu m_p} \Rightarrow \mu = \frac{\rho}{n m_p}$, where n is the total # density

② for fully ionized gas

\# of proton = # of neutron

$$n = \underbrace{2\cdot\rho\cdot X/m_p}_{n_H+n_{e^-}} + \underbrace{3\rho\,Y/4m_p}_{n_{He}+n_{e^-}} + \underbrace{\rho\,Z/2m_p}_{n_Z+n_{e^-}}$$

$$= \frac{\rho}{m_p}\left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right) \Rightarrow \mu = \left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right)^{-1}$$

① for neutral gas

$$n = \underbrace{\rho X/m_p}_{n_H} + \underbrace{\rho Y/4m_p}_{n_{He}} + \underbrace{\rho Z/Am_p}_{N_Z}$$

$$= \frac{\rho}{m_p}\left(X + \frac{1}{4}Y + Z/A\right)$$

Equation of radiative energy transport.

$P_{rad} = \frac{4\sigma}{3c}T^4$, $a = \frac{4\sigma}{c}$

(a) radiative force on a thin shell @ r=r , $P_{rad} = \frac{a}{3}T^4 \Rightarrow dP_{rad} = \frac{4a}{3}T^3 dT$

$$F_{rad} = [P_{rad}(r) - P_{rad}(r+dr)]\cdot 4\pi r^2 = -\frac{16\pi}{3}ar^2T^3dT$$

(b) $F_{rad} = \frac{L(r)}{c}d\tau = -\frac{L(r)}{c}\rho(r)\kappa(r)dr$

Combining (a) & (b)

$$\frac{dT}{dr} = -\frac{3\rho(r)\,\kappa(r)\,L(r)}{16\pi a c\, T(r)^3 r^2}, \quad \text{where } a = \frac{4\sigma}{c}$$

$$= -\frac{3\rho(r)\,\kappa(r)\,L(r)}{64\pi \sigma_{SB}\, T(r)^3 r^2}$$

we can estimate the mean T gradient based on T_c, T_{surf} & $R_\odot$

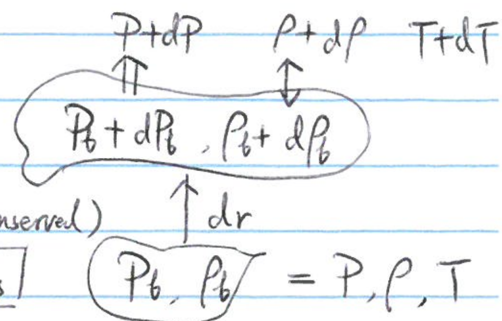
$$\left\langle \frac{dT}{dr} \right\rangle = -\frac{T_c}{R_\odot} \simeq -20\,\mathrm{K\,km^{-1}}$$

we can also estimate $L_\odot$, $\frac{\Delta T}{\Delta r} = -\frac{T_c}{R_\odot} = -\frac{3\langle\kappa\rangle \rho_\odot L_\odot}{64\pi\sigma_{SB}(T_c/2)^3 (R_\odot/2)^2}$

$$L_\odot \simeq \frac{2\pi\sigma_{SB}\cdot T_c^4 R_\odot}{3\langle\kappa\rangle\rho} \simeq 3\times10^{27}\,\mathrm{W}\left(\frac{\langle\kappa\rangle}{1\,\mathrm{m^2\,kg^{-1}}}\right)^{-1}$$

$$L_{\odot,obs} = 3.9\times10^{26}\,\mathrm{W} \Rightarrow \langle\kappa\rangle = 8\,\mathrm{m^2\,kg^{-1}}$$

Equation of convective energy transport



condition for stability against convection:

$$\rho_b + d\rho_b > \rho + d\rho \Rightarrow d\rho_b > d\rho$$

assume adiabatic process (no exchange of heat, entropy conserved)

$$PV^\gamma = \text{const}, \quad \boxed{\gamma = 5/3 \text{ for atomic/ionized gas}}$$

hydrostatic equilibrium $P_b + dP_b = P + dP \Rightarrow dP_b = dP$

$$V \propto \frac{1}{\rho}$$

$$PV^\gamma = \text{const} \Rightarrow P\rho^{-\gamma} = \text{const} \Rightarrow dP = \gamma\cdot\rho^{\gamma-1}\cdot\text{const}\cdot d\rho$$

$$\Rightarrow \frac{d\rho_b}{\rho_b} = \frac{1}{\gamma}\frac{dP_b}{P_b}$$

$$\Rightarrow \left.\begin{aligned} d\rho_b &= \frac{\rho}{\gamma}\frac{dP}{P} = \frac{\rho}{\gamma P}\frac{dP}{dr}\cdot dr \\ d\rho &= \frac{d\rho}{dr}\cdot dr \end{aligned}\right\} + \begin{gathered}\text{Stability condition} \\ d\rho_b > d\rho \Rightarrow\end{gathered}$$

$$\boxed{\frac{\rho}{\gamma P}\cdot\frac{dP}{dr} > \frac{d\rho}{dr}}$$

$d\rho/dr$ now we need to convert this into T gradient (dT/dr)

for ideal gas $P = nkT = \rho kT/\mu m_p$, $\mu = \frac{\rho}{n m_p}$ (the mean molecular mass)

$$\frac{dP}{dr} = nk\frac{dT}{dr} + \frac{kT}{\mu m_p}\frac{d\rho}{dr} = \frac{P}{T}\frac{dT}{dr} + \frac{P}{\rho}\frac{d\rho}{dr}$$

the stability condition $d\rho_b > d\rho \Rightarrow \frac{1}{\gamma P}\frac{dP}{dr} > \frac{1}{\rho}\frac{d\rho}{dr}$

$$\Rightarrow \frac{1}{\gamma P}\frac{dP}{dr} > \frac{1}{P}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr}$$

$$\underbrace{-\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}}_{\text{adiabatic T gradient}} > \underbrace{-\frac{dT}{dr}}_{\text{actual T gradient}} \Leftrightarrow \left|\frac{dT}{dr}\right|_{\text{adiabatic}} > \left|\frac{dT}{dr}\right|_{\text{actual}}$$

when actual T gradient is greater than the adiabatic T gradient, convection ensues; convection then brings T gradient back to adiabatic T gradient. Thus the equation of convective energy transport is:

$$\frac{dT}{dr} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} \Leftrightarrow \frac{T\,dP}{P\,dT} < \frac{\gamma}{\gamma-1}$$

Equation of energy generation:

$$dL/dr = 4\pi r^2 \rho(r) \underset{\text{rate of energy production}}{\epsilon(r)}$$

Timescales:

Kelvin-Helmholtz $\quad \tau_{KH} = \frac{GM_\odot^2/R_\odot}{L_\odot} \simeq 50\ \text{Myr}$

Nuclear Fusion $\quad \tau_{fus} = \frac{N_H/4 \cdot \Delta E}{L_\odot} \simeq 100\ \text{Gyr}$

$$\tau_{fus} \simeq 10\ \text{Gyr} \cdot (M/M_\odot)^{-3}$$

Equations of Stellar Structure

$$\begin{cases} \frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} & \frac{dM}{dr} = 4\pi r^2 \rho(r) \\ \frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{64\pi\sigma_{SB} r^2 T^3} \quad \text{or} & \frac{dT}{dr} = \left(1-\frac{1}{\gamma}\right)\frac{T(r)}{P(r)}\frac{dP}{dr} \\ \frac{dL}{dr} = 4\pi r^2 \rho(r)\epsilon(r) & P(r) = \frac{k\rho(r)T(r)}{\mu(r) m_p} \end{cases}$$

Plus $\mu(\rho,T)$, $\kappa(\rho,T)$, & $\epsilon(\rho,T)$ ← ~~constituent~~ constitutive relations

And boundary condition at surface ⇒ $P(r)$, $\rho(r)$, $T(r)$, $L(r)$, & $M(r)$

Vogt-Russell Theorem: Mass & Composition determines $R_\odot$, $L_\odot$, $\rho(r)$, $T(r)$, $P(r)$ & evolution

Mixing length theory $\rightarrow$ convective flux & convective velocity (skipped)

Derivation of Adiabatic temperature gradient

① ideal gas law $P = \frac{\rho}{\mu m_p} \cdot kT \Rightarrow \frac{dP}{dr} = \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$

② adiabatic gas law $P = K \cdot \rho^\gamma \Rightarrow \frac{dP}{dr} = \gamma \frac{P}{\rho}\frac{d\rho}{dr}$

combining ① & ②, eliminate $d\rho/dr$

$$\left.\frac{dT}{dr}\right|_{adiabatic} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

further, we can apply ideal gas law & the equation of hydrostatic equilibrium

$$\left.\frac{dT}{dr}\right|_{adiabatic} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2} = -\frac{g_r}{C_p}$$

recall $\gamma = C_p/C_v$, $nR = \frac{k}{\mu m_H}$, $C_p - C_v = nR$

$$\frac{\mu m_H}{k}\left(1 - \frac{1}{\gamma}\right) = \frac{1}{nR} \cdot \left(\frac{C_p - C_v}{C_p}\right) = \frac{1}{C_p}$$

Polytropic Models: ① $\frac{dP}{dr} = -G\frac{M_r \cdot \rho}{r^2}$ ② $dM = 4\pi r^2 \rho\, dr$

③ $P = K \cdot \rho^\gamma$, $\gamma = (n+1)/n \Rightarrow n = \frac{1}{\gamma - 1}$

differentiate ①, $\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM_r}{dr} = -4\pi r^2 G\rho$

i.e. $\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$

plug in the polytrope relation ③,

$$\frac{\gamma K}{r^2}\frac{d}{dr}\left[r^2 \rho^{\boxed{\gamma-2}}\frac{d\rho}{dr}\right] = -4\pi G\rho, \quad \gamma - 2 = \left(\frac{1}{n} - 1\right) = \frac{1-n}{n}$$

we can simplify this equation by using $\rho = \rho_c [D_n(r)]^n$, $r = \lambda_n \xi = \lambda_n x$

where $\lambda_n = \left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]^{1/2}$ surface condition

$\rightarrow$ Solution $D_1(x) = \frac{\sin x}{x}$, $x_1 = \pi$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{dD_n}{d\xi}\right] = -D_n^n \quad \leftarrow \text{Lane-Emden Equation}$$

Total mass of the star $M = 4\pi\int_0^R r^2\rho\, dr = 4\pi\int_0^{\xi_1}\lambda_n^2\xi^2\rho_c D_n^n\, d(\lambda_n\xi)$

$M = 4\pi\lambda_n^3\rho_c\int_0^{\xi_1}\xi^2 D_n^n\, d\xi = -4\pi\lambda_n^3\rho_c\left[\xi_1^2 \left.\frac{dD_n}{d\xi}\right|_{\xi_1}\right]$ because of the LE Equation

Stellar Energy Sources : Gravity vs. Nuclear Fusion

Kelvin-Helmholtz Timescale:

gravitational potential energy of dm shell located at r:

$$dU_g = -G\frac{M_r}{r}\cdot 4\pi r^2 \rho\, dr$$

$$\Rightarrow \quad U_g = -4\pi G \int_0^R M_r \rho r\, dr$$

assuming constant density $\rho = M/\frac{4}{3}\pi R^3$, $M_r = \frac{4}{3}\pi r^3 \rho$

$$\Rightarrow \quad U_g = -4\pi G \int_0^R \frac{4}{3}\pi r^3 \cdot \frac{9M^2}{16\pi^2 R^6}\cdot r\, dr = -3GM^2 \int_0^R r^4 dr \cdot \frac{1}{R^6}$$

$$= -\frac{3}{5}\frac{GM^2}{R}$$

applying virial theorem $\langle E\rangle = \frac{1}{2}\langle U\rangle = -\frac{3}{10}\frac{GM^2}{R}$, total energy

KE timescale $= \frac{-\langle E\rangle}{L_\odot} = \frac{3}{10}\frac{GM_\odot^2}{R_\odot L_\odot} \sim 10^7\ yr = 10\ Myr$

Nuclear timescale $= \frac{E_{nuclear}}{L_\odot} = \frac{10\% \times 0.007 \times M_\odot c^2}{L_\odot} \simeq 10\ Gyr$

$4H \rightarrow He$ releases the binding energy of $He = 0.007 \cdot m_H \cdot c^2$

$1u = \frac{1}{12} m({}^{12}_{6}C) = 931.5\ MeV/c^2$

$4H = 4.0313\,u$, $1\,He = 4.002603\,u \Rightarrow \Delta m = 0.0287\,u = 0.7\% 4H$

Importance of Quantum Mechanical Tunnelling

Condition for fusion to happen is to bring two protons within the size of the proton which requires the kinetic energy to be greater than the potential energy at $r \sim 10^{-15} m$

$$\frac{1}{2}\mu_m \bar{v}^2 = \frac{3}{2}kT \geq \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r} \Rightarrow T \geq 10^{10} K \gg T_c \sim 10^7 K$$

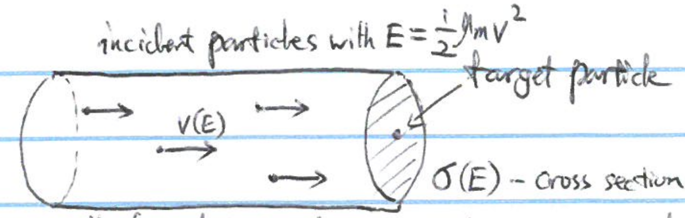
Taking into account quantum tunnelling, the kinetic energy only needs to overcome the Coulomb barrier at ~ de Broglie wavelength, $\lambda = h/p$

$$\frac{1}{2}\mu_m \bar{v}^2 = \frac{3}{2}kT = \frac{p^2}{2\mu_m}$$, where $\mu_m = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_p}{2}$, the reduced mass

$$\frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{\lambda} = \frac{(h/\lambda)^2}{2\mu_m}, \Rightarrow \lambda = \frac{2\pi\epsilon_0 h^2}{Z_1 Z_2 e^2 \mu_m}$$

$$\frac{3}{2}kT \geq \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{\lambda} = \frac{Z_1^2 Z_2^2 e^4 \mu_m}{8\pi^2 \epsilon_0^2 h^2} \Rightarrow T \geq \frac{Z_1^2 Z_2^2 e^4 \mu_m}{12\pi^2 \epsilon_0^2 h^2 k} \sim 10^7 K$$

Nuclear Reaction Rate



① Maxwell - Boltzmann Distribution

$$\underbrace{n_E\, dE}_{} = \frac{2n}{\sqrt{\pi}} \frac{1}{(kT)^{3/2}} E^{1/2} e^{-E/kT} dE$$

density of all particles with energy [E, E+dE]

of incident particles that can hit target particle

$$dN_E = \underbrace{\sigma(E)\, v(E)\, dt}_{\text{volume of cylinder}} \cdot \underbrace{n_{iE}\, dE}_{\text{number density of incident particles within } (E,\, E+dE)}$$

② # of incident particles per target particle over dt period, dN_E

$$\frac{dN_E}{dt} = \sigma(E)\, v(E)\, n_{iE}\, dE = \sigma(E)\, v(E) \cdot \frac{n_i}{n} \cdot n_E\, dE$$

where $n_i = \int n_{iE}\, dE$, $n = \int n_E\, dE$ are the total number densities of incident particles & all particles.

$$\frac{dN_E}{dt} = \text{\# of reactions per target particle per unit time}$$

③ # of reactions per unit volume per unit time (reaction rate per volume)

$$r_{ix} = \int n_x \cdot \frac{dN_E}{dt} = \int_0^\infty n_x \cdot n_i\, \sigma(E)\, v(E) \cdot \frac{n_E}{n} dE$$

④ cross section 's dependency on energy

Barrier potential height

$$\sigma(E) = S(E) \cdot \frac{1}{E} \cdot e^{-b/\sqrt{E}} \leftarrow \text{tunnelling probability} \propto e^{-2\pi^2 U_c/E}$$

$$\sigma(E) \propto \pi \lambda^2 = \pi \left(\frac{h}{p}\right)^2 \propto \frac{1}{E} \quad \text{de Broglie wavelength}$$

$$\frac{U_c}{E} = \frac{Z_1 Z_2 e^2 / 4\pi\epsilon r}{\mu v^2/2}$$

$r = \lambda = h/p = \frac{h}{\mu_m v}$ ← de Broglie wavelength

$$\frac{U_c}{E} = \frac{Z_1 Z_2 e^2}{2\pi \epsilon_0 h v} \propto \frac{1}{\sqrt{E}}$$

⑤ substituting Maxwell distribution & cross section to r_{ix}

$$r_{ix} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{(\mu_m \pi)^{1/2}} \int_0^\infty S(E)\, \underline{e^{-bE^{-1/2}}\, e^{-E/kT}}\, dE$$

Gamov peak occurres because the function

$$f(E) = e^{-bE^{-1/2}} \cdot e^{-E/kT} = e^{-\frac{b}{\sqrt{E}} - \frac{E}{kT}}$$

peaks at $\cancel{E_0 = \sqrt{\frac{bkT}{2}}}$ $E_0 = \left(\frac{bkT}{2}\right)^{2/3}$ where $b = \frac{\pi \mu_m^{1/2} Z_i Z_x e^2}{\sqrt{2} \cdot \epsilon_0 h}$

for p-p collision/fusion, $E_0 = 6\ \text{keV} \left(\frac{T}{10^7 K}\right)^{2/3}$

recall $1\ \text{eV} \sim 10^5 K \cdot k$, $6\ \text{keV} \sim 6 \times 10^8 K \cdot k$

⑥ Power-law approximations : $r_{ix} \simeq r_0\, X_i\, X_x\, \rho^{\alpha'}\, T^{\beta}$

⑦ ϵ : energy generation rate per unit mass $dL = \epsilon \cdot dm$

$\rho \cdot \epsilon_{ix} = r_{ix} \cdot \mathcal{E}_0$, where $\mathcal{E}_0$ is the energy generated per reaction

$\Rightarrow \epsilon_{ix} = \left(\frac{\mathcal{E}_0}{\rho}\right) \cdot r_{ix} = \epsilon_0 X_i X_x \rho^{\alpha'-1} T^{\beta}$ [W/kg]

Now it's interesting to think about $\epsilon_{gravity} = \frac{1}{M} \cdot \frac{dU_g/2}{dt}$

for homogeneous stars $\frac{1}{2}U_g = E = \frac{-3}{10}\frac{GM^2}{R}$

$$\epsilon_{gravity} = \frac{3}{10}\frac{GM}{R^2}\frac{dR}{dt}$$

Neutrinos : $m_\nu < 2.2\ eV/c^2$, $m_{e^-} = 0.5\ MeV/c^2$, $m_p = 1\ GeV/c^2$

PP chain : $4\,{}^1_1H \rightarrow {}^4_2He + 2e^+ + 2\nu_e + 2\gamma$ [How neutrino is produced? $p^+ \rightarrow n + e^+ + \nu_e$] β decay

low mass MS $\epsilon_{pp} \simeq 10^{-12}\ W\,m^3\,kg^{-2} \cdot \rho\, \boxed{X^2 \cdot T_6^4}$ @ $T_6 \sim 15$

$\epsilon_{pp} = 0.24\,\rho\, X^2\, T_6^{-2/3}\, e^{-33.8\,T_6^{-1/3}}$ $W\,kg^{-1}$

CNO cycle : $\epsilon_{CNO} \simeq 8\times10^{-31}\ W\,m^3\,kg^{-2} \cdot \rho \cdot \boxed{X} \cdot X_{CNO}\, \boxed{T_6^{19.9}}$ @ $T_6 \sim 15$

high mass MS $\epsilon_{CNO} = 8.67\times10^{20}\,\rho\, X \cdot X_{CNO} \cdot T_6^{-2/3}\, e^{-152.28\,T_6^{-1/3}}$ $W\,kg^{-1}$

Triple α : $2\,{}^4_2He \Leftrightarrow {}^8_4Be$, ${}^8_4Be + {}^4_2He \rightarrow {}^{12}_6C + \gamma$

Horizontal branch $\epsilon_{3\alpha} \simeq \epsilon'_{0,3\alpha}\,\rho^2 \boxed{Y^3\, T_8^{41}}$ [three body interaction, $r_{ix} \propto (\rho Y)^3$]

$\epsilon_{3\alpha} = 51\,\rho^2\, Y^3\, T_8^{-3}\, e^{-44\,T_8^{-1}}$ $W\,kg^{-1}$ @ $T_8 \simeq 1$

α capturing ${}^{12}_6C + {}^4_2He \rightarrow {}^{16}_8O + \gamma$

produces O, Ne, Na, Mg, Si, P, S

binding energy per nucleon : $\frac{E_b}{A} = \frac{\Delta m c^2}{A} = [Z m_p + (A-Z)m_n - m_{nucleus}]c^2/A$

peaks at ${}^{56}_{26}Fe$ at ~ 9 MeV/nucleon

CNO cycle : ${}^{12}_6C + {}^1_1H \rightarrow {}^{13}_7N + \gamma$

${}^{13}_7N \rightarrow {}^{13}_6C + e^+ + \nu_e$

${}^{13}_6C + {}^1_1H \rightarrow {}^{14}_7N + \gamma$

${}^{14}_7N + p^+ \rightarrow {}^{15}_8O + \gamma$

${}^{15}_8O \rightarrow {}^{15}_7N + e^+ + \nu_e$

${}^{15}_7N + p^+ \rightarrow {}^{12}_6C + {}^4_2He$

total reaction $4p^+ \rightarrow {}^4_2He + 2e^+ + 2\nu_e + 3\gamma$

PP chain : $p^+ + p^+ \rightarrow {}^2_1H + e^+ + \nu_e$, ${}^2_1H + p^+ \rightarrow {}^3_2He + \gamma$, $2\,{}^3_2He \rightarrow {}^4_2He + 2p^+$

Faint Young Sun Paradox

Stars become more luminous as they evolve along/within the main sequence (H-burning)

Start with the Lane-Emden equation for polytropic models

$$\frac{\gamma \cdot K}{r^2} \frac{d}{dr}\left[r^2 \rho^{\gamma-2} \frac{d\rho}{dr}\right] = -4\pi G \rho$$

the solution is $\rho(r, \gamma, K)$, where $K = P \cdot \rho^{-\gamma}$ is a constant

during the MS evolution, γ & K remain constant, so the density structure of the star remains essentially the same, i.e., the core density ρ_c is constant

Given $P = \frac{\rho}{\mu m_p} \cdot kT$ & $P = K \cdot \rho^{\gamma}$

we have $kT = \mu m_p \cdot K \cdot \rho^{\gamma-1}$ or $T \propto \mu \cdot \rho^{\gamma-1}$

so the core temperature should increase as $\mu \nearrow$ because $4\,{}^1_1H \rightarrow {}^2_4He + 2e^+ + 2\nu$

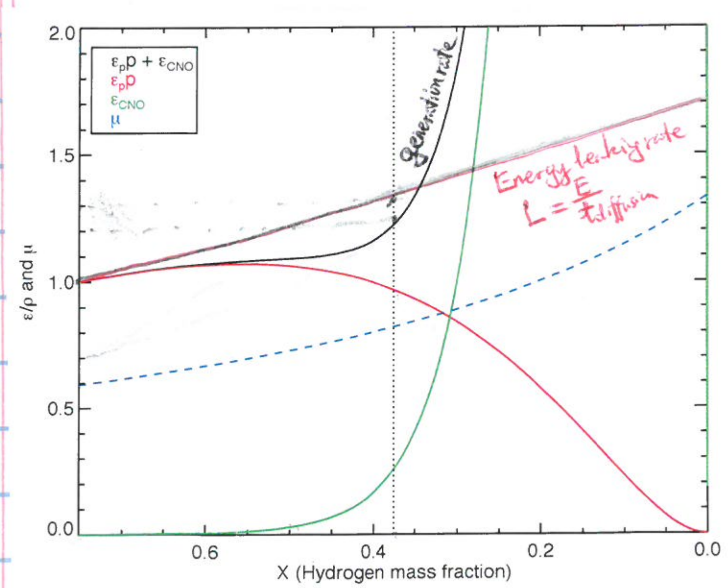
For pp chain, $\epsilon_{pp} \propto \cancel{\rho X^2 T_6^4}\ \rho X^2 \cdot T_6^{-2/3} e^{-33.8 T_6^{-1/3}}$ W kg^{-1}

$T_6 = 15\,(\mu/0.8) = T/10^6$K because currently $\mu_0 = 0.8$ at the core

$\mu \simeq (2X + \frac{3}{4}Y)^{-1} = \frac{4}{5X+3}$

Plug the above two to ϵ_{pp}, we can calculate ϵ_{pp} as a function of X

CNO cycle becomes important as $X \searrow$ and $T \nearrow$, so we compute ϵ_{CNO} as well



see ms_evolution.pro

since $\frac{dL}{dM} = \epsilon$, if the core mass does not change, the solar luminosity has increased by 20% since it became a MS star 4.5 Gyrs ago

More sophisticated models predicts a change of 30%

Another way to explain the MS evolution is to assume the core T does not evolve, but the star becomes less opaque due to loss of e^- density

Ken Grayley: from radiative energy transport: $\frac{T_c}{R} = \frac{3\rho\kappa L}{64\pi\sigma_{SB} T_c^3 R^2} \Rightarrow L \propto \frac{T_c^4 \cdot R}{\kappa}$, $\kappa \propto n_e = \frac{\rho}{m_p}(X + \frac{1}{2}Y) = \frac{\rho}{2m_p}(1+X)$

Virial theorem & L-M relation of stars (Ken Gayley)

virial theorem $K = -\frac{1}{2}U \Rightarrow \frac{M}{\mu m_p} \cdot kT = \frac{GM^2}{R} \Rightarrow kT = \frac{\mu m_p \cdot GM}{R}$

luminosity of a star is determined by radiative diffusion rate

$$L = \frac{E}{t_{diffusion}} = \frac{aT^4 \cdot R^3}{R/(c/\tau)}, \quad \tau = \kappa \rho R = \kappa \frac{M}{R^3} \cdot R$$

$$\Rightarrow L \propto T^4 R^4 / M \cdot \kappa \propto \frac{\mu^4 \cdot M^4}{R^4} \cdot \frac{R^4}{M \cdot \kappa} \propto \frac{\mu^4}{\kappa} \cdot M^3$$

$\Rightarrow L \propto M^3$ with no R dependency & T dependency

Proving $t_{diffusion} = R/[c/\tau]$, effective speed of light $= c/\tau$

① random walk with mean free path l

$D = l \cdot \sqrt{N} \Rightarrow N = \left(\frac{R}{l}\right)^2$ is the number of scatting required

② definition of optical depth & mean free path ($\kappa \rho l = 1$)

$$\tau = \kappa \rho \cdot R = \frac{R}{l}$$

$$\Rightarrow t_{diffusion} = \frac{N l}{c} = \frac{\tau^2 \cdot l}{c} = \tau^2 \cdot \frac{R}{\tau} \cdot \frac{1}{c} = R/[c/\tau] = \frac{\tau \cdot R}{c}$$

$$\Rightarrow L = E/t_{diffusion} \propto \frac{1}{\tau} \propto \frac{1}{\kappa}$$

If the Sun was 30% less luminous 4 Ga, what was the surface temperature of the Earth?

Solar input energy: $P_{in} = \pi R_\oplus^2 \cdot S(1-\text{albedo})$, $S = L_\odot / 4\pi d^2$

Earth's radiative energy: $P_{out} = 4\pi R_\oplus^2 \cdot \epsilon \sigma_{SB} T_\oplus^4$, ϵ is surface emissivity

Equalibrium condition $P_{in} = P_{out}$

$$\Rightarrow \quad T_\oplus{}^4 = \frac{S(1-\text{albedo})}{4\epsilon \cdot \sigma_{SB}} = \frac{L_\odot(1-\text{albedo})}{16\pi\epsilon\sigma_{SB} \cdot d^2}$$

Green house gas increases the surface temperature by $\left(1+\frac{3\tau}{4}\right)$, τ the optical depth

$$\Rightarrow \quad T_\oplus{}^4 = \frac{L_\odot(1-\text{albedo})}{16\pi\epsilon\sigma_{SB} \cdot d^2}\left(1+\frac{3\tau}{4}\right)$$

$T_\oplus = 288\text{ K}$ (15 °C) at the present $L_\odot$

$\Rightarrow T_\oplus = 263\text{ K}$ when the Sun had 70% of $L_\odot$ today, which is below freezing

Geological evidence for liquid water 4 Gyrs ago.

(1) Oxygen isotopes in Jack Hills zircons (dated 4.4 - 4.3 Ga), suggests parent rock interactions with liquid water

(2) 3.8 Ga old sedimentary rocks from West Greenland suggests large & deep ocean.

Kelvin-Helmholtz timescale of the Earth $\rightarrow$ 20 Myr too short to maintain an ocean

$$\tau = \frac{GM^2}{R\sigma 4\pi R^2 \cdot T^4} \propto \frac{M^2}{R^3 T^4}, \quad \tau_\odot = 10^7\text{ yr}$$

$M_\odot = 2\times10^{30}\text{ kg}$, $R_\odot = 7\times10^8\text{ m}$, $T_\odot = 5800\text{ K}$ $\Rightarrow \tau_\odot = 10^7\text{ yr}$

$M_\oplus = 6\times10^{24}\text{ kg}$ $R_\oplus = 6.4\times10^6\text{ m}$, $T_\oplus = 288\text{ K}$ $\Rightarrow \tau_\oplus = 2\times10^7\text{ yr}$

Ocean freezing timescale $\rightarrow$ 100 years, really short compared to other timescales.

$V = 4\pi R_\oplus^2 \cdot h$, $A = 4\pi R_\oplus^2$, $\tau \propto h$ the depth of the ocean

for a bottle of water with diameter of 5 cm, it takes 0.5 hr to freeze:

$V = \frac{\pi D^2}{4} \cdot H$, $A = \pi D \cdot H$ $\Rightarrow$ $\tau \propto \text{Diameter} \simeq 1\text{ hr}/10\text{ cm}$

suppose the ocean is 10 km deep. $\tau = \frac{10\text{ km}}{10\text{ cm}}\text{ hr} \simeq 10^5\text{ hr} \simeq 10\text{ yrs}.$

Chap 12: ISM & Star Formation

Closed box system

$$ISM \xrightarrow{\text{Star formation}} Stars$$

stellar evolution

Interstellar Extinction:

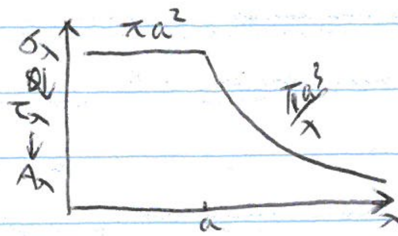
$$m_\lambda = M_\lambda + 5\log d - 5 + A_\lambda \iff m_{\lambda,obs} = m_{\lambda,int} + A_\lambda$$

radiative transfer for pure absorption $\quad F_{\lambda,obs} = F_{\lambda,int} \cdot e^{-\tau_\lambda}$

given the definition of magnitude $\quad m_\lambda = -2.5\log F_\lambda / F_{\lambda,ref}$

$$A_\lambda = m_{\lambda,obs} - m_{\lambda,int} = -2.5\log F_{\lambda,obs}/F_{\lambda,int} = 2.5\cdot\tau_\lambda\log e = 1.086\,\tau_\lambda$$

$$\tau_\lambda = \int_0^s n_d(s')\,\sigma_\lambda\,ds' \simeq \underbrace{\sigma_\lambda}_{\text{cross section}}\ \underbrace{N_d}_{\text{column density}}$$



The Mie Theory

$\sigma_g = \pi a^2 \rightarrow$ geometric cross section

define extinction coeff $\quad Q_\lambda \equiv \sigma_\lambda/\sigma_g \begin{cases} \text{when } \lambda \gtrsim a, & Q_\lambda \sim a/\lambda \\ \text{when } \lambda \lesssim a, & Q_\lambda \sim \text{const} \end{cases}$

$$\Rightarrow \sigma_\lambda \propto a^3/\lambda \ (\lambda \gtrsim a) \ / \ a^2 \ (\lambda \ll a)$$

Application:

Estimate the amount of extinction (A_λ) @ ~~0.5 μm~~ @ 1.0 μm from interstellar dust with $\bar{a} \sim 0.2\,\mu m$, $\bar{n} \sim 10^{-13}\,cm^{-3}$, and a column/distance of 1 kpc.

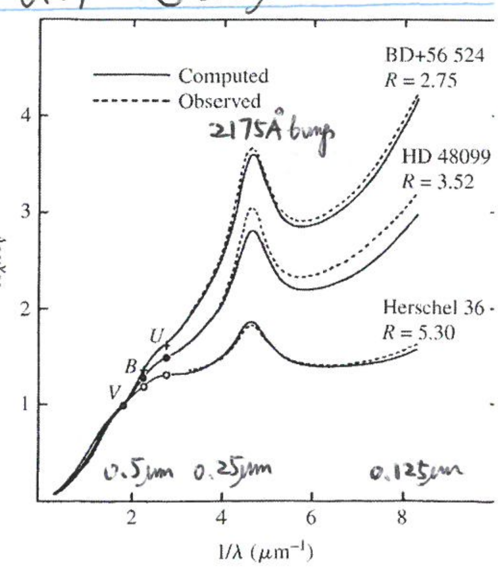
$$A_\lambda = 1.086\,\tau_\lambda = 1.086\cdot\pi a^3/\lambda\cdot\bar{n}\cdot d = 0.17 \quad @\,0.5\,\mu m$$

Observed extinction curves

color excess $\quad E(B-V) = -(B-V)_{intrinsic} + (B-V)_{obs}$

Page 402, wrong definition $\quad = -A_V + A_B = A_B - A_V$

2175Å bump → graphite or PAHs

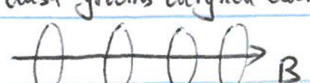


Polarization of scattered light

a few percent polarization & level of Polarization depends on λ, this has two implications:

① non-spherical dust grains

② dust grains aligned along a unique direction by B



long axis perpendicular to B direction, ~~polarization angle aligned with B~~

Gas component of ISM

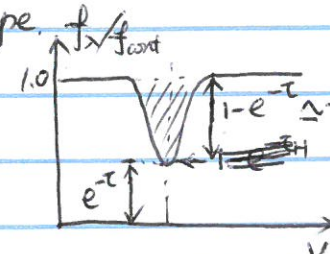
H I 21 cm line, transition from alighed spin to anti-aligned spin

Spontaneous emission timescale $\sim 10^6$ yrs

collisional time scale in low density medium $\sim 10^2$ yrs, $\tau = \frac{1}{\pi n \cdot v \cdot a_0^2}$

here we assumed $v \simeq 10$ km/s, $n = 1\ cm^{-3}$, $a_0 = 5\times10^{-11}$ m (Bohr radius)

This line is mostly optically thin and has a Gaussian shape.



Based on the linear part of the Curve of Growth, we have

$EW \propto N_H \cdot f_{ground}$, f_{ground} is the fraction in ground level

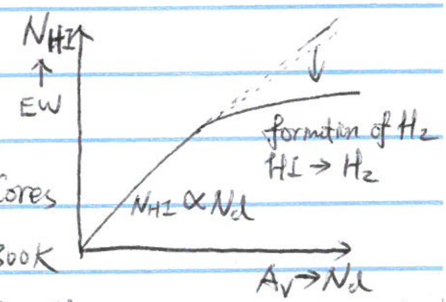
For a Gaussian, we have $EW = 1.064 \times Peak \times FWHM$

where $Peak = 1 - e^{-\tau_H} \simeq \tau_H$ at optically thin limit, τ_H is the opt depth at line cent

$\Rightarrow \quad \tau_H \propto \frac{N_H \cancel{f_{ground}}}{FWHM} \propto \frac{N_H}{T \cdot FWHM}$ because $f_{ground} \propto \frac{1}{T}$

To prove $f_{ground} \propto \frac{1}{T}$ we need Boltzmann distribution

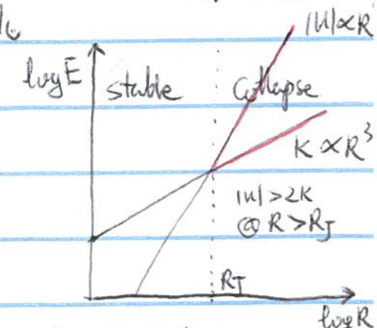
$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu/kT} \Rightarrow f_{ground} = \frac{n_1}{n_1+n_2} = \frac{1}{1+e^{-\Delta E/kT}} = \begin{cases} 1.0 & @T=0 \\ 0.5 & @T\to\infty \end{cases}$$



Molecular clouds $\to$ traced by CO

	GMC	Complexes	Clumps	Cores (SF sites)	Hot Cores
T	15K	10K	10K	10K	100–300K
M	$10^5 M_\odot$	$10^4 M_\odot$	$30 M_\odot$	$10 M_\odot$	10–$3000 M_\odot$
d	50pc	10pc	1pc	0.1pc	0.1pc
n	$10^8 m^{-3}$	5×10^8	$\boxed{10^9}$	$\cancel{10^9}\ 10^{10}$	10^{13}–10^{15}

$\rho = 2 m_p \cdot n$, $m_p = 1.7\times10^{-27}$ kg $\quad \boxed{3\times10^{-18}\ kg/m^3} \Rightarrow M_J \simeq 24 M_\odot$



Formation of Protostars

in comparison. $\rho_{air} = 1.2\ kg/m^3$

Jeans mass: condition for collapse $2K < |U| = -U$, i.e. $2K + U < 0$

$$U = -\frac{3}{5}\frac{GM^2}{R}, \quad K = \frac{3}{2} N kT = \frac{3}{2}\frac{M}{\mu m_p} \cdot kT$$

$$2K < -U \Rightarrow \frac{3MkT}{\mu m_p} < \frac{3}{5}\frac{GM^2}{R}, \quad R = \left(\frac{3M}{4\pi\rho_0}\right)^{1/3} \Leftrightarrow M = \frac{4}{3}\pi\rho_0 R^3$$

$$\Rightarrow M > \left(\frac{5kT}{G\mu m_p}\right)^{3/2}\left(\frac{3}{4\pi\rho_0}\right)^{1/2} \equiv M_J \quad \text{or} \quad R > \left(\frac{15kT}{4\pi G\mu m_p \rho_0}\right)^{1/2} \equiv R_J$$

Jeans Mass $M_J = \left(\frac{5kT}{G\mu m_p}\right)^{3/2}\left(\frac{3}{4\pi\rho_0}\right)^{1/2} = \boxed{8\,M_\odot\left(\frac{T/10\text{K}}{\mu/2}\right)^{3/2}\cdot\left(\frac{3\times10^{-17}\text{kg/m}^3}{\rho_0}\right)^{1/2}}$

$$= \frac{5^{3/2}}{G^{3/2}}\left(\frac{kT}{\mu m_p}\right)^{4/2}\left(\frac{\mu m_p}{kT}\frac{1}{\rho_0}\right)^{1/2}\cdot\left(\frac{3}{4\pi}\right)^{1/2}$$

$$= \frac{5.46\ V_T^4}{G^{3/2}\cdot P_0^{1/2}}$$ where $V_T = \sqrt{\frac{kT}{\mu m_p}}$ is the isothermal sound speed

Sound speed $v_s = \sqrt{\gamma P/\rho}$. for isothermal gas $P = \frac{\rho}{\mu m_p}kT \propto \rho^1$ so $\gamma = 1$

Homogeneous collapse: (in the absence of pressure supported by rotation, turbulence & B)

free fall motion: $\frac{d^2r}{dt^2} = -G\frac{M_r}{r^2} = -\left(\frac{4\pi}{3}G\rho_0 r_0^3\right)\frac{1}{r^2}$

$$\Rightarrow \frac{dr}{dt}\cdot\frac{d^2r}{dt^2} = -\left(\frac{4\pi}{3}G\rho_0 r_0^3\right)\frac{1}{r^2}\frac{dr}{dt}$$

integrate over dt on both side , $\int\frac{dr}{dt}\frac{d^2r}{dt^2}dt = \frac{1}{2}\int\frac{d\left(\frac{dr}{dt}\right)^2}{dt}\cdot dt = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + C$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \left(\frac{4\pi}{3}G\rho_0 r_0^3\right)\frac{1}{r} + C_1 ,$$

evaluate @ initial condition $dr/dt = 0$ when $r = r_0$

$$\frac{dr}{dt} = -\left[\frac{8\pi}{3}G\rho_0 r_0^2\left(\frac{r_0}{r}-1\right)\right]^{1/2}$$

substitute $\theta = r/r_0$ & $\chi = \sqrt{\frac{8\pi}{3}G\rho_0}$ into the above, $\left(\chi \propto \frac{1}{t}\right)$

$$\frac{d\theta}{dt} = -\chi\left(\frac{1}{\theta}-1\right)^{1/2}$$

substitute $\theta = \cos^2\beta = r/r_0 \le 1$ (we can do this because $\theta \le 1$)

$$\frac{d\cos^2\beta}{dt} = -\chi\left(\frac{1-\cos^2\beta}{\cos^2\beta}\right)^{1/2} \Rightarrow 2\cos\beta\sin\beta\frac{d\beta}{dt} = \chi\cdot\frac{\sin\beta}{\cos\beta}$$

Trigonometric identity

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos 2A = 2\cos^2 A - 1$

$\Rightarrow \cos^2\beta\frac{d\beta}{dt} = \frac{\chi}{2}$, now integrate over dt, replace $\cos^2\beta = \frac{1}{2}[1+\cos(2\beta)]$

$\frac{\beta}{2} + \frac{1}{4}\sin 2\beta = \frac{\chi}{2}t$. ; note that when $t=0$, $\beta=0$, $\theta=1$, $r=r_0$

when the free fall completes, $r=0$, $\beta = \pi/2$ $t = t_{ff}$

$$\Rightarrow \frac{\pi}{4} = \frac{\chi}{2}t_{ff} \Rightarrow t_{ff} = \frac{\pi}{2\chi} = \left(\frac{3\pi}{32}\frac{1}{G\rho_0}\right)^{1/2}$$

$$\boxed{t_{ff} = 3.8\times10^5\,\text{yr}\left(\frac{3\times10^{-17}\text{kg/m}^3}{\rho_0}\right)^{1/2}}$$

Free Fall timescale & Jeans Length (alternative derivation, RP17.1)

Kepler's 3rd law : $P^2 = \frac{4\pi^2}{GM} a^3$

a free fall particle @ r_0 from the center must follow this law even though $e \simeq 1$

so $t_{ff} = \frac{P}{2}$, $a = \frac{r_0}{2}$, we also know $M = \frac{4}{3}\pi r_0^3 \cdot \rho_0$

$$\Rightarrow 4 t_{ff}^2 = \frac{4\pi^2}{G} \cdot \frac{3}{4\pi r_0^3 \cdot \rho_0} \cdot \frac{r_0^3}{8} \Rightarrow t_{ff} = \sqrt{\frac{3\pi}{32 G\rho_0}} = 4\times10^4 yr \left(\frac{\rho_0}{3e\text{-}15 k}\right.$$

$\rho_0 = 3\times10^{-15}\ kg/m^3 \Leftrightarrow n_{H_2} = \frac{\rho}{2m_p} = 10^{12}\ m^{-3}$ [dense cores]

Jeans criterion for a cloud that is $\boxed{\text{stable}}$ against purturbations:

$\boxed{t_{ff} > t_{sound}}$ free-fall time longer than sound travel time

$$\left(\frac{3\pi}{32 G\rho_0}\right)^{1/2} > \frac{r_0}{c_s} = \frac{r_0}{(\gamma kT/\mu m_p)^{1/2}}, \quad \gamma \text{ is the adiabatic index}$$

$$\Rightarrow r_0 < r_J = \left(\frac{3\pi\gamma kT}{32 G\rho_0 \mu m_p}\right)^{1/2} = 2000\ AU \left(\frac{T}{10K}\right)^{1/2} \left(\frac{\rho_0}{3\times10^{-15}\ kg\,m^{-3}}\right)^{-1/2}$$

Jeans mass is simply the mass enclosed within r_J

$$\Rightarrow M < M_J = \frac{4}{3}\pi r_J^3 \cdot \rho_0 = 0.2 M_\odot \left(\frac{T}{10K}\right)^{3/2} \left(\frac{\rho_0}{3\times10^{-15}\ kg\,m^{-3}}\right)^{-1/2}$$

After cloud fragmentation (see next page), talk about the formation of disks

Collapse stopped by rotation $\longrightarrow$ Protoplanetary Disk

conservation of angular momentum $v_0 r_0 = v_f r_f$

the cloud would stop falling when it forms a rotationally supported disk

the disk would have a radius of r_f & velocity v_f, & gravity balanced by rotatic

$$\frac{GM}{r_f^2} = \frac{v_f^2}{r_f} \Rightarrow r_f = \frac{v_f^2 r_f^2}{GM} = \frac{v_0^2 r_0^2}{GM}$$

$$\Rightarrow r_f \simeq 200\ AU \left(\frac{v_0}{0.1\ km/s}\right)^2 \left(\frac{r_0}{4000\ AU}\right)^2 \left(\frac{M}{1 M_\odot}\right)^{-1}$$

How to reduce angular momentum?

① viscous torques from outer disk ② magnitized protostellar wind

Cloud fragmentation: $M_J \propto T^{3/2} \rho_0^{-\frac{1}{2}}$ so M_J decreases as ρ_0 increases if T is consta[nt]

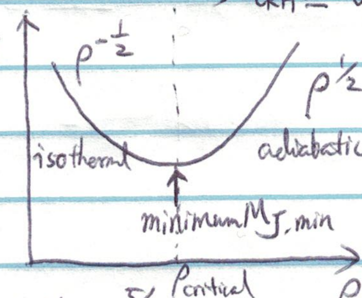
This causes a cascading collapse ~~of~~ leading to formation of large # of smaller objects if T remains constant, i.e., $\boxed{\text{isothermal}}$ collapse.

But if the collapse is $\boxed{\text{adiabatic}}$ $T = K' \cdot \rho^{\gamma-1}$, then $M_J \propto \rho^{(3\gamma-4)/2}$

γ: heat capacity ratio $\gamma = C_p/C_v = 1 + \frac{2}{dof}$

for monoatomic gas, $\gamma = 5/3$, $M_J \propto \rho^{\frac{1}{2}}$ $\rightarrow t_{KH} \geq t_{ff}$

for diatomic gas, $\gamma = 7/5$, $M_J \propto \rho^{0.1}$



Now estimate the minimum Jeans mass

to remain isotherml, the energy generated due to gravity must be released within the t_{ff} time scale

$$L_{ff} = \frac{\Delta(U_g+K)}{t_{ff}} = \frac{\frac{3}{10} G \frac{M_J^2}{R_J}}{\left(\frac{3\pi}{32}\frac{1}{G\rho_0}\right)^{1/2}} \sim G^{3/2}\left(\frac{M_J}{R_J}\right)^{5/2}$$

$\rho_{critical} \propto T^{\frac{8}{3}}$

but the proto star can only radiate away energy through its surface

$$L_{rad} = 4\pi R_J^2 e \sigma T^4, \quad \text{where } 0 < e < 1$$

Heating $\geq$ Cooling

the transition from isotherml to adiabatic collapse occures when

$\boxed{t_{ff} \leq t_{KH}} \Longleftrightarrow \boxed{L_{rad} \leq L_{ff}}$, radiation becomes insufficient to remove gravitational energy

or $M_J^{5/2} \geq \frac{4\pi}{G^{3/2}} R_J^{9/2} e \sigma T^4$ → now we need to eleminate R_J

Since $2 \cdot \frac{3M_J}{2\mu m_p} \cdot kT = \frac{3}{5}\frac{GM_J^2}{R_J}$ $[2K = -U]$ virial theorem

we have $R_J = \frac{GM_J}{5kT}\mu m_p$, plug this into the above, solve for M_J

$$M_J \geq 0.03 M_\odot \left(\frac{T^{1/4}}{e^{1/2}\mu^{9/4}}\right) = M_{J,min}$$

e.g. $M_{J,min} = \boxed{0.5 M_\odot} (T/1000K)^{1/4} \cdot (0.1/e)^{\frac{1}{2}} \cdot (1/\mu)^{9/4}$

$R_{J,min} = \frac{\mu m_p}{5kT} \cdot GM_{J,min} = \boxed{10\ AU} (T/1000K)^{-3/4} \cdot (0.1/e)^{\frac{1}{2}} \cdot (1/\mu)^{5/4}$

Estimating MW's SFR: For a core with mass above M_J, the SFR is simply M_J/t_{ff}

$SFR_{core} = \frac{8M_\odot}{4\times10^5 yr}\left(\frac{T/10K}{\mu/2}\right)^{3/2} = 2\times10^{-5}\ M_\odot/yr$ for a single core with $8M_\odot$ mass

Heyer & Dame 15 → total mass in H_2 is $(1\pm0.3)\times10^9\ M_\odot$, if all within cores, $SFR = \frac{10^9}{8} \times 2\times10^{-5} = 2\times10^3\ M_\odot/yr$

current $SFR_{observed} \sim 1\ M_\odot/yr \ll SFR_{estimated}$

How to resolve this apparent contradiction?

Explaining MW & all local galaxies' low SF efficiency

$$SFR_{MW} = \frac{\epsilon \cdot M_{H_2}}{M_{Jeans}} \cdot \frac{M_{Jeans}}{t_{SF}} = \frac{\epsilon\, M_{H_2}}{t_{SF}}$$

① the freefall timescale $t_{ff} = 4\times10^5\,\mathrm{yr}\cdot\sqrt{\frac{10^{10}\,\mathrm{m^{-3}}}{n}}$ is too short to describe the SF process, instead, SF proceeds at almost the KH timescale, $t_{KH} \sim 10^7\,\mathrm{yr}$

② the fraction of H_2 that is actively forming stars is $\epsilon \sim 10^{-2} = 1\%$

99% of H_2 mass are in structures that are stable againsts gravitational collapse

This doesn't make sense when the Jeans mass is only $140\,M_\odot$ for a GMC with $10^5\,M_\odot$

③ requires turbulence support to increase Jeans mass

Kinetic energy $\frac{1}{2}\mu m_p v^2 = \frac{3}{2} k T_{kinetic} \Rightarrow T_K = 10\,\mathrm{K}\cdot\left(\frac{v}{0.3\,\mathrm{km/s}}\right)^2$

sound speed $c_s = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma k T}{\mu m_p}} = 540\,\mathrm{m/s}\left(\frac{\gamma}{7/5}\right)^{1/2}\left(\frac{2}{\mu}\right)^{1/2}\left(\frac{T}{50\mathrm{K}}\right)^{1/2}$

supersonic turbulence can increase T significantly, causing the Jeans mass increase

$$M_J = 8\,M_\odot\left(\frac{T}{10\mathrm{K}}\right)^{3/2}\cdot\left(\frac{10^{10}\,\mathrm{m^{-3}}}{n}\right)^{1/2}$$

$$= 10^4\,M_\odot\left(\frac{\sigma_T}{3\,\mathrm{km/s}}\right)^3\cdot\left(\frac{10^{10}\,\mathrm{m^{-3}}}{n}\right)^{1/2}$$ for a turbulence v dispersion of 3 km/s

④ turbulence increases in larger clouds (Larson's Law [1981])

$\sigma = 3\,\mathrm{km/s}\left(\frac{R}{20\,\mathrm{pc}}\right)^{0.4}$, $\sigma \sim c_s(50\mathrm{K})\ 0.5\,\mathrm{km/s}$ @ 0.23 pc

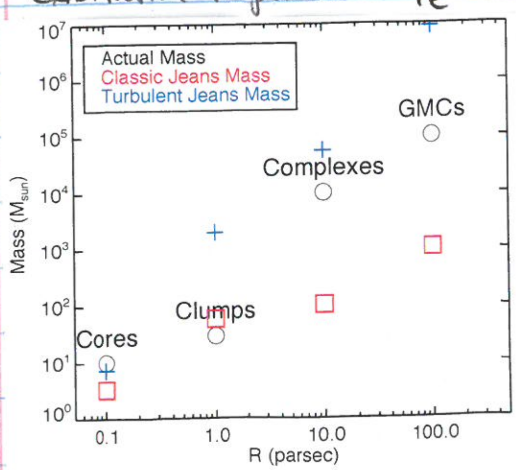
$\sigma \sim c_s(10\mathrm{K}) \sim 0.24\,\mathrm{km/s}$ @ 0.04 pc

comparable to core size ~0.1 pc

kinetic temperature $T_k = \frac{\mu m_p \sigma^2}{3k}$

excitation temperature $T_e = \frac{\mu m_p}{\gamma k} c_s^2$

} so when $\sigma \sim c_s$, $T_k \sim T_e \sim T$ and Jeans mass' original definition becomes valid



Kelvin-Helmholtz timescale

$$t_{KH} \sim 10^7\,\mathrm{yr}\left(\frac{M}{M_\odot}\right)^2\cdot\left(\frac{R}{R_\odot}\right)^{-3}\left(\frac{T}{T_\odot}\right)^{-4}$$

$R_\odot = \frac{1}{215}\,\mathrm{AU}$, $T_\odot = 5800\,\mathrm{K}$

Cloud fragmention reaches minimum Jeans mass & Jeans ~~rad~~ length

denser clouds collapse faster, $t_{ff} \propto \rho^{-\frac{1}{2}}$, and decouple from parent cloud start to fragment as Jeans mass decreases, $M_J \propto T^{3/2} \rho^{-\frac{1}{2}}$ until the smallest Jeans mass, $M_{J,min}$, determined by $t_{ff} = t_{KH}$

$$\left(\frac{3\pi}{32}\frac{1}{G\rho}\right)^{1/2} = \frac{\frac{3}{10} G M^2/R}{4\pi R^2 e \sigma T^4}, \quad \rho = \frac{M}{\frac{4}{3}\pi R^3} \Rightarrow \frac{M^{\frac{5}{2}}}{R^{3/2}} = A \cdot R^2 T^4$$

plus the virial theorem $2K = -U$

$$2 \cdot \frac{3}{2} kT \cdot \frac{M_J}{\mu m_p} = \frac{3}{5}\frac{GM^2}{R} \Rightarrow R = \frac{GM}{5kT}\mu m_p$$

$$\Rightarrow \quad M_{J,min} = 0.01 M_\odot \left(\frac{T}{10K}\right)^{1/4} \cdot e^{-\frac{1}{2}} \cdot \left(\frac{\mu}{2}\right)^{-9/4}$$

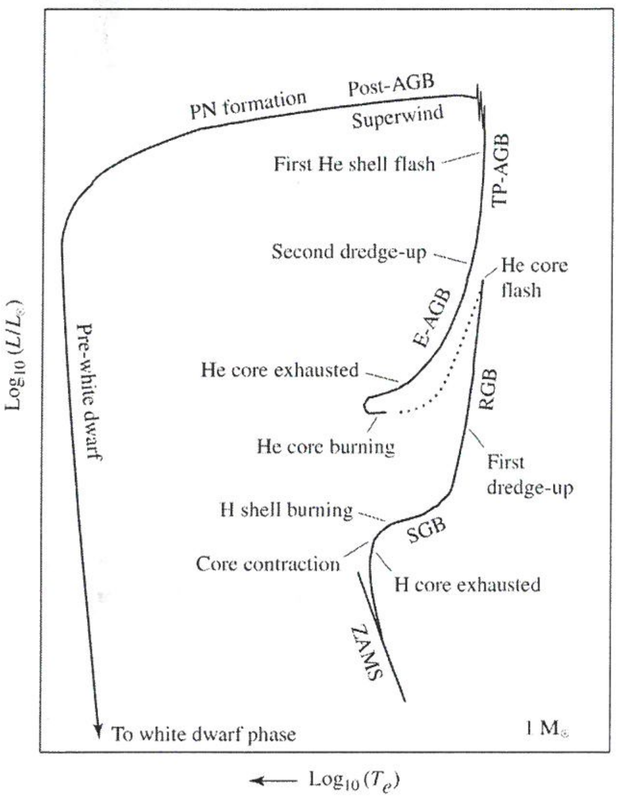
$$R_{J,min} = 42\ AU \left(\frac{T}{10K}\right)^{-\frac{3}{4}} \cdot e^{-\frac{1}{2}} \left(\frac{\mu}{2}\right)^{-5/4}$$

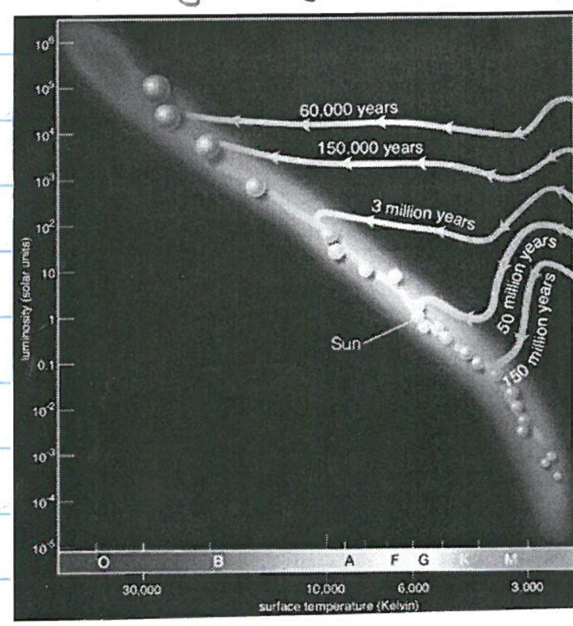
$$\rho_{crit} = \frac{3M_{J,min}}{4\pi R_{J,min}^3} = 1.9\times10^{-11}\ kg\,m^{-3} \left(\frac{T}{10K}\right)^{\frac{5}{2}} \cdot e \cdot \left(\frac{\mu}{2}\right)^{3/2}$$

$$n_{crit} = \frac{\rho_{crit}}{\mu m_p} = 5.7\times10^{15}\ m^{-3} \left(\frac{T}{10K}\right)^{5/2} e \cdot \left(\frac{\mu}{2}\right)^{1/2}$$

Evolution of a low-mass star (e.g. the Sun) [RP §17.2]

Protostar	$t \sim 50$ Myr	gravity powered
MS	$t \sim 10$ Gyr	$H \to He$ in the core, $0.7 L_\odot \to 2.2 L_\odot$ [3.5 Gyr from runaway greenhouse effect]
RGB	$t \sim 1$ Gyr	$H \to He$ in a shell, e^- degenerate He core
Helium Flash	$t \sim 10$ hrs	$He \to C$ in e^- degenerate core ($T = 10^8 - 3.5 \times 10^8$ K
Horizontal Branch	$t \sim 100$ Myr	$H \to He$ shell + $He \to C$ core
AGB	$t \sim 20$ Myr	$H \to He$ shell + $He \to C$ shell + e^- degenerate core
PN (post-AGB)	$t \sim 50$ kyr	e^- degenerate C core radiates UV photons
WD	$t \sim \infty$	eternal cooling through radiation





$15 M_\odot$

9

3

$1 M_\odot$

$0.5 M_\odot$

Hayashi tracks

Core Contraction & Schönberg-Chandrasekhar Limit

When the core depletes H, it no longer generates L, so it becomes isothermal $\frac{dT}{dr} \propto \frac{-L_r}{T^3 r^2} = 0$

external pressure helps gravity to fight kinetic energy

The core follows the virial theorem $4\pi R_{ic}^3 P_{ic} - 2K_{ic} = U_{ic}$, P_{ic} is pressure at interface

$$P_{ic} = \frac{3}{4\pi R_{ic}^3}\left(\frac{M_{ic} k T_{ic}}{\mu_{ic} m_H} - \frac{1}{5}\frac{G M_{ic}^2}{R_{ic}}\right)$$

, where we assumed isothermal

$\frac{dP_{ic}}{dM_{ic}} = 0$ gives $P_{ic,max} = \frac{375}{64\pi} \frac{1}{G^3 M_{ic}^2}\left(\frac{k T_{ic}}{\mu_{ic} m_p}\right)^4$

Pressure from the envelop $P_{ic,env} = \int_0^{P_{ic,env}} dP \simeq \frac{G}{8\pi \langle r^4 \rangle}(M^2 - M_{ic}^2) \sim \frac{GM^2}{4\pi R^4}$

$\Rightarrow \frac{M_{ic}}{M} \simeq 0.37 \left(\frac{\mu_{env}}{\mu_{ic}}\right)^2$

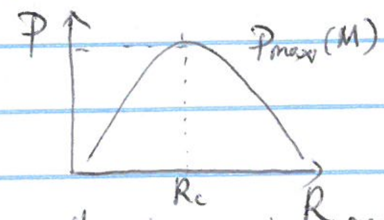
$\simeq 8\%$ for solar abundance

Schönberg - Chandraskher limit

start from the virial theorem with external pressure P

$$4\pi R^3 P = 2K + U = \frac{3MkT}{\mu m_p} - \frac{3}{5}\frac{GM^2}{R}$$

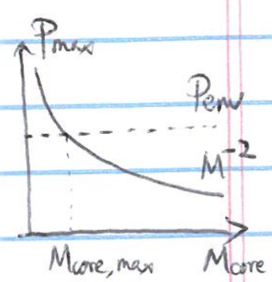
$$\Rightarrow P = \frac{3}{4\pi}\left(\frac{MkT}{\mu m_p R^3} - \frac{GM^2}{5R^4}\right)$$



suppose the mass of the isothermal core is fixed,

P increases first then starts to decline as R increases, so there is a maximum pressure the core can support, and it occurs @ $\frac{\partial P}{\partial R} = 0$

$$\Rightarrow -\frac{3kT}{\mu m_p}\frac{M}{R^4} + \frac{4GM^2}{5R^5} = 0 \Rightarrow R_c = \frac{4GM\mu m_p}{15kT}$$



$$\Rightarrow P_{max}(M) = P(M, R_c) = 3.15 \cdot \frac{1}{G^3 M^2}\left(\frac{kT}{\mu m_p}\right)^4 = A \cdot \frac{1}{G^3 M^2}\left(\frac{kT}{\mu m_p}\right)^4$$

To derive the bottom pressure of the envelop, we integrate the hydrostatic equation

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho \quad \& \quad dM_r = 4\pi r^2 \rho\, dr$$

$$\Rightarrow \frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} \qquad \Rightarrow P_{env} = \int dP = -\int_M^{M_{ic}} \frac{GM_r}{4\pi r^4} dM_r \sim \frac{GM^2}{4\pi R^4}$$

based on the ideal gas law, $P_{env} = \frac{\rho k T}{\mu m_p} \sim \frac{\alpha 3MkT}{4\mu m_p R^3 \pi} \sim \frac{GM^2}{4\pi R^4}$, $\alpha > 1$

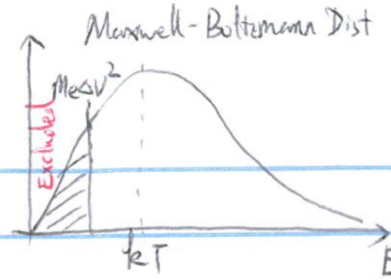
$$\Rightarrow R \sim \frac{1}{3\alpha}\frac{GM}{T}\frac{\mu m_p}{k}$$, plug this into $P_{env}(M,R)$, we have

$$P_{env} \simeq \frac{GM^2}{4\pi R^4} \sim \frac{81\alpha^4}{4\pi}\frac{1}{G^3 M^2}\left(\frac{kT_0}{\mu m_p}\right)^4 = B \cdot \frac{1}{G^3 M^2}\left(\frac{kT}{\mu m_p}\right)^4$$

the condition for core collapse is $P_{env} \geq P_{max}$

$$B \cdot \frac{1}{M^2 \mu_{env}^4} \geq A \cdot \frac{1}{M_{ic}^2 \mu_{ic}^4}$$

$$\Rightarrow \frac{M_{ic}}{M} \geq \sqrt{\frac{A}{B}}\left(\frac{\mu_{env}}{\mu_{ic}}\right)^2 = 0.7\left(\frac{\mu_{env}}{\mu_{ic}}\right)^2 \text{ vs. } 0.37\left(\frac{\mu_{env}}{\mu_{ic}}\right)^2$$



Pressure from Electron Degenerate Cores

e^- degeneracy pressure (non-relativistic)

Heisenberg uncertainty principle $\Delta x \Delta p \geq \hbar$

$$\Delta x \sim V^{1/3} \sim n_e^{-1/3} \Rightarrow \Delta p \sim \hbar n_e^{1/3} \rightarrow \Delta v = \frac{\Delta p}{m_e} \sim \frac{\hbar n_e^{1/3}}{m_e}$$

Thermal Pressure : $P_{th} = n_e kT \sim n_e m_e v_{th}^2$

Degeneracy Pressure : $P_{degen} \sim n_e m_e \Delta v^2 \sim \hbar^2 \frac{n_e^{5/3}}{m_e}$

The electrons are degenerate when $P_{degen} > P_{th}$, $\boxed{\hbar^2 \frac{n_e^{5/3}}{m_e} > n_e kT}$

or $\frac{\Delta p^2}{m_e} > kT$, $\hbar^2 \frac{n_e^{2/3}}{m_e} > kT$

Mass-Radius Relation from non-relativistic degeneracy

Hydrostatic Equilibrium : $\frac{dP}{dr} = -\frac{GM}{r^2} \cdot \rho \Rightarrow P_c \sim \frac{GM^2}{R^4}$ since $\rho \sim \frac{M}{r^3}$

$$P_{degen} \sim \hbar^2 \frac{n_e^{5/3}}{m_e} \sim \hbar^2 \frac{\rho^{5/3}}{(\mu m_p)^{5/3} \cdot m_e} \sim \frac{M^{5/3}}{R^5}$$, $\mu \sim 2$ for fully ionized C/O

Given $P_c \sim P_{degen} \Rightarrow R \sim \frac{\hbar^2}{G m_e m_p^2} \left(\frac{M}{m_p}\right)^{-1/3} \sim 0.01 R_\odot \left(\frac{M}{0.7 M_\odot}\right)^{-1/3}$

Chandrasekhar Limit :

Relativistic degenerate electrons : $E_{rel} \gg m_e c^2$, similar to massless particles (photons)

$$E_{rel}^2 = p^2c^2 + (mc^2)^2 \Rightarrow E_{rel} \sim pc \sim \hbar n_e^{1/3} c \quad \left[E_{non\text{-}rela} \sim \frac{p^2}{m_e} \sim n_e^{2/3}\right]$$

$$P_{rel,degen} = \frac{1}{3} u_{rel} = \frac{1}{3} E_{rel} \cdot n_e \sim \hbar c\, n_e^{4/3}/3 \quad \left[\text{vs. } P_{degen} \sim \hbar^2 \frac{n_e^{5/3}}{m_e}\right]$$

Hydrostatic Equilibrium : $P_c \sim GM^2/R^4$

$P_{rel,degen} \sim \frac{\hbar c}{3} \left(\frac{\rho}{\mu m_p}\right)^{4/3} \propto \frac{M^{4/3}}{R^4}$

$\Rightarrow G \frac{M^2}{R^4} \sim \frac{\hbar c}{m_p^{4/3}} \frac{M^{4/3}}{R^4} \underbrace{\left[\frac{1}{3}\left(\frac{1}{\mu}\right)^{4/3}\right]}_{\text{ignored}}$

$$\Rightarrow M_c = \left(\frac{\hbar^3 c^3}{G^3 m_p^4}\right)^{1/2} \sim 2 M_\odot \longrightarrow \text{The Chandrasekhar Limit}$$

Because $\frac{P_c}{P_{rel,degen}} \propto M^{2/3}$, $\frac{P_c}{P_{degen,rel}}$ goes above unity at $M \geq M_c$

Hydrostatic equilibrium would require a central pressure exceeding the relativistic electron degenerate pressure. So the core will collapse to a neutron star or undergoing a thermonuclear runaway — a carbon flash [type I SNe]

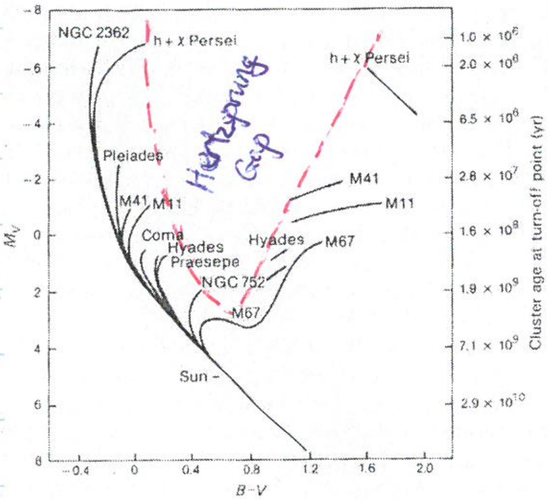
Hertzsprung Gap in CMD of Star Clusters [CO §13.3]

Absence of stars between MS and red giant region

A common feature in young clusters

Due to rapid evolution when the He core exceeds the Schönberg-Chandrasekhar limit

Evolution timescale ~ Kelvin-Helmholtz



CO, Fig 13.19

Spectroscopic Parallax = MS fitting ⇒ distance to clusters

MS lifetime: $\tau = \frac{\epsilon M c^2}{L} = 30\ \mathrm{Myr}\left(\frac{M}{7M_\odot}\right)^{-3} = 10\ \mathrm{Gyr}\left(\frac{M}{1M_\odot}\right)^{-3}$

Isochrones vs. evolutionary tracks ⇒ Age of clusters

Stellar Pulsation — Pulsating variable stars [RP §17.3]

① Acoustic oscillation period

$$P = \frac{2R}{c_s} = 2R\left(\frac{\mu m_p}{\gamma k \langle T\rangle}\right)^{1/2}$$

$\langle T\rangle \simeq \frac{1}{2}T_c$, half of central temperature

$\frac{kT_c}{\mu m_p} = \frac{P_c}{\rho} \Leftarrow$ $T_c = P_c \frac{\mu m_p}{\rho k}$, ideal gas law

$\frac{dP}{dr} = -g\rho \Rightarrow$ $\frac{P_c}{R} \sim \frac{G(M/2)\rho}{(R/2)^2}$, hydrostatic equilibrium

$$\Rightarrow P \simeq \left(\frac{4R^3}{\gamma G M}\right)^{1/2} = \left(\frac{3}{\pi\gamma G\langle\rho\rangle}\right)^{1/2} \propto \left(\frac{1}{\langle\rho\rangle}\right)^{1/2}$$

interestingly $\frac{P/2}{t_{ff}} \simeq \frac{2}{\pi}\left(\frac{2}{\gamma}\right)^{1/2} \simeq 0.7$

☆ P-L relation, more luminous stars have longer period.

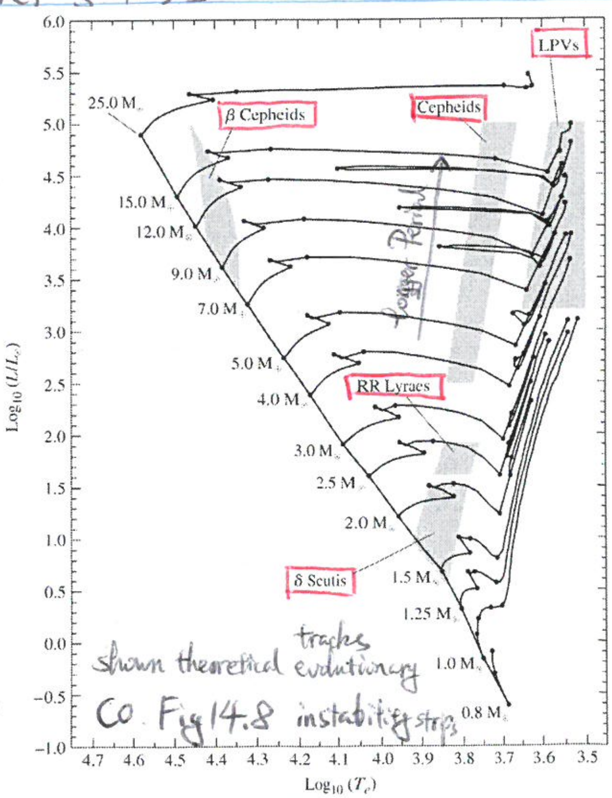
② Expansion & Contraction Period of T~40kK layer

ⓐ T~40,000 K, $He^+ \rightarrow He^{++} + e^-$

ⓑ opacity increases rapidly due to e^-

ⓒ layer expands & cools, $He^{++} e^- \rightarrow He^+$ ~~...~~

ⓓ opacity plummets, layer contracts & reheat to 40,000 K



CO. Fig 14.8

$\boxed{\text{Cepheid P-L relation} \quad \overline{M}_V = -2.76\log(P/10\ \mathrm{days}) - 4.16}$

$\boxed{\log(d/10\,\mathrm{pc}) = \frac{m-M}{5}}$

③ When P[acoustic] matches P[Expansion/contraction], the acoustic oscillations are driven & amplified