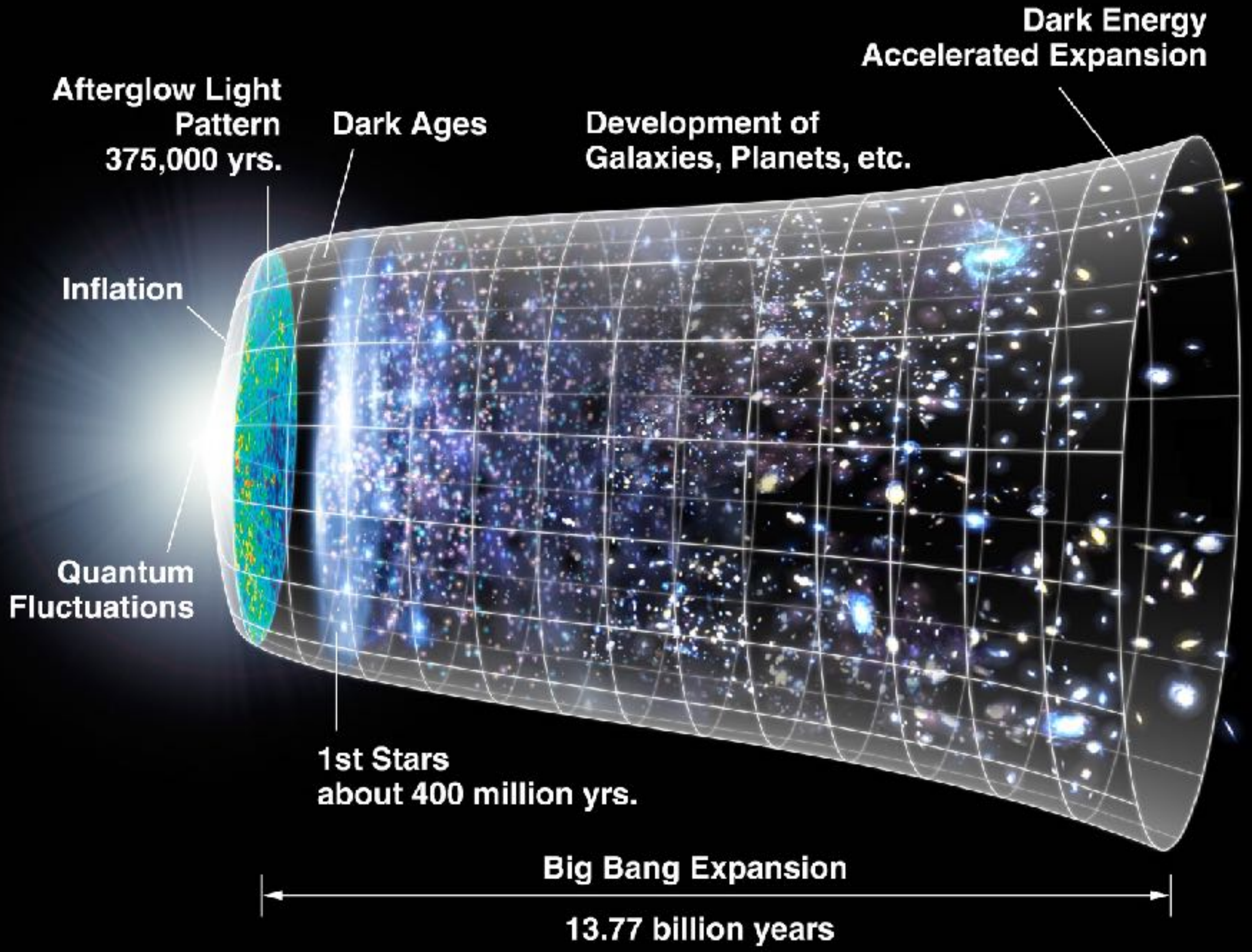
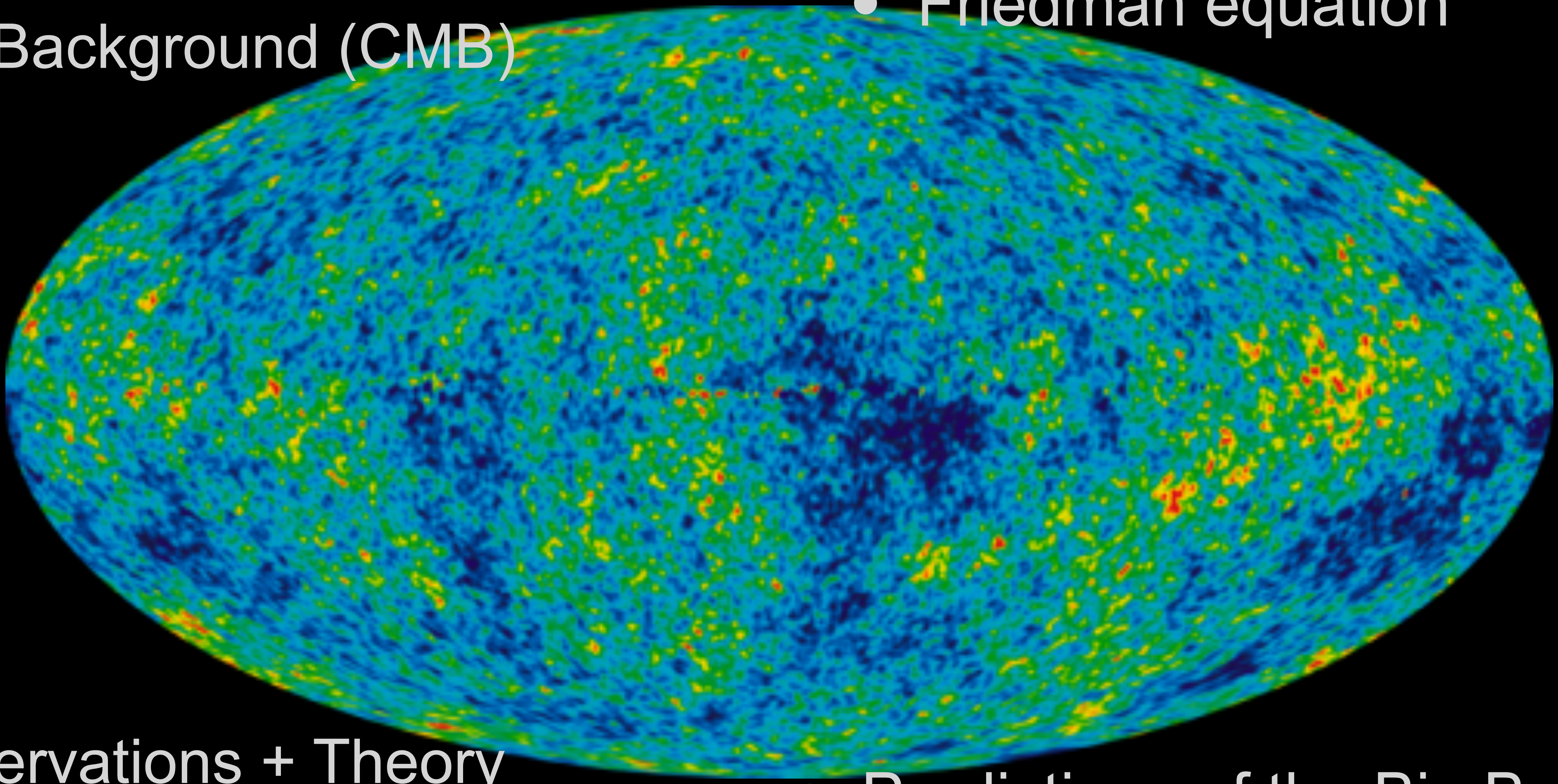
Chap 6 - The Expanding Universe



Chap 6: The Expanding Universe - Outline



- Observations (facts)
  - Hubble's Law & the Hubble "constant"
  - The Cosmic Microwave Background (CMB)
- Interpretations (theories)
  - The cosmological principle
  - Robertson-Walker metric
  - Friedman equation
- Observations + Theory
  - Accelerating Expansion: Evidence of dark energy
  - The cosmic composition
- Predictions of the Big Bang theory: how everything began?

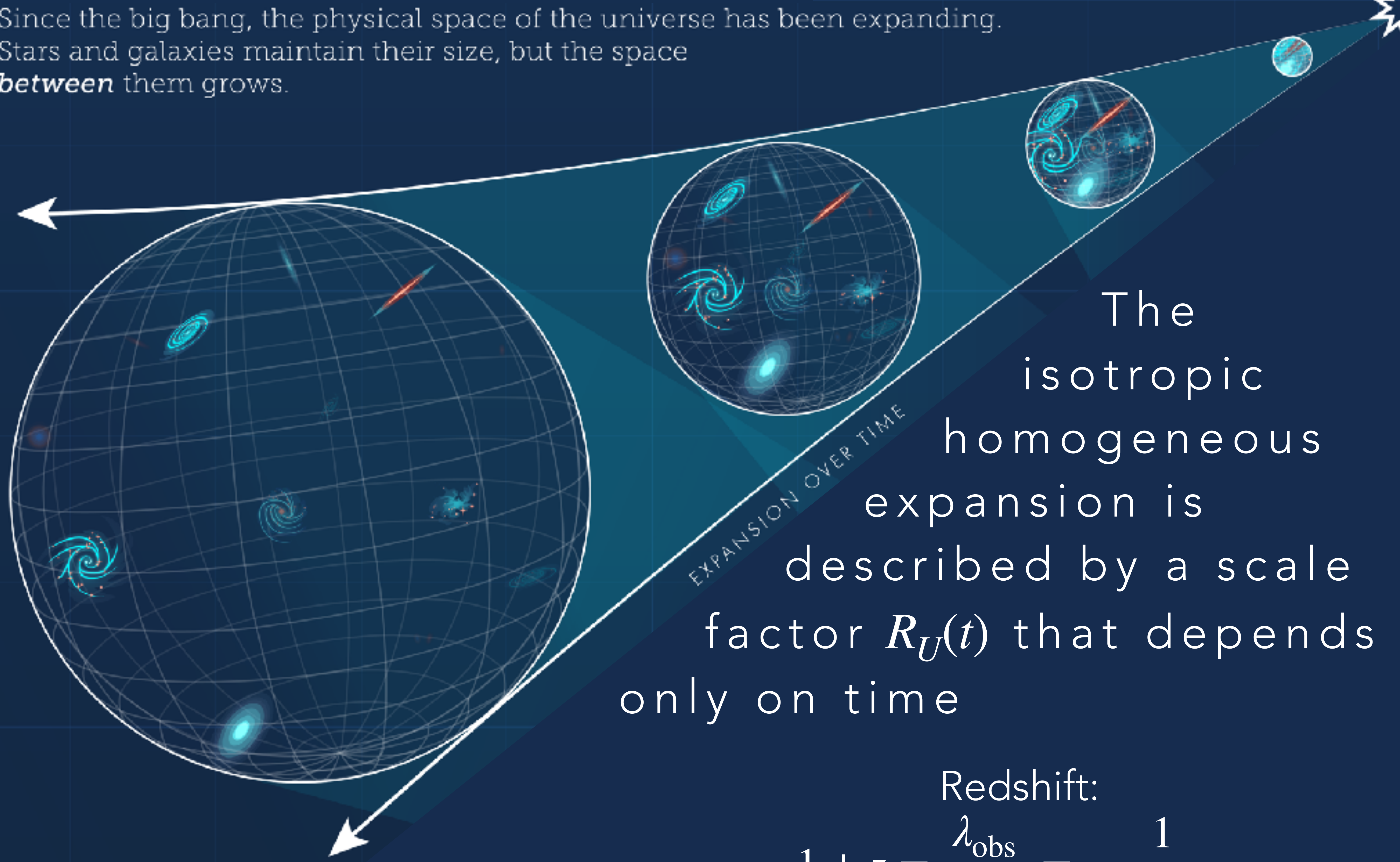
Observing distant galaxies is like watching an old movie

The Kid (1921), Chaplin

What is cosmological redshift?

It is NOT Doppler shift,
It is due to an expanding universe

We live in an expanding universe



Since the big bang, the physical space of the universe has been expanding. Stars and galaxies maintain their size, but the space ***between*** them grows.

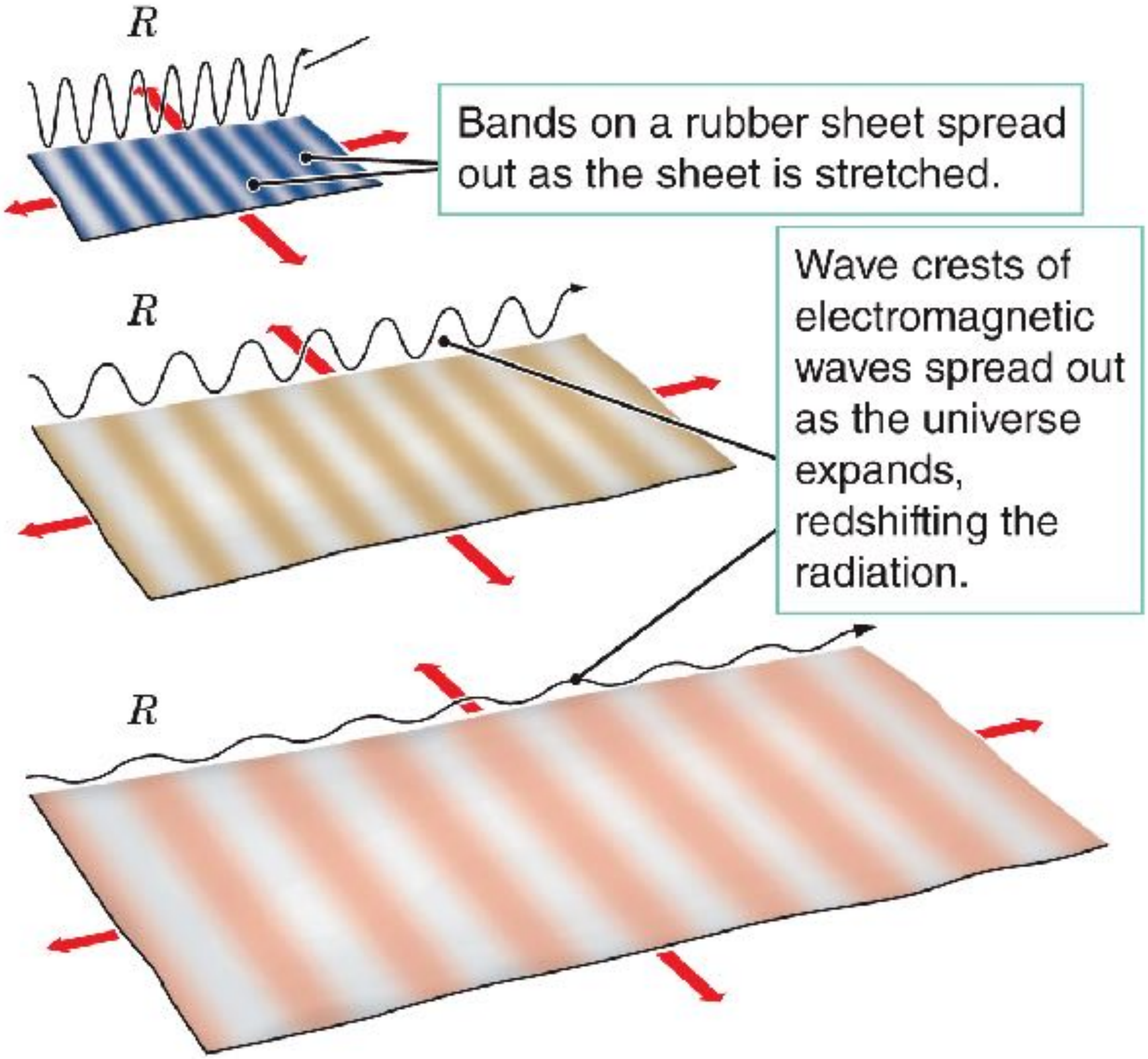
The isotropic homogeneous expansion is described by a scale factor $R_U(t)$ that depends only on time

Redshift:

$$1 + z = \frac{\lambda_{\mathrm{obs}}}{\lambda_{\mathrm{emit}}} = \frac{1}{R_U(t_{\mathrm{emit}})}$$

Redshift and the Expansion

- Redshifts of galaxies are **not** due to Doppler shifts (relative motions)
- Instead, the light is “stretched out” as it travels through the expanding universe: this is known as **cosmological redshift**.
- The wavelength of light is getting longer over time because the scale factor is increasing.
- A higher redshift indicates a smaller scale factor. Light emitted at high redshift will be very stretched out.



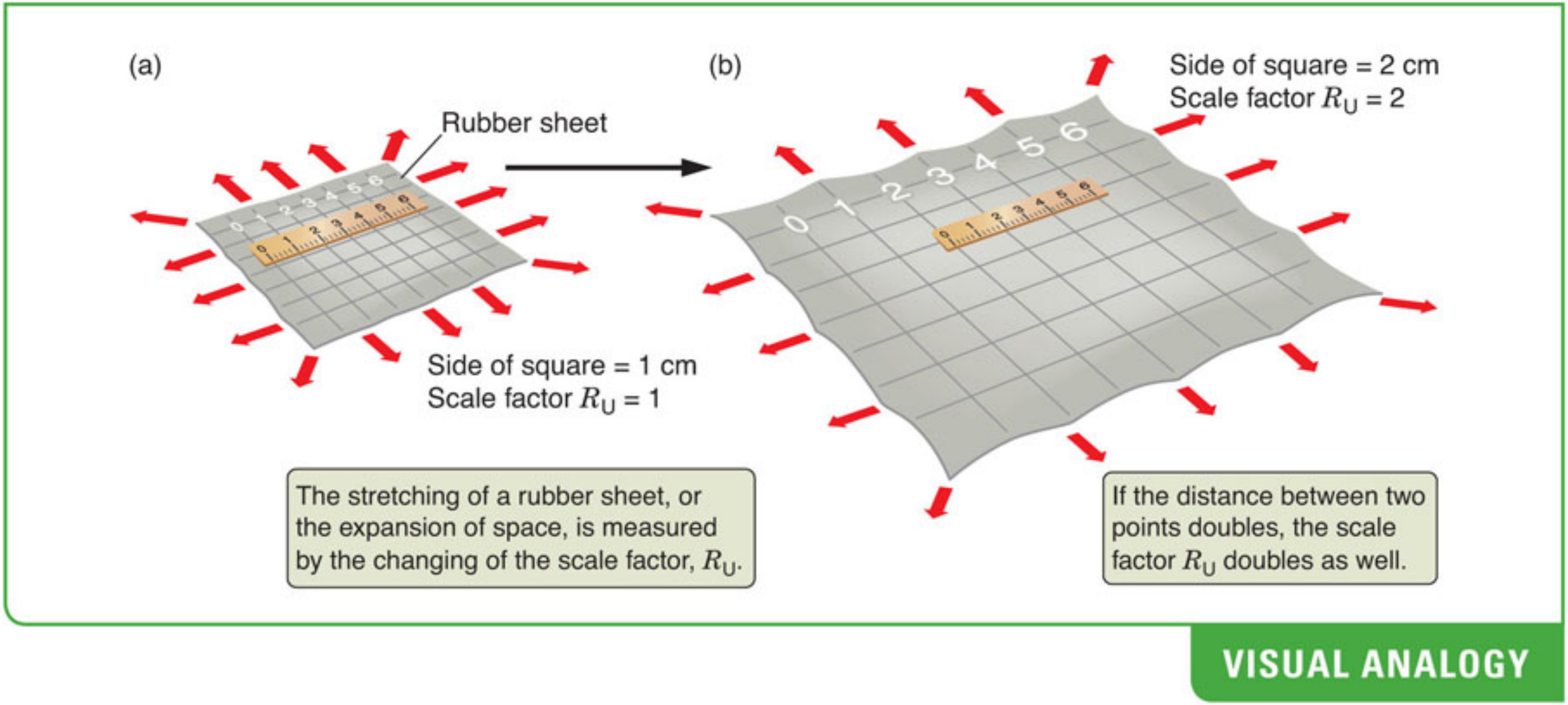
Bands on a rubber sheet spread out as the sheet is stretched.

Wave crests of electromagnetic waves spread out as the universe expands, redshifting the radiation.

Cosmological Redshifts measure the Scale Factor of the Universe

- **Cosmological redshift** should be understood as a ratio of **scale factors**:

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_0} = \frac{R_U(t_0)}{R_U(t_{\text{emit}})} = \frac{1}{R_U(t_{\text{emit}})}$$

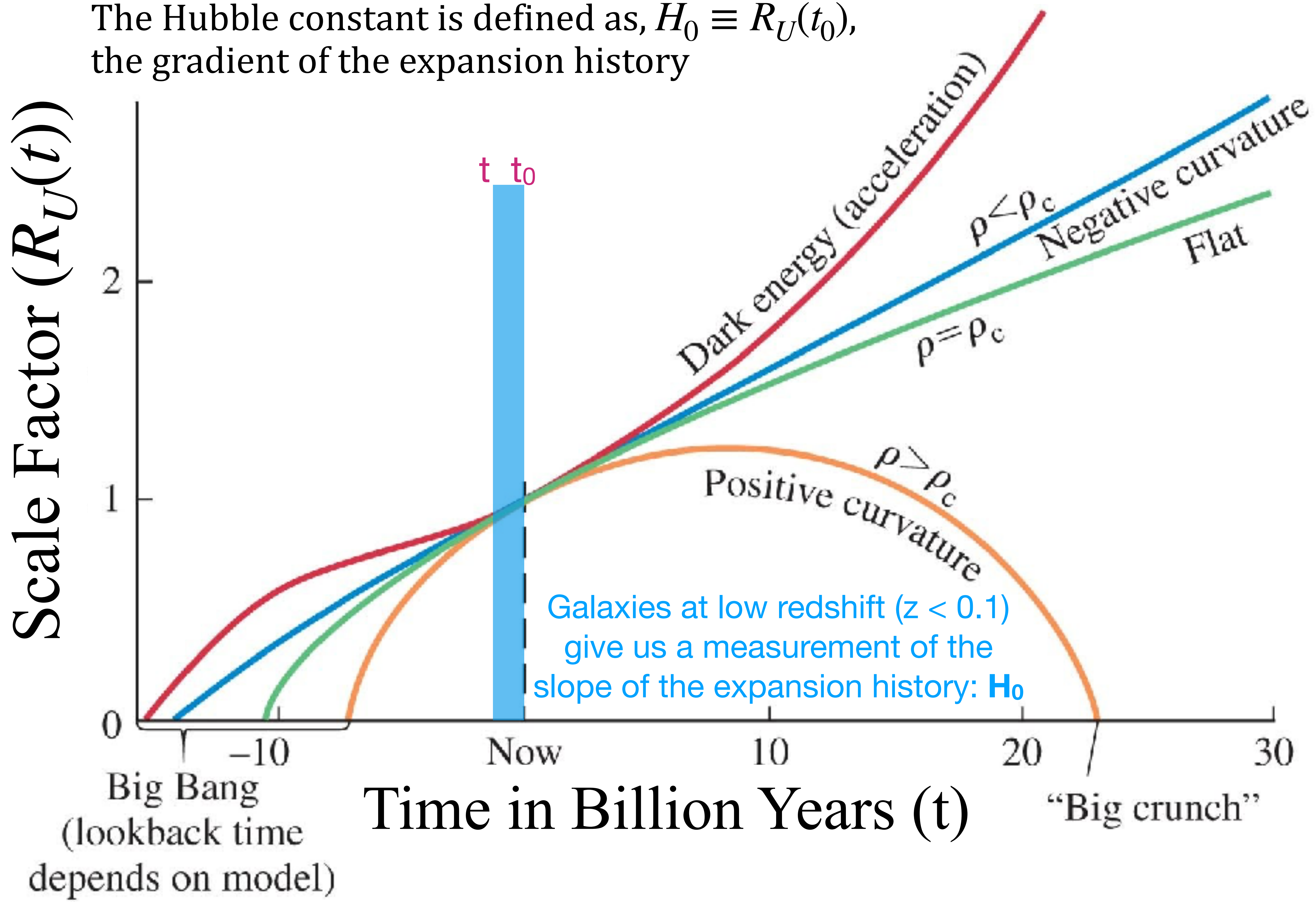


The stretching of a rubber sheet, or the expansion of space, is measured by the changing of the scale factor, R_U.

If the distance between two points doubles, the scale factor R_U doubles as well.

H_0: the time derivative of the scale factor

The Hubble constant is defined as, $H_0 \equiv \dot{R}_U(t_0)$,
the gradient of the expansion history



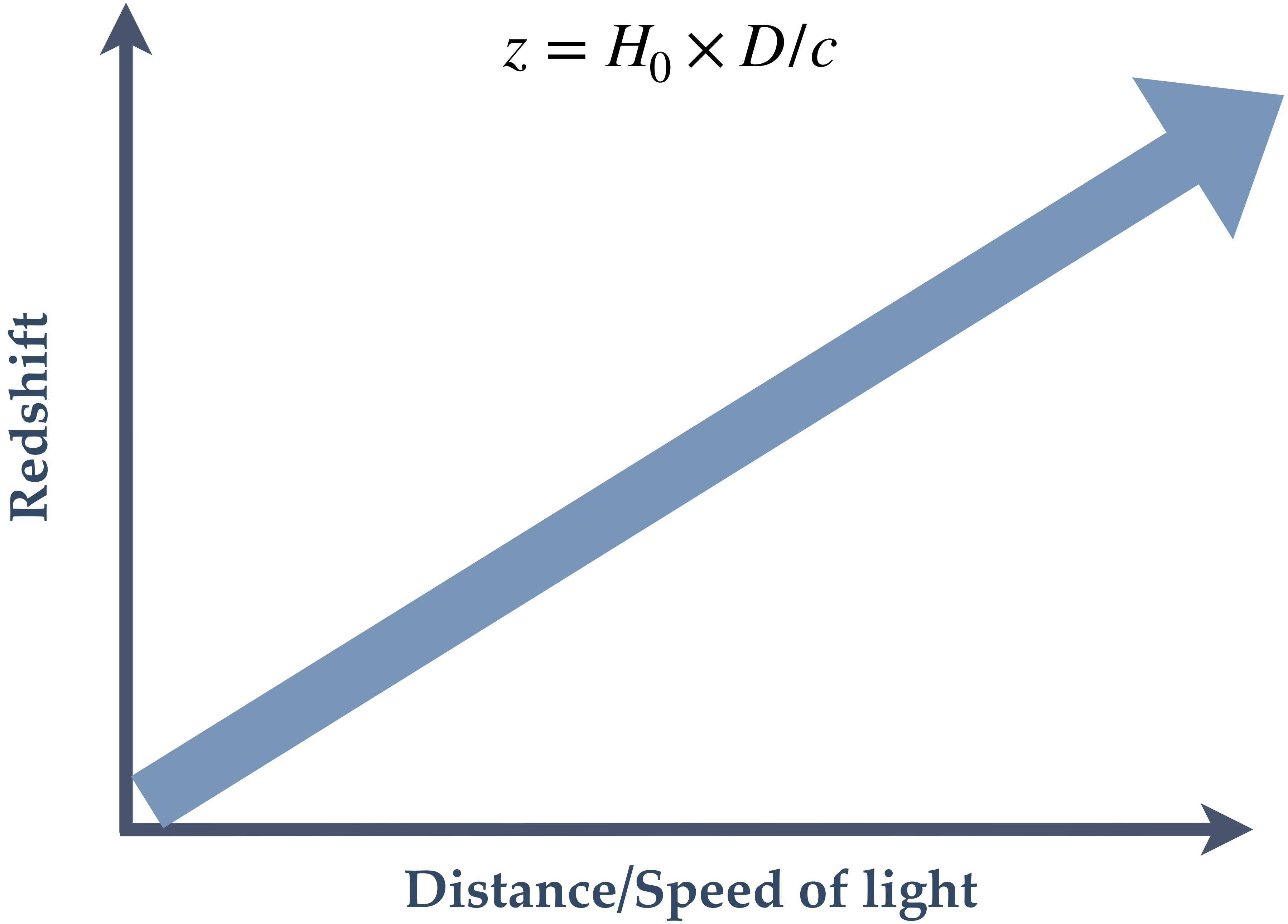
Galaxies at low redshift ($z < 0.1$) give us a measurement of the slope of the expansion history: **H_0**

Theoretical Derivation of the Redshift-Distance Relation

- The expansion is described by the **scale factor**: $R_U(t)$
- Its time derivative today is **Hubble constant:** $H_0 \equiv \dot{R}_U(t_0)$
- When observing *relatively nearby* galaxies, t is in the recent past ($t \lesssim t_0$), so we can make two approximations:
  - **Cosmological redshift**: $$z = \frac{1}{R_U(t)} - 1 \approx R_U(t_0) - R_U(t) = \delta R_U$$
  - **Light travel time**: $\delta t = t_0 - t \approx D/c$
- Combining the above, we arrive at a linear relation called **Hubble's law** that connects a cosmological parameter (H_0) with two observable quantities (z and D):
  - $H_0 \approx \dfrac{\delta R_U}{\delta t} = \dfrac{z}{D/c}$ or $cz = H_0 D$
- Unit: **km/s/Mpc** where **Mpc** is 10^6 parsec (or 3 million lyr) (e.g., our distance to Andromeda galaxy is 0.77 Mpc)

Redshift-Distance Relation Expected from an Expanding Universe

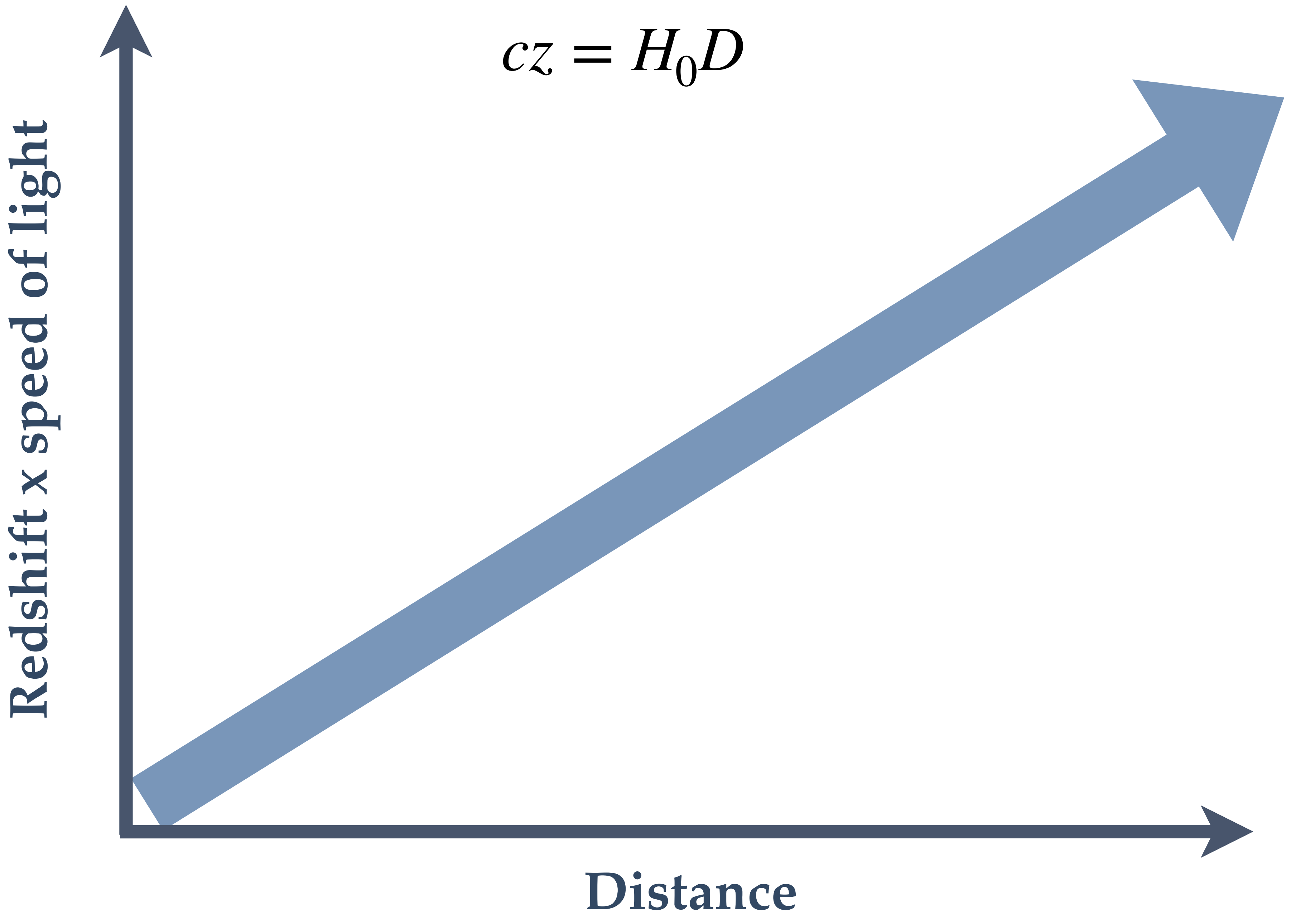
As derived previously, we would expect a linear correlation between redshift and light travel time, and the **slope** of the correlation is ***H_0***.



$$z = H_0 \times D/c$$

Redshift-Distance Relation Expected from an Expanding Universe

Instead of dividing distance by speed of light, we can multiply it to redshift. The **slope** of the correlation is still ***H_0***.



The Unit of Hubble Constant: km second^{-1} Mpc^{-1}

- **Distance (D)**
  - Astronomical Unit (AU) is the mean Earth-Sun distance (1 AU ~ 150 million km)
  - 1 parsec = 206,265 AU, 1 Mpc = 1e6 pc
  - Distance to M31: 0.77 Mpc

- **Cosmological Redshift (z_a)**
  - $1 + z = \dfrac{1}{R_U(t_\mathrm{emit})}$, where $R_U(t_\mathrm{emit})$ is the dimensionless scale factor of the universe at the time when light was emitted (t_emit); $R_U(t_0) = 1$ at present-day (t_0). In an expanding universe $R_U(t) < 1$ when $t < t_0$, leading to z > 0.

- **Hubble Constant (H_0)**
  - Present-day expansion rate: $H_0 = \dot{R}_U(t_0)/R_U(t_0) = \dot{R}_U(t_0)$ since $R_U(t_0) \equiv 1$
  - Connection w/ observables: $H_0 \approx \dfrac{\delta R_U}{\delta t} = \dfrac{z}{D/c} = \dfrac{cz}{D}$, unit: km/s/Mpc

Doppler Shift vs. Cosmological Expansion

- The **classical Doppler shift formula,**

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_0} = 1 + \frac{v_r}{c}$$

gives recession velocity:

$$v_r = cz$$

- The **relativistic Doppler shift formula,**

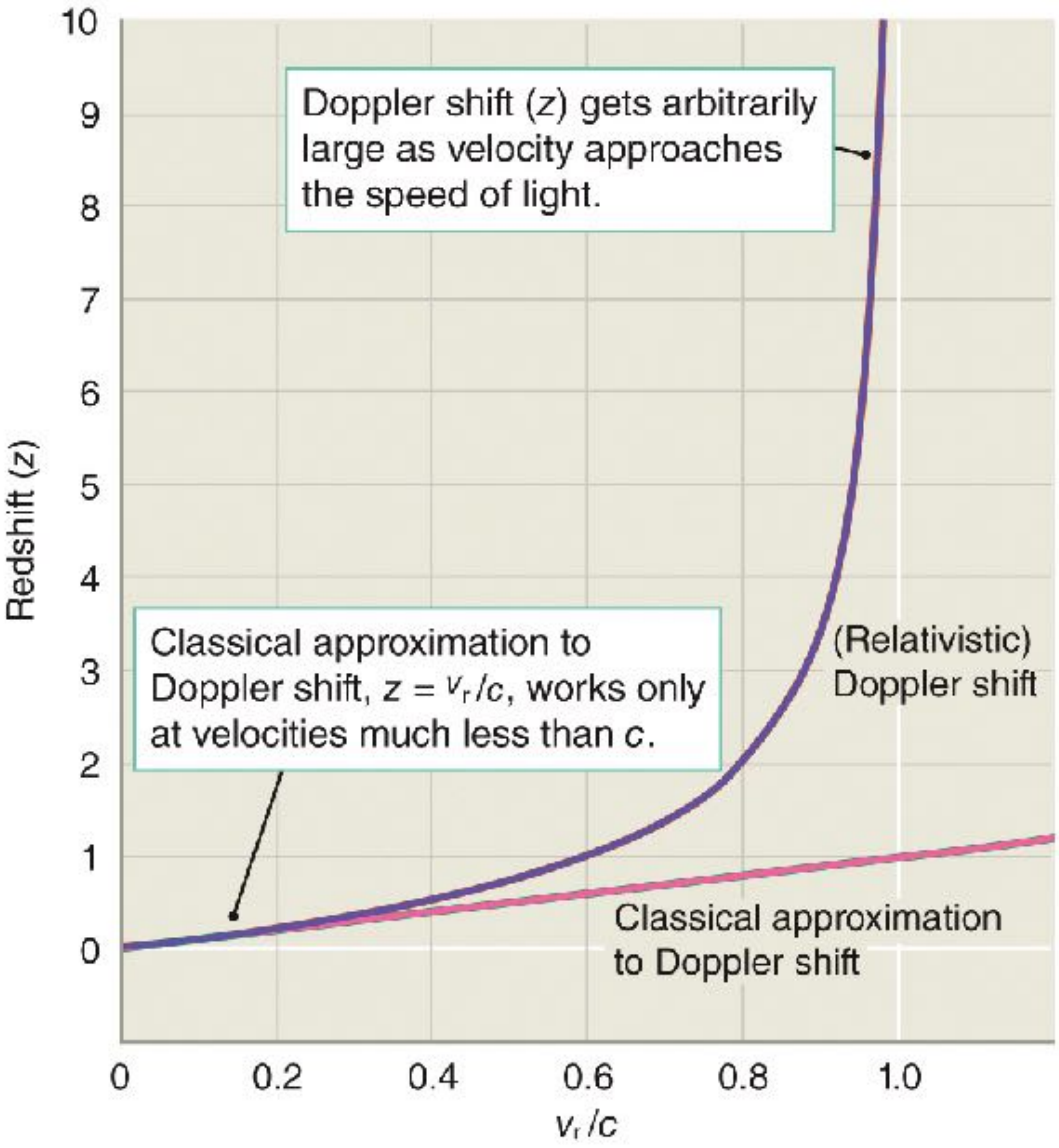
$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_0} = \sqrt{\frac{1 + v_r/c}{1 - v_r/c}}$$

gives recession velocity:

$$v_r = c\frac{(1+z)^2 - 1}{(1+z)^2 + 1}$$

- But, **cosmological redshift** should be understood as a ratio of **scale factors**:

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_0} = \frac{1}{R_U(z)}$$


Doppler shift (z) gets arbitrarily large as velocity approaches the speed of light.

Classical approximation to Doppler shift, $z = v_r/c$, works only at velocities much less than c.

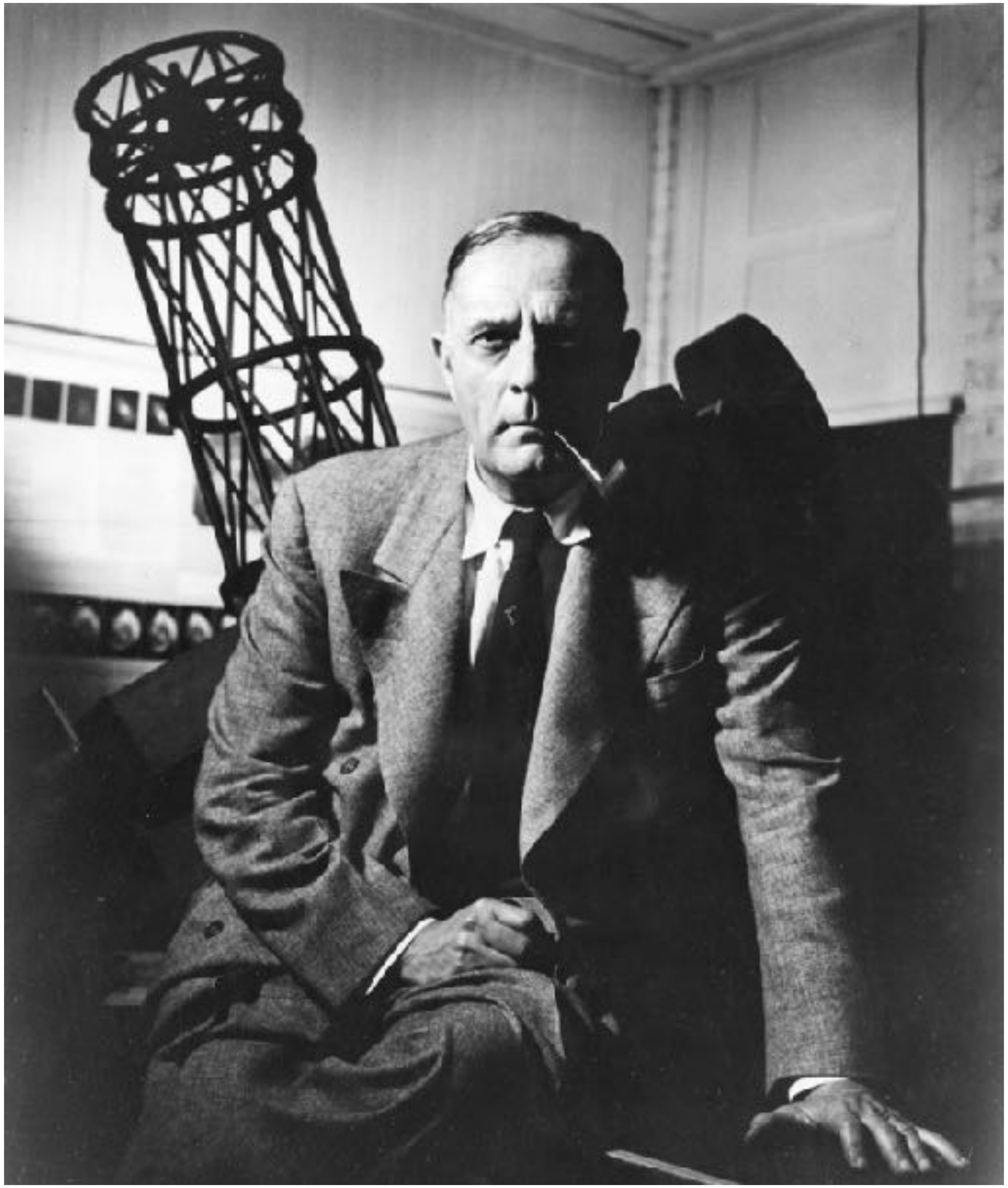
(Relativistic) Doppler shift

Classical approximation to Doppler shift

Evidence for an expanding Universe:

Discovery of Hubble's Law: redshift-distance relation at $z < 0.2$

Edwin Hubble (1889-1953)



- Born in **Marshfield, Missouri**
- B.S. (1910) & Ph.D. (1917), **University of Chicago**
- Key accomplishments:
  - **Extragalactic distances**: “island universes” instead of nebulae inside the Galaxy
  - **Hubble’s Law**: the beginning of Cosmology
  - **The age of the Crab nebula** and its association with the supernova in 1054 AD.

University Of Chicago Maroons Basketball Team (1906-1910)

Hubble played Forward & Center positions in the **U of Chicago basketball team** that won **four** consecutive **Big Ten Conference champions.**


1908


1907

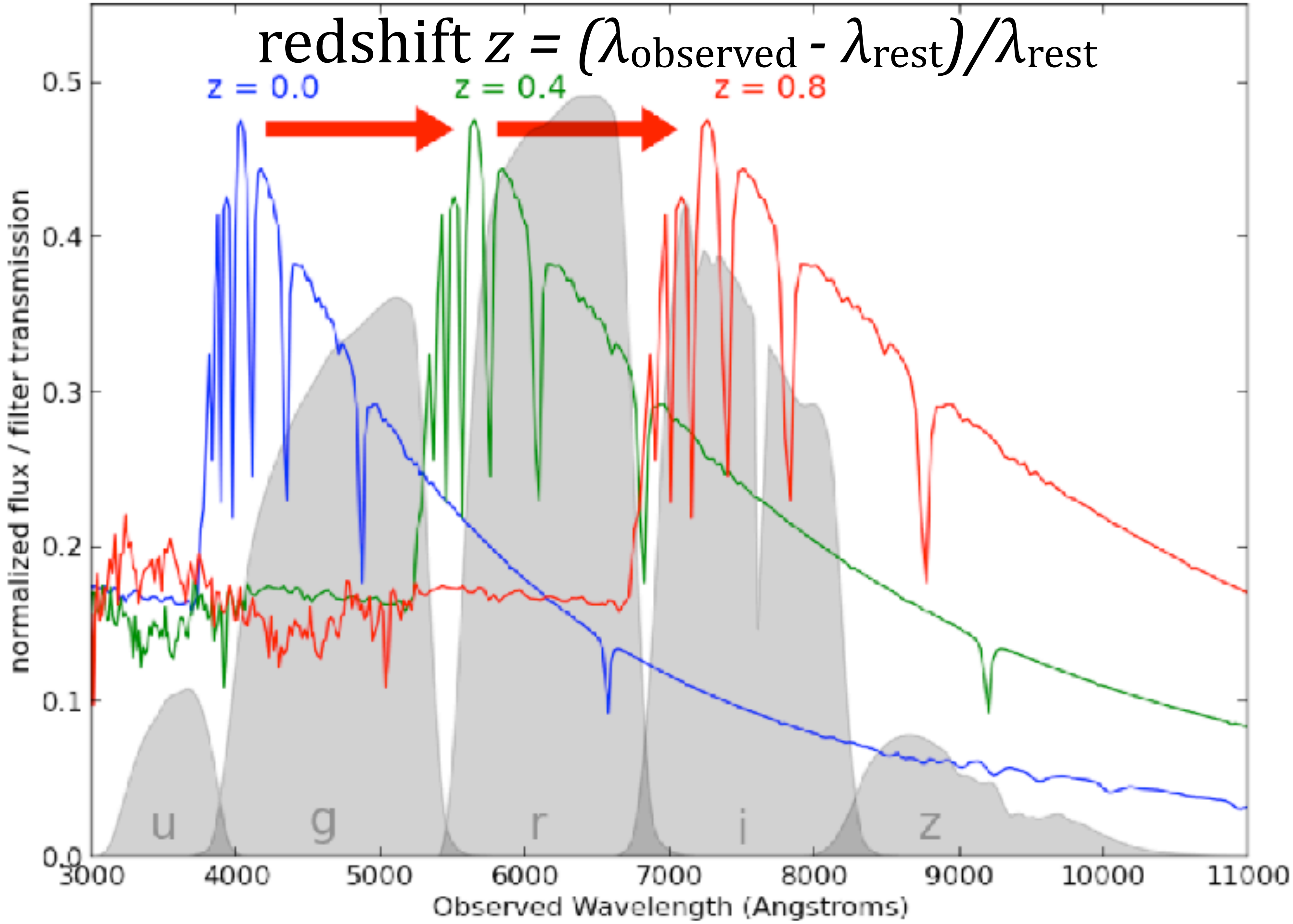
THE BASKETBALL SQUAD

Top row—Dr. Raycroft Page Buhlig Carter Georgen
Middle row—Falls Schommer Houghton (Capt.) McKeag, Henry
Bottom row—Hubble Harris Hoffman


1909

Spectroscopy: radial velocity measurements to trace the flow

- Instead of finding similar numbers of *blueshifted* and *redshifted* galaxies, Hubble found that most of the galaxies are *redshifted* — i.e., they appear to be moving away from us.



redshift $z = (\lambda_{\mathrm{observed}} - \lambda_{\mathrm{rest}})/\lambda_{\mathrm{rest}}$

Vesto Slipher (1875-1969)

Vesto Melvin Slipher. Courtesy of the National Academy of Sciences.

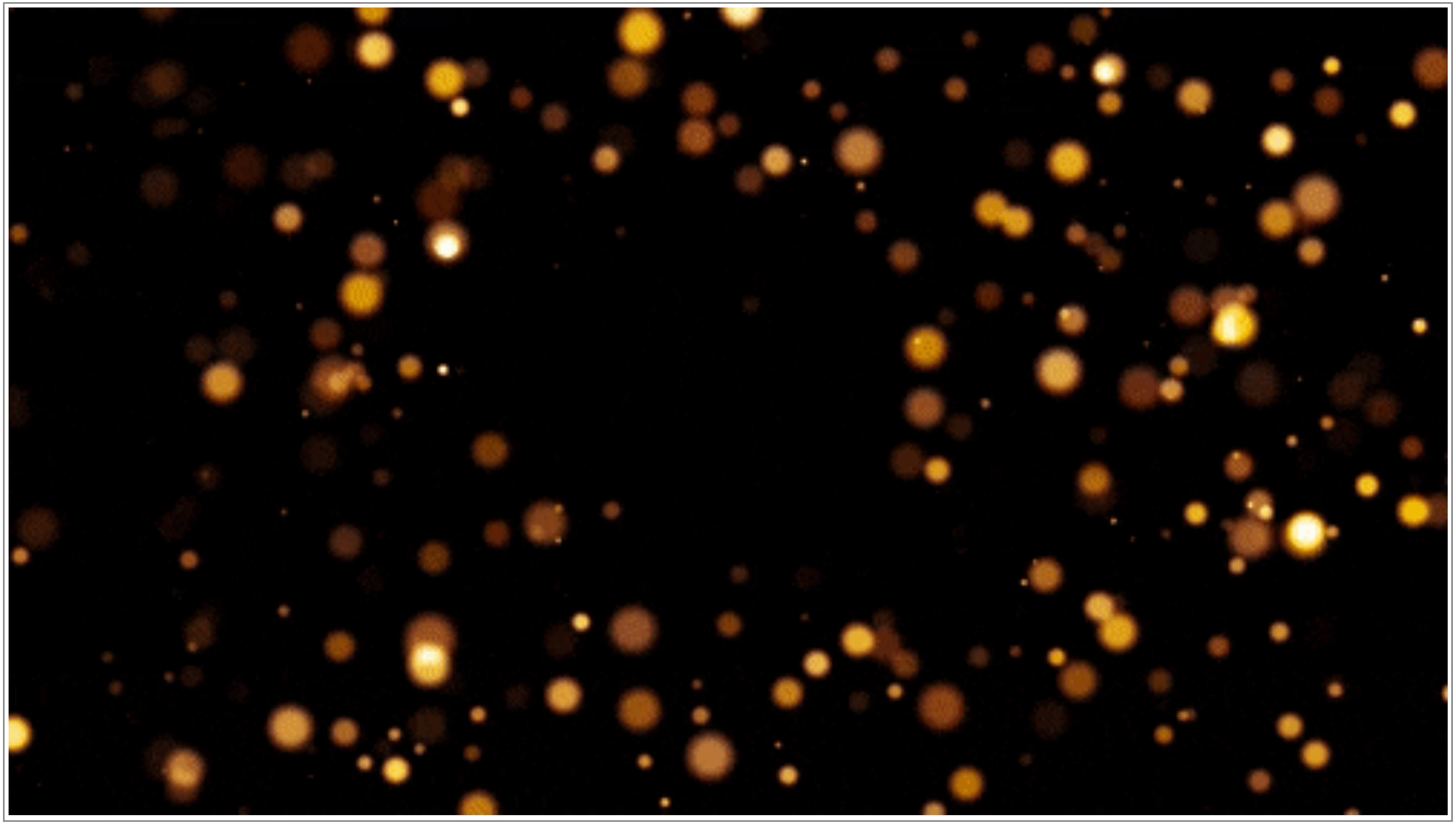
Vesto Slipher provided key spectroscopic measurements of the galaxies on the original Hubble's diagram.

He was born on the family farm in Mulberry, Indiana. As biographer William Graves Hoyt noted, Slipher's early life on the farm "helped him develop the strong, vigorous constitution that later stood him in good stead for the more strenuous aspects of observational astronomy." Slipher received Ph.D (1909) in Astronomy from Indiana University.

Lowell Observatory. Courtesy of the Wall Street Journal/State of Arizona.

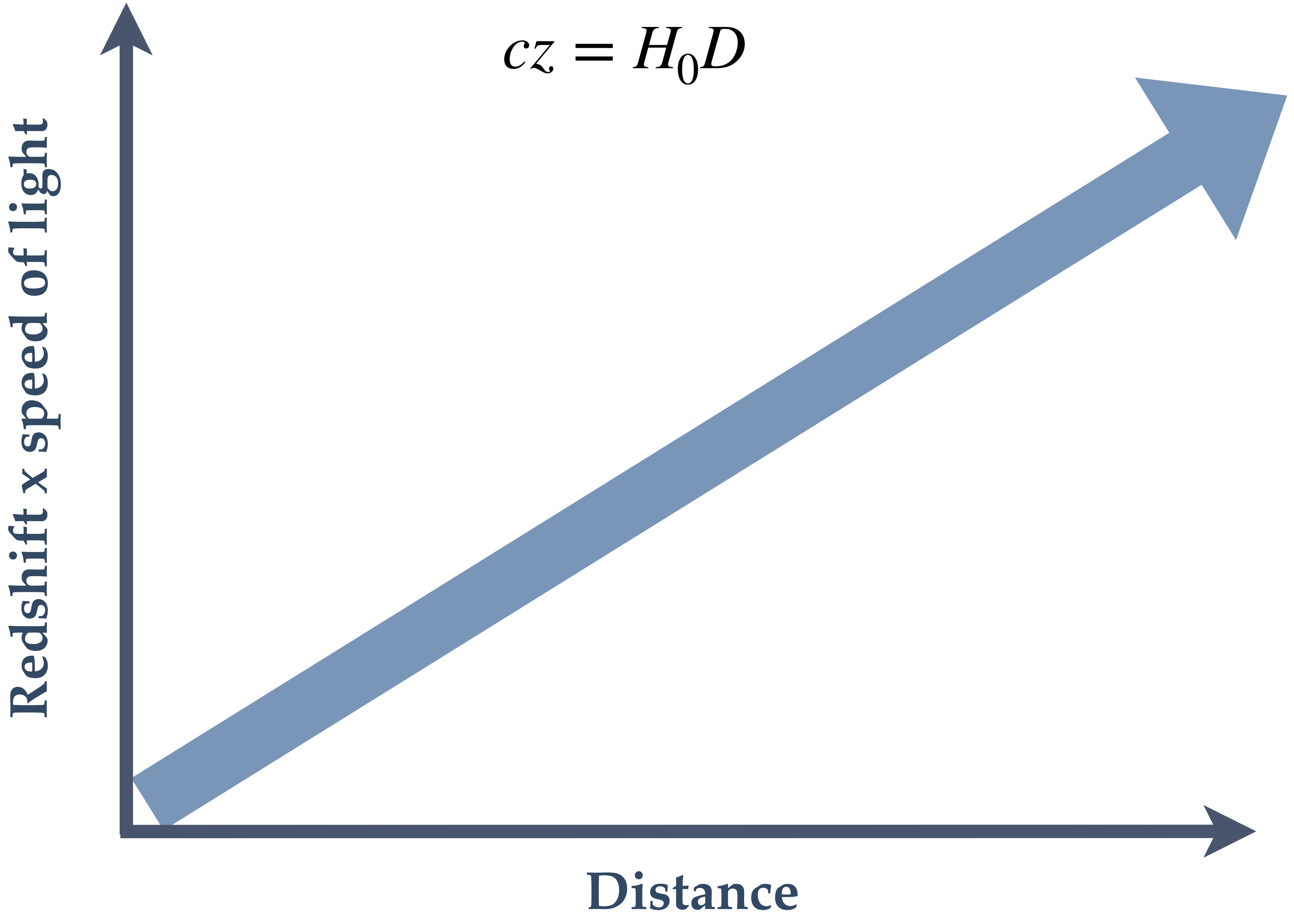
Spectroscopy: radial velocity measurements to trace the flow

- Use galaxies as “massless” particles to trace kinematics.
- Before Hubble’s discovery, pure gravitational motions are expected.



Redshift-Distance Relation Expected from an Expanding Universe

As derived previously, we would expect a linear correlation between redshift and light travel time, and the **slope** of the correlation is ***H_0***.

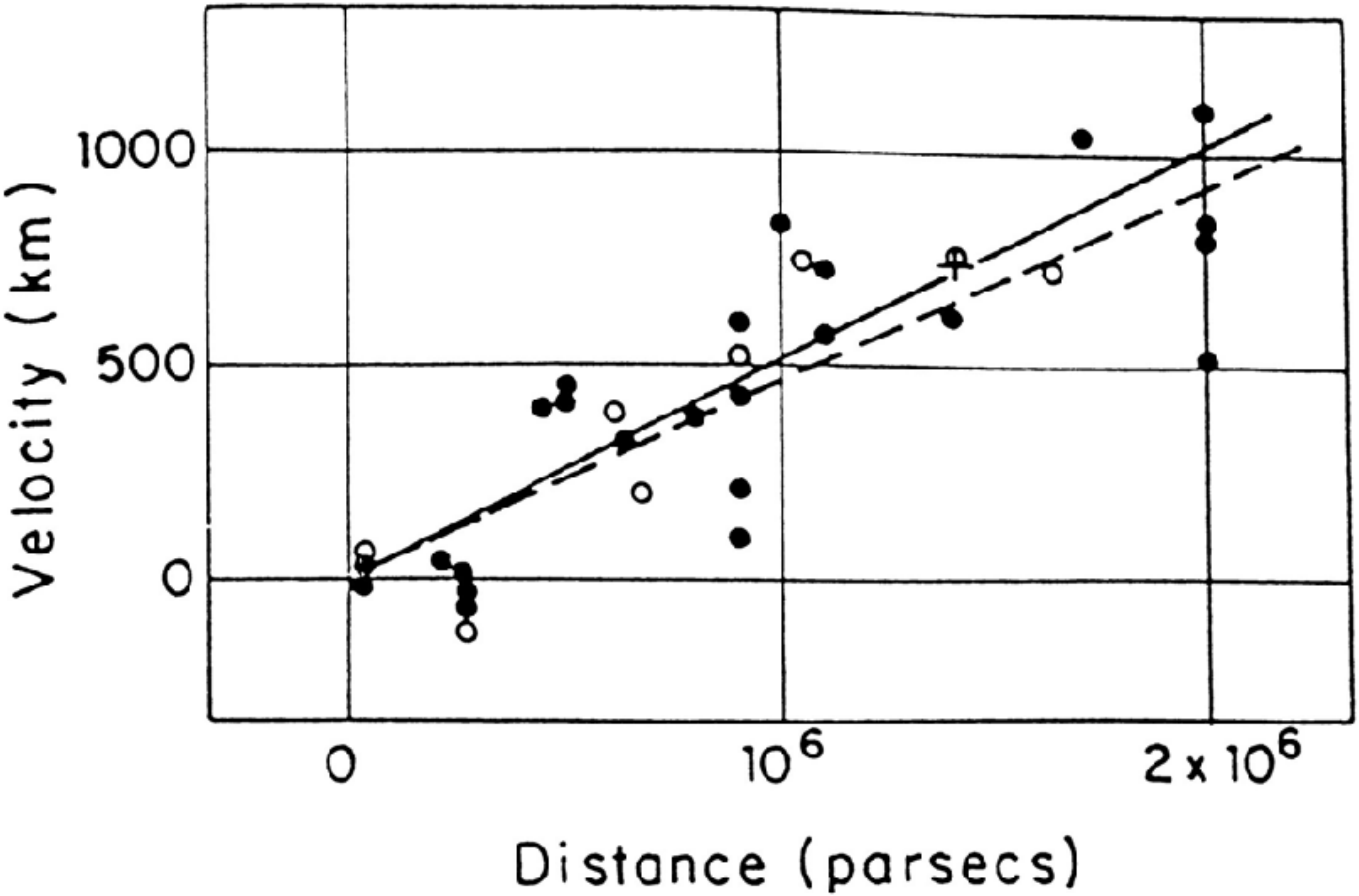


$$cz = H_0 D$$

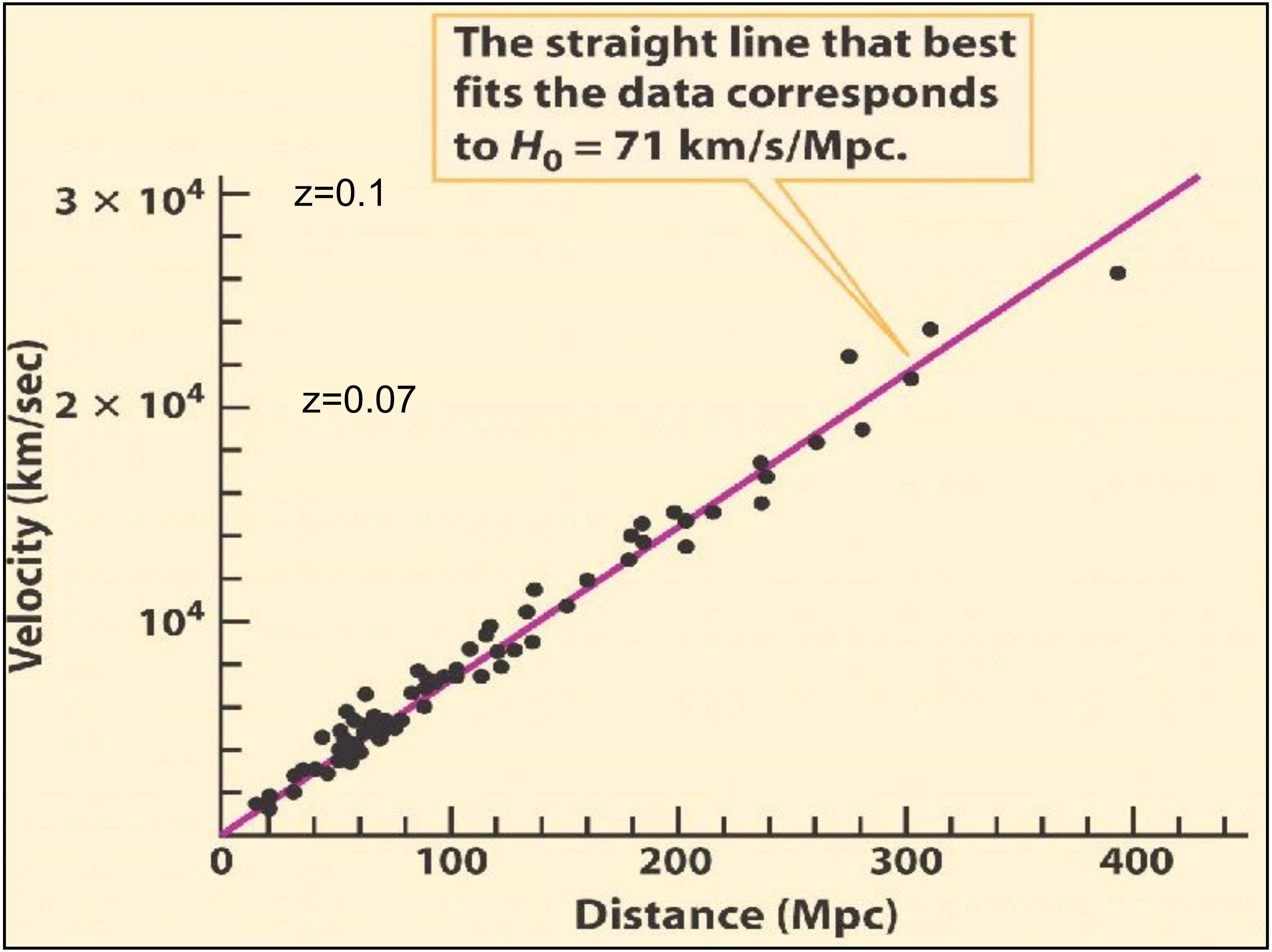
Hubble’s Law: discovered in 1929

The slope of the **redshift-distance relation** measures the expansion rate of the universe, and it’s called the **Hubble constant (H_0).** In 1929, Hubble measured a value that is 10x too high at 500 km/s/Mpc.

Hubble (1929): “*A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae*”

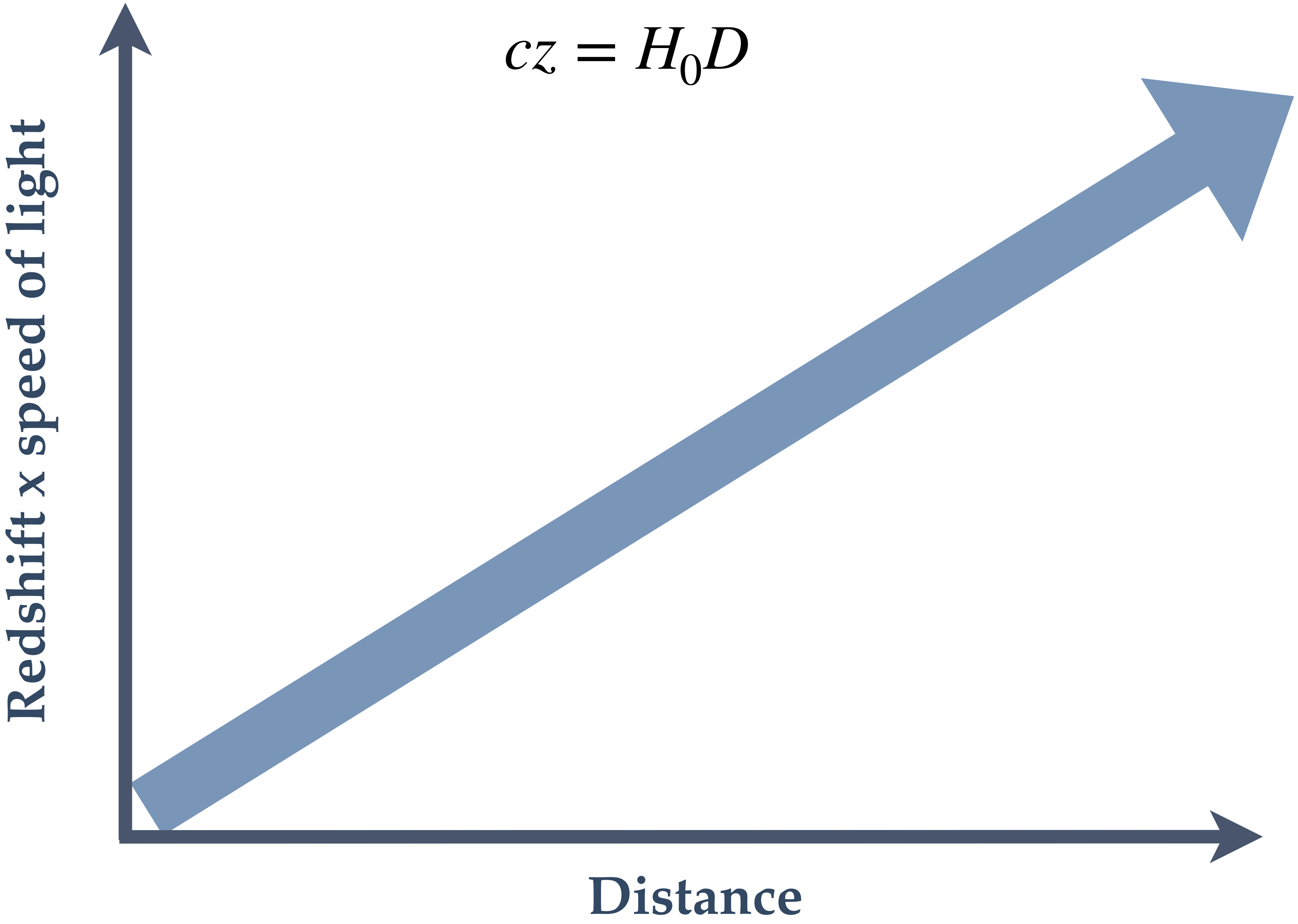


Modern Hubble Diagram using Distances from SNe Ia



Hubble’s Law: an empirical relation between redshift and distance

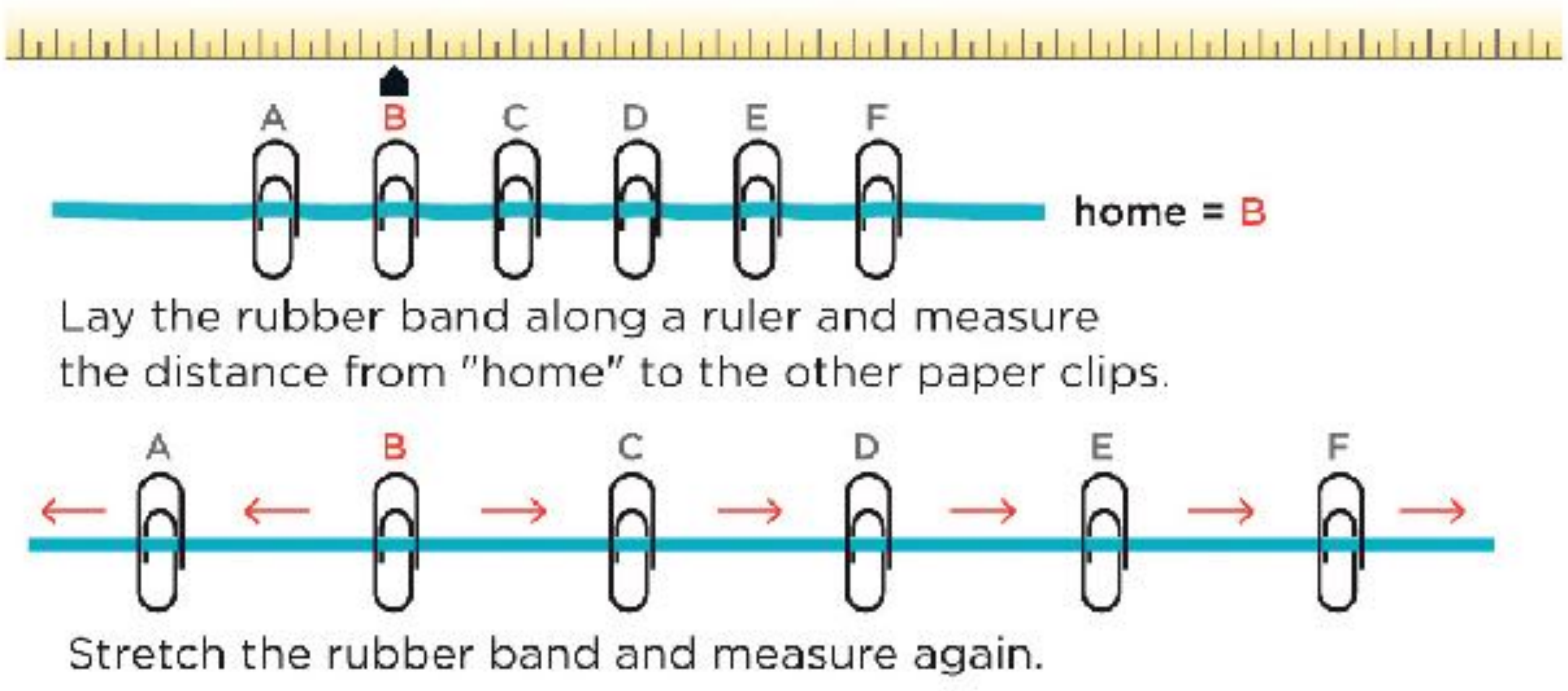
The cosmic expansion is very slow and its effect can only be seen over great distances. e.g., between Andromeda galaxy and the Milky Way galaxy (2.5 Mlyr), space expands by only 50 km every second. Still, H_0 can be measured from the slope of the redshift-distance relation.



$$cz = H_0 D$$

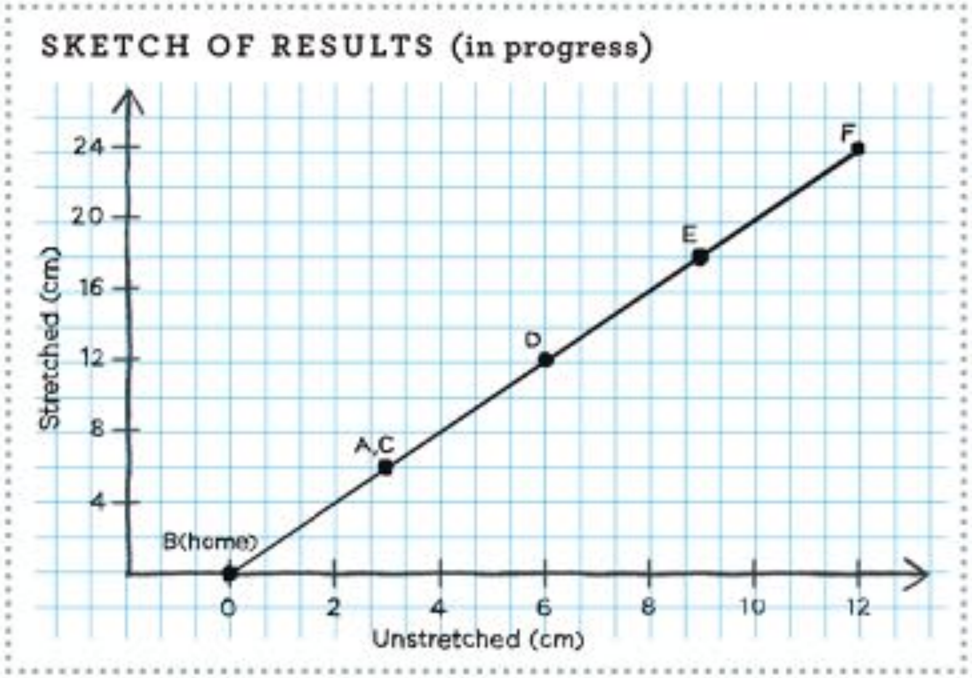
Hubble Flow: Visualizing Expansion in 1D

- Simple expansion model: paper clips on a rubber band
- As the rubber band stretches, an ant riding on clip B:
  - observes itself as stationary
  - observes clip F moving away twice as fast as clip D
  - observes clip A and C moving away at the same speed
- An ant on any paper clip would make similar observations.



Lay the rubber band along a ruler and measure the distance from "home" to the other paper clips.

Stretch the rubber band and measure again.



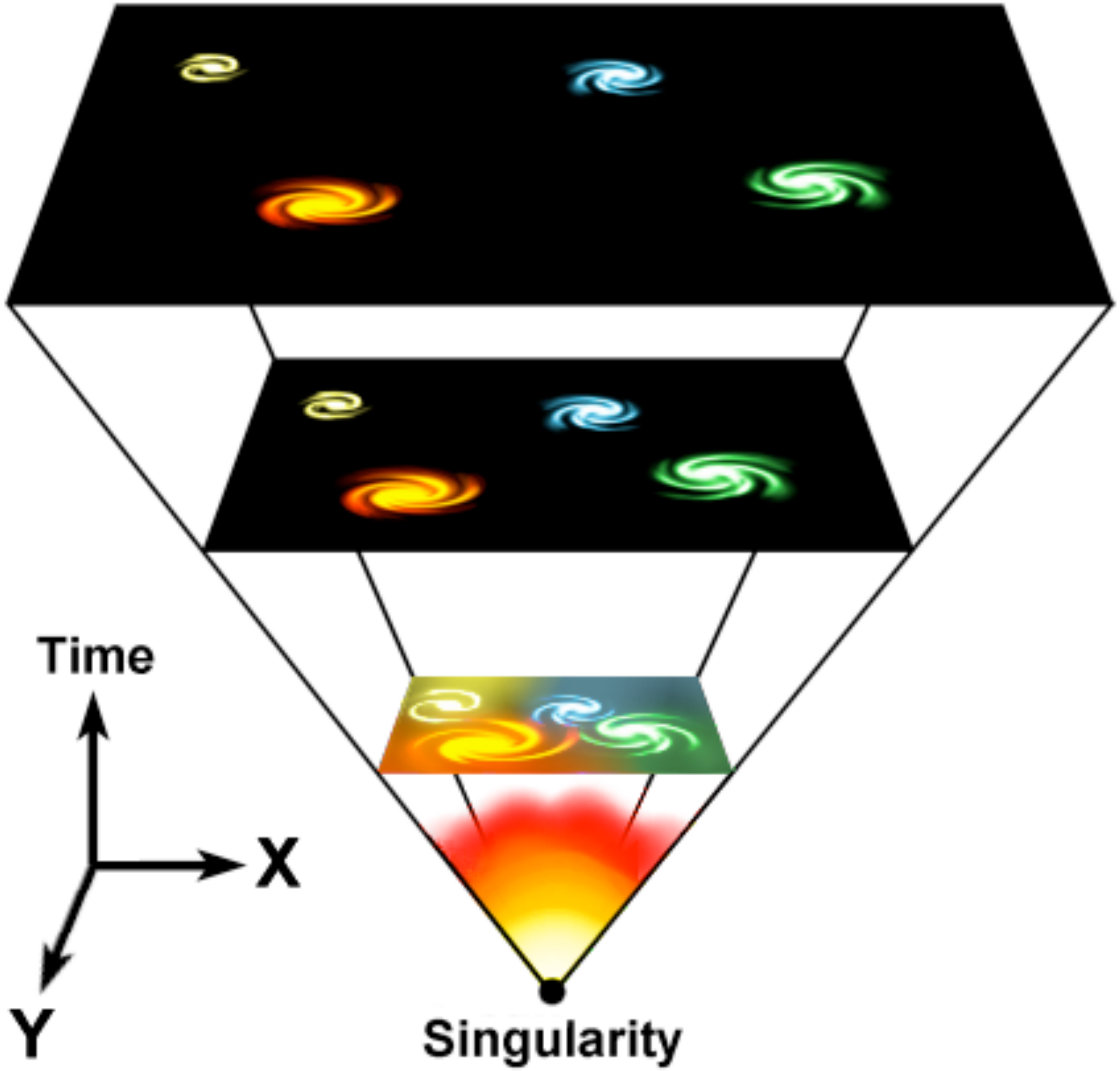
SKETCH OF RESULTS (in progress)

The Hubble Time ($t_H = 1/H_0$)

An estimate of the age of the Universe

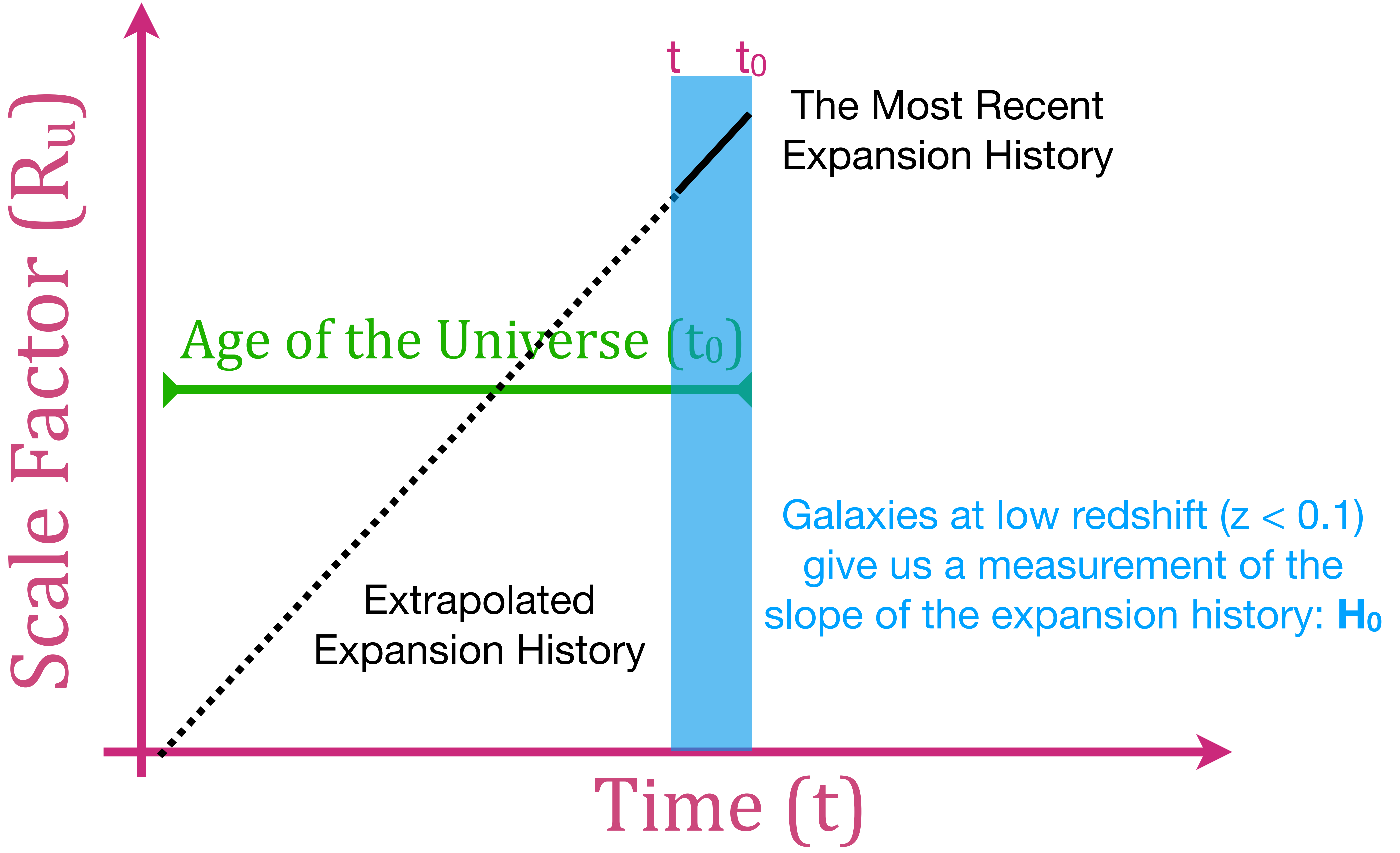
Expansion of space means that Universe was very small in distant past

- If galaxies are getting farther apart now, they must be closer together in the past. Hubble's law implies that the entire Universe started from a single point. **Can we estimate when the Universe was a single point?**



Derivation of the Hubble time using Scale Factor

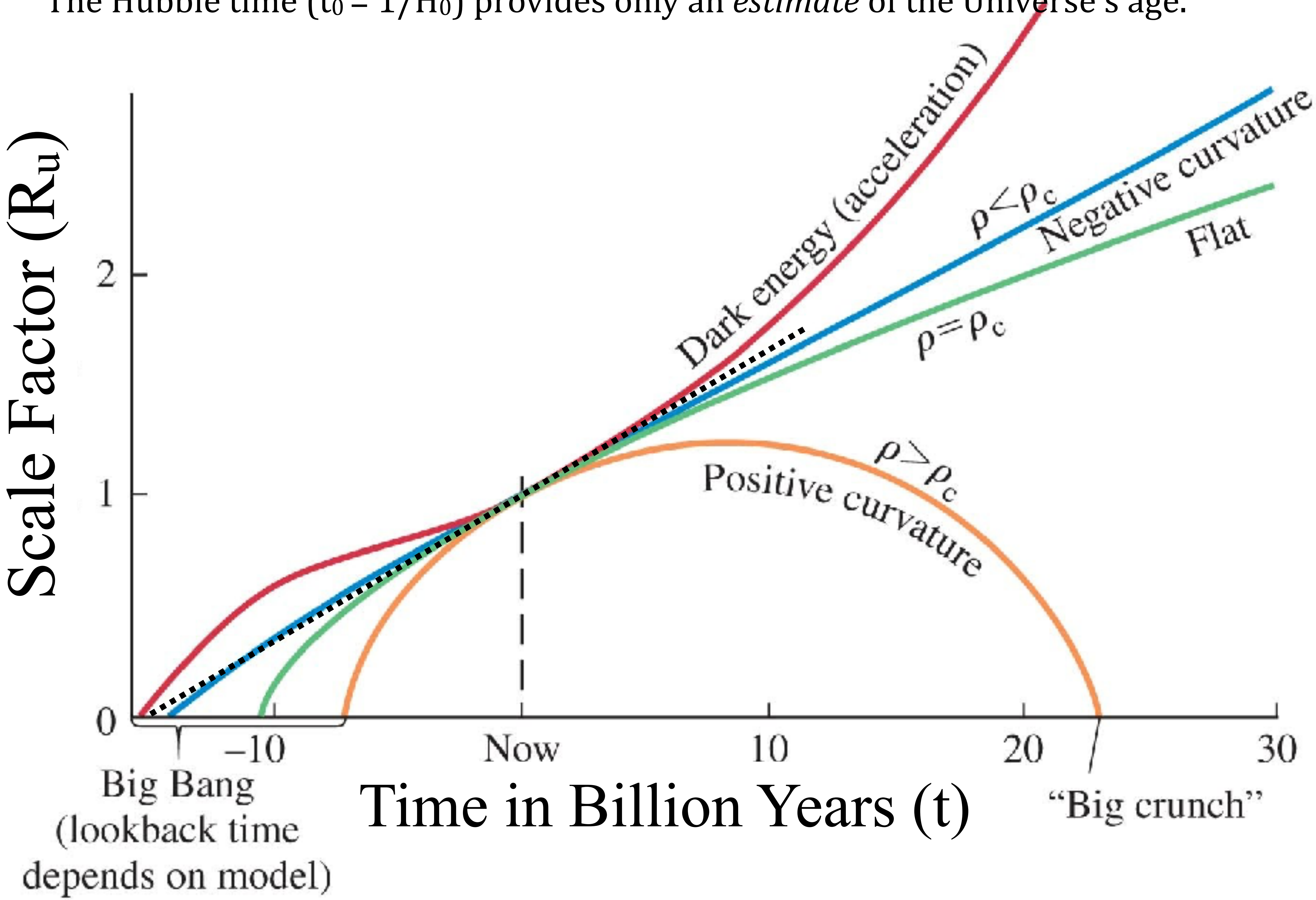
- **Hubble's law**: $cz = H_0 D = H_0 [c (t_0 - t)]$, where we used the ***lookback time*** times c to replace distance D. Canceling c on both sizes, we have $\boldsymbol{z = H_0 (t_0 - t)}$
- **Scale factor**: $R_u = 1/(1+z) \approx 1 - z$ for $z \ll 1$
- which give the *most recent expansion history* of the Universe: $\boldsymbol{R_u = 1 - H_0 (t_0 - t)}$,
- Extrapolating the relation to $\boldsymbol{R_u(t=0) = 0}$, we solve for *Hubble time:* $\boldsymbol{t_0 = 1/H_0}$



Galaxies at low redshift ($z < 0.1$) give us a measurement of the slope of the expansion history: $\mathbf{H_0}$

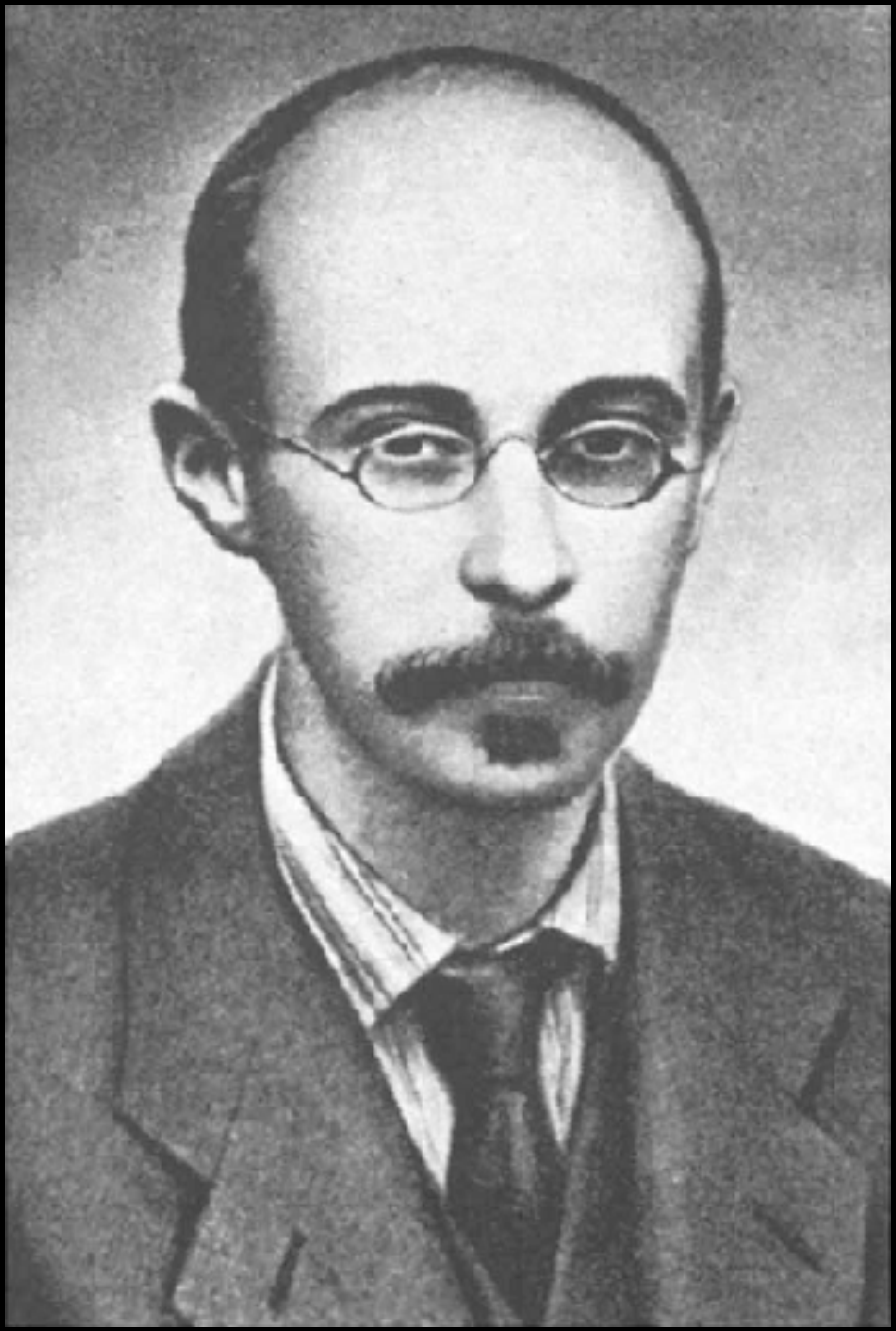
The various possible expansion histories of the Universe

The Hubble time ($t_0 = 1/H_0$) provides only an *estimate* of the Universe's age.



Mathematical description of the dynamics of the Universe:

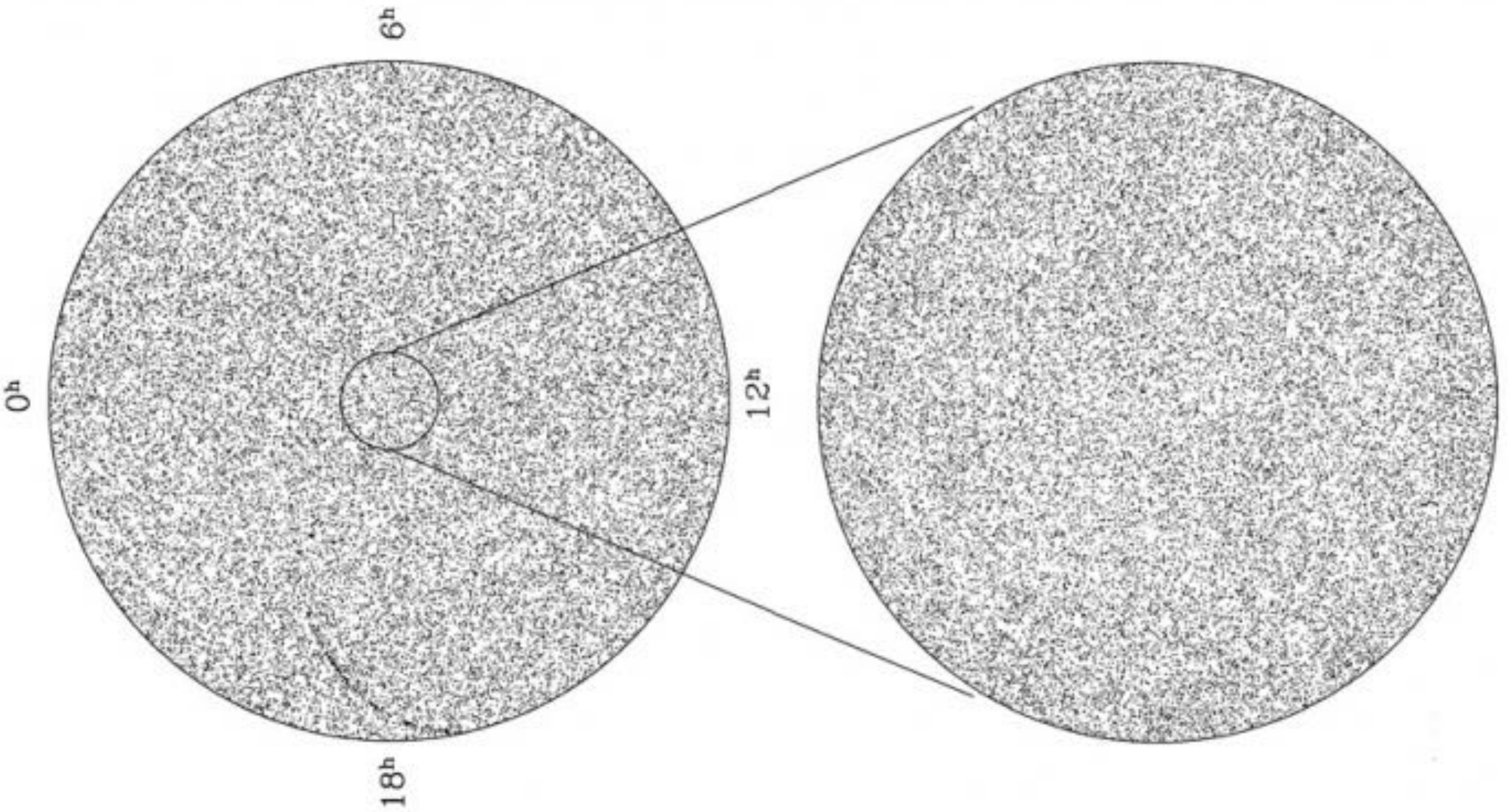
The Friedmann Equation



Alexander Friedmann (1888 – 1925) was a Russian physicist and mathematician. Fought in WWI as an aviator. Died at age 37 from typhoid fever.

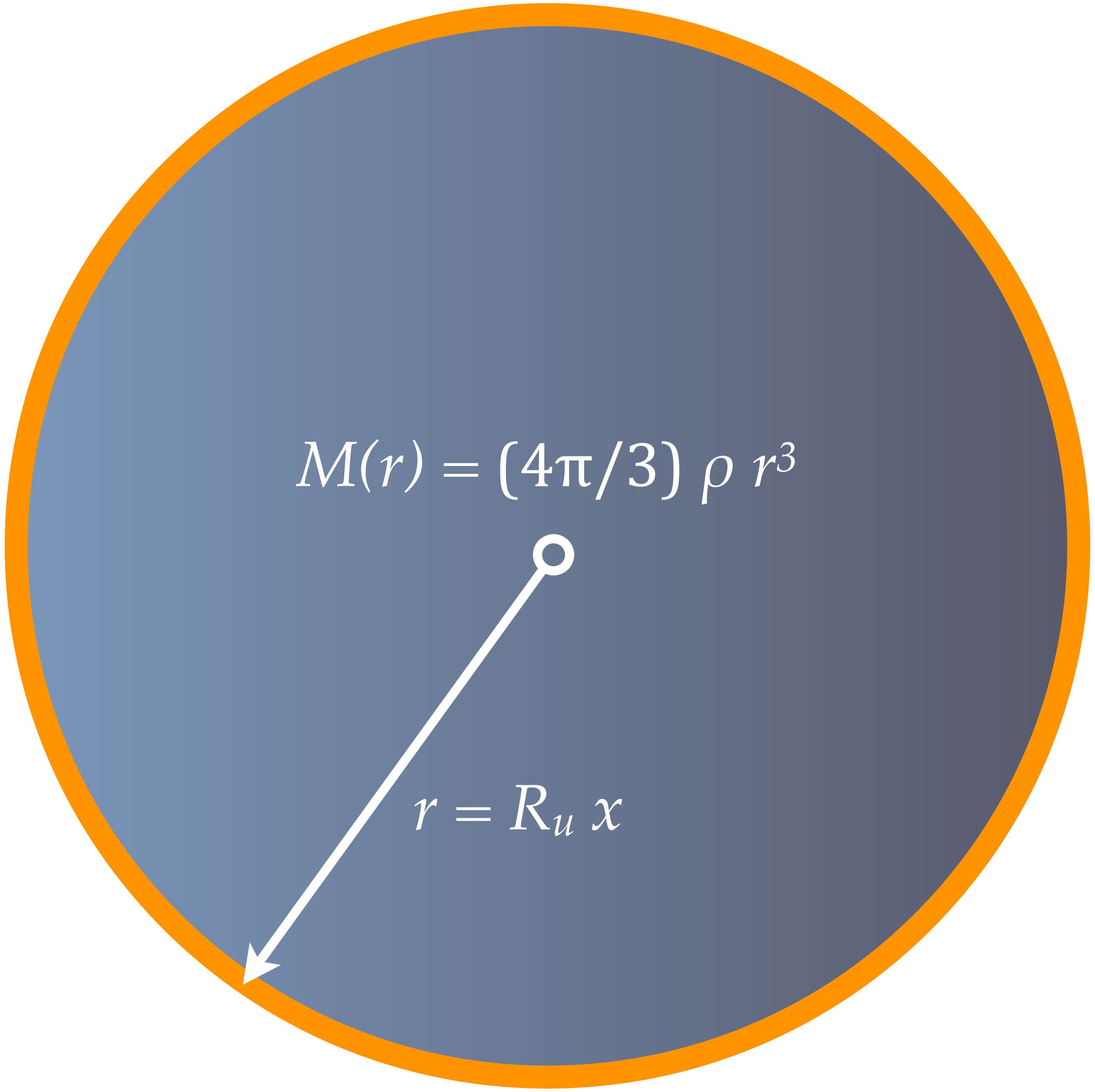
Fundamental Assumption: The Cosmological Principle

- Although galaxies tend to clump, on the largest cosmic scales, the Universe is both **homogeneous** and **isotropic**
  - **Homogeneous:** there is no preferred **location** in the Universe
  - **Isotropic**: there is no preferred **direction** in the Universe



Friedmann Equation: Classic Derivation based on Energy Conservation

- Imagine a **spherical shell** with **unit mass** in a **matter-only universe** with a **comoving radius** of **x**, as the universe expands:
  - its **physical radius** at time t is $r(t) = R_U(t)\,x$,
  - its **expanding velocity** is $v(t) = \dot{R}_U(t) \cdot x$, and
  - the **mass enclosed** in the shell is $$M(r) = \frac{4\pi}{3}[R_U(t)x]^3 \cdot \rho(t)$$



Friedmann Equation: Newtonian Derivation from Energy Conservation

- Imagine a **spherical shell** with **unit mass** in a **matter-only universe** with a **comoving radius** of x, as the universe expands:
  - its **physical radius** at time *t* is $r(t) = R_U(t)\, x$,
  - its **expanding velocity** is $v(t) = \dot{R}_U(t) \cdot x$, and
  - the **mass enclosed** in the shell is $M(r) = \frac{4\pi}{3}[R_U(t)x]^3 \cdot \rho(t)$
- We can write down the **kinetic + gravitational potential energy** for the unit-mass spherical shell:

$$E = \frac{1}{2}v^2 - \frac{GM(r)}{r} = \frac{1}{2}\dot{R}_U(t)^2x^2 - \frac{4\pi}{3}G\rho(t)R_U(t)^2x^2$$

- This **energy per unit mass** must be the same for every shell with the same comoving radius **x**, and it is related to the geometry of the Universe, which is defined by **comoving curvature** $k = \kappa/R^2$ (where $\kappa = 0, \pm 1$ and R is comoving curvature radius):

$$E \equiv -\frac{1}{2}kc^2x^2$$

- Combining the two Eqs. and cancel out x^2 on both sides, we obtain:

$$\left(\frac{\dot{R}_U^{\,2}}{R_U^2} - \frac{8}{3}\pi G\rho\right) R_U^2 = -kc^2$$

Friedmann Equation: Classic Derivation based on Energy Conservation

- Define **Hubble parameter** (whose present-day value is H_0):

$$H(t) \equiv \dot{R}_U / R_U$$

- define a new parameter called **critical density**:

$$\rho_c = \frac{3H^2}{8\pi G}$$

note that because H varies, the critical density is not a constant.

- we can now rewrite the energy conservation

$$\left(\frac{\dot{R}_U^{\ 2}}{R_U^2} - \frac{8}{3}\pi G \rho \right) R_U^2 = -kc^2$$

as:

$$H^2 \left(1 - \frac{\rho}{\rho_c} \right) R_U^2 = -kc^2$$

- next, define the density ratio as a dimensionless **density parameter** called Omega:

$$\Omega_m \equiv \rho_m / \rho_c$$

- Finally, we have the Friedmann equation in a **matter-only universe**:

$$H^2 \ (1 - \Omega_m) \ R_U^2 = -kc^2$$

Calculating the Critical Density in Today's Universe

- The critical density, ρ_c, varies as the universe evolves, just like the Hubble parameter. Its value **Today**, $\rho_c(t_0)$, can be calculated from the Hubble constant:

$$\rho_c = \frac{3H_0^2}{8\pi G}$$

- If we rewrite $H_0 = 70\ \mathrm{km/s/Mpc}$ as $H_0 = 2.3 \times 10^{-18}/\mathrm{s}$ by converting Mpc to km, then the critical density of Today's universe is:

$$\rho_c = \frac{3 \times \left(2.3 \times 10^{-18}/\mathrm{s}\right)^2}{8 \times \pi \times \left[6.67 \times 10^{-20}\mathrm{km}^3/\left(\mathrm{kg\ s^2}\right)\right]}$$

$$\rho_c = 9.5 \times 10^{-27}\mathrm{kg/m^3}$$

- Given that $\rho = m_H n$ for hydrogen, this is equal to a number density of **5.7 hydrogen atoms per cubic meter** (m_H = 1.67e-27 kg).
- It seems small, but the observed mass density of ordinary matter, averaged over large volumes, is **less than one hydrogen atom per cubic meter** (n < 1 m^{-3} = 1e-6 cm^{-3}).

Expansion histories predicted by the Friedmann Equation

Part I: matter-only universe

Solving the Friedmann Equation in a Matter-Only Universe

- The **Friedmann Equation (FE)** in a **matter-only universe** is:

  $$H^2\ (1-\Omega_m)\ R_U^2 = -\, kc^2$$

  where the original terms were replaced by three key parameters:
  - **Hubble parameter**: $H(t) \equiv \dot{R}_U / R_U,$
  - **critical density**: $\rho_c = \frac{3H^2}{8\pi G}$, and
  - **density parameter**: $\Omega_m \equiv \rho_m / \rho_c$,
- To solve the Friedmann Equation, we need
  - the **boundary condition** at t = t$_0$**:**

    $H = H_0$, $R_U = 1$, thus $H_0^2(1-\Omega_{m,0}) = -\, kc^2$
  - the **density relation** for matter:

    $$\frac{\Omega_m}{\Omega_{m,0}} = \frac{\rho_m \rho_{c,0}}{\rho_{m,0}\rho_c} = \frac{\rho_m}{\rho_{m,0}} \frac{H_0^2}{H^2} = \frac{1}{R_U^3} \frac{H_0^2}{H^2}$$
- Replacing $-kc^2$ and Ω_m with the above two relations, we arrive at a solution of the **Hubble parameter** as a function of **redshift** or scale factor:

  $$H^2 = \frac{H_0^2}{R_U^2}[(1-\Omega_{m,0}) + \Omega_{m,0}/R_U]$$

Solution of the 1st Friedmann Equation for matter-only universes

- Plug in the boundary condition and the density parameter relation:

$$H^2 = \frac{H_0^2}{R_U^2}[(1-\Omega_{m,0}) + \Omega_{m,0}/R_U]$$

- The above is the solution for the Hubble parameter. For example, for

$$\Omega_{m,0} = 1 \Rightarrow H = H_0/R_U^{3/2} = H_0(1+z)^{3/2}$$

- To solve for the *time* evolution, we need to express $H(t) \equiv \dot{R}_U/R_U$:

$$\left(\frac{1}{R_U}\frac{dR_U}{dt}\right)^2 = \frac{H_0^2}{R_U^2}[(1-\Omega_{m,0}) + \Omega_{m,0}/R_U]$$

- The simplest case is $\Omega_{m,0} = 1$ ***(Einstein-de Sitter universe)***. Separate time *t* and scale factor R_U to two sides:

$$dt = \frac{\sqrt{R_U}dR_U}{H_0}$$

- Then integrate the differential equation from $R_U = 0$ (i.e., t = 0) to $R_U = 1/(1+z)$ [i.e., t(z)], we can solve for time as a function of scale factor or redshift:

$$t = \frac{2}{3H_0}R_U^{3/2} \text{ or } t(z) = \frac{2}{3}t_H(1+z)^{-3/2} \text{ or } R_U = (3H_0t/2)^{2/3}$$

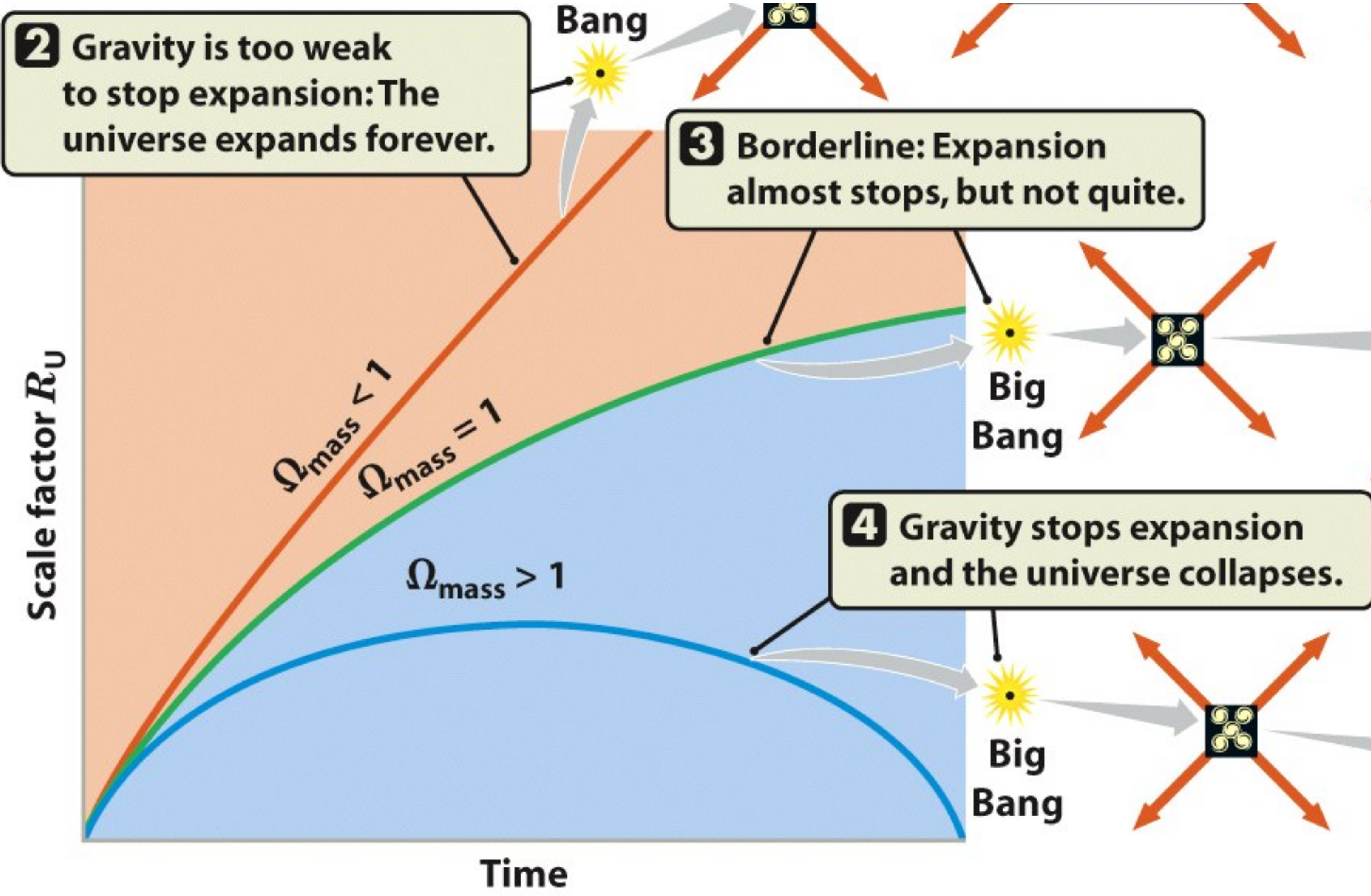
Predicted Expansion History if Only Matter Is Involved

$$H^2 = \frac{H_0^2}{R_U^2}[(1-\Omega_{m,0}) + \Omega_{m,0}/R_U]$$

$$t = t_H \int_0^{R_U} \frac{dR_U}{\sqrt{(1-\Omega_{m,0}) + \Omega_{m,0}/R_U}}$$

The expansion history depends on $\Omega_{m,0}$, while H_0 sets the overall scale.

- $\Omega_{m,0} < 1$: *sub-critical, expanding forever.*
- $\Omega_{m,0} = 1$: *critical, expanding forever, but expansion rate approaches zero as time goes. This critical universe is called* ***Einstein-de Sitter*** *universe*
- $\Omega_{m,0} > 1$: *super-critical expansion stops and the universe collapse.* ***(Big Crunch)***



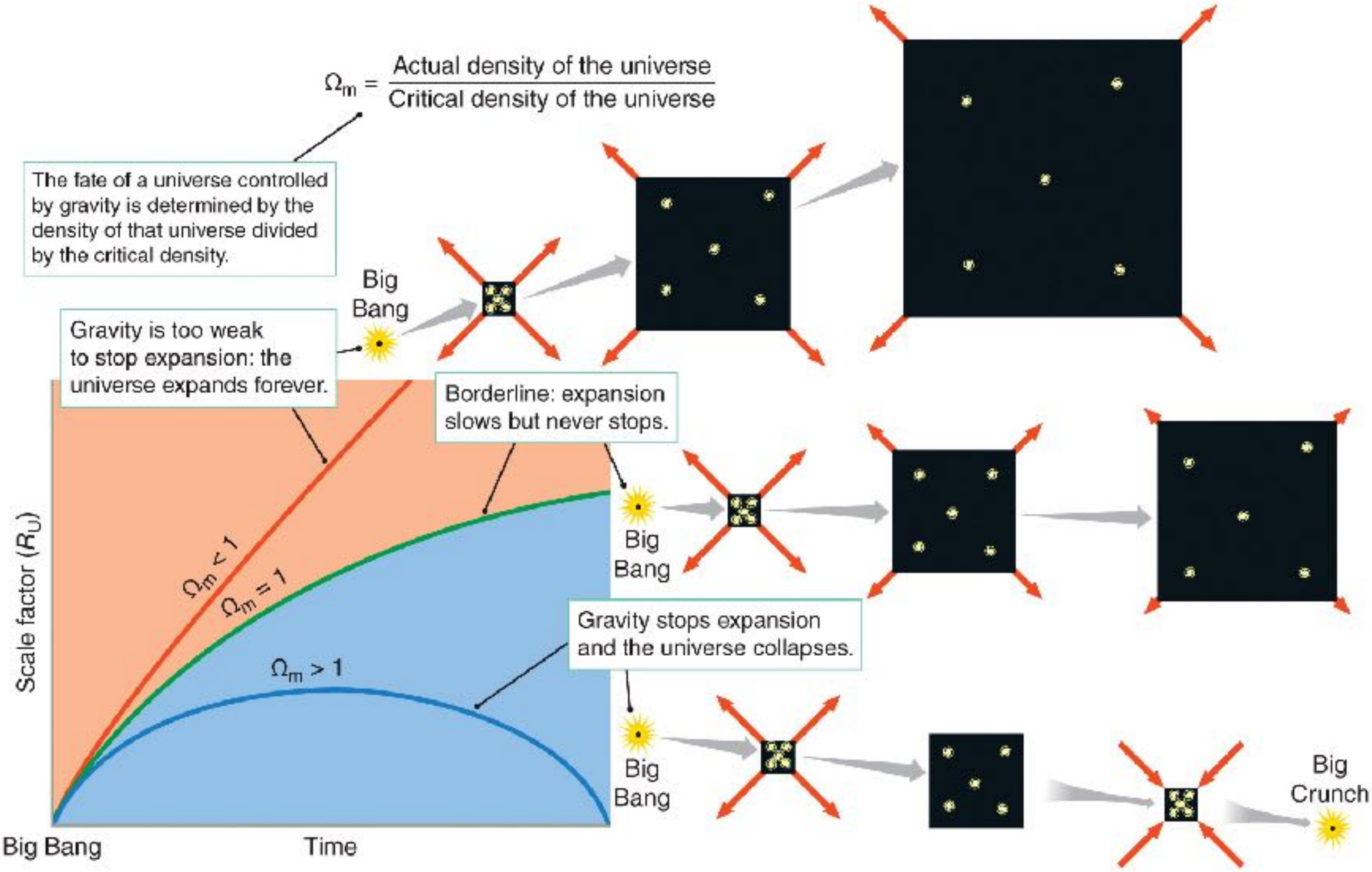
2 Gravity is too weak to stop expansion: The universe expands forever.

3 Borderline: Expansion almost stops, but not quite.

4 Gravity stops expansion and the universe collapses.

The Total Normal Matter Density is Sub-Critical

- Ordinary matter in galaxies, IGM, and ICM: $\Omega_{m,0} = 4.5\%$ (today).
- Dark matter increases $\Omega_{m,0}$ to 32% (today).



Expansion histories predicted by the Friedmann Equation

Part II: matter + dark energy universe

The Complete Friedmann Equation with matter and dark energy

- By defining a new parameter called **critical density**: $\rho_c = \dfrac{3H^2}{8\pi G}$, we have derived the Friedmann Equation for **matter-only** universe:

$$H^2 R_U^2 \left(1 - \frac{\rho_m}{\rho_c}\right) = -\,kc^2$$

- Introduce **relativistic matter** and **dark energy terms** in the energy equation:

$$\frac{1}{2}\dot{R_U}^2 - \frac{4\pi G}{3}(\rho_m + \rho_\gamma)R_U^2 - \frac{\Lambda c^2}{6}R_U^2 = -\,\frac{1}{2}kc^2$$

- The full General Relativity version of the Friedmann Equation is:

$$H^2 R_U^2 \left[1 - \left(\frac{\rho_m}{\rho_c} + \frac{\rho_\gamma}{\rho_c} + \frac{\Lambda c^2/8\pi G}{\rho_c}\right)\right] = -\,kc^2$$

- There are now three **density parameters**, i.e., define three Omega's:
  - $\Omega_m \equiv \rho_m/\rho_c$, **ordinary matter** (baryons and dark matter)
  - $\Omega_\gamma \equiv \rho_\gamma/\rho_c$, **relativistic matter** (light and neutrinos)
  - $\Omega_\Lambda \equiv \Lambda c^2/(8\pi G\rho_c)$, **dark energy** ($\Lambda$ is the cosmological constant and has the same physical unit as the curvature constant k)

- Replacing those, we have the final form of the **Friedmann Equation**:

$$H^2 R_U^2[1 - (\Omega_m + \Omega_\gamma + \Omega_\Lambda)] = -\,kc^2$$

How to Solve the Complete Friedmann Equation? H(z) solution

- **Boundary Condition at t = t_0:**

  H = H_0, R_U = 1, thus $H_0^2(1-\Omega_0) = -kc^2$

- **Density parameter relations**:

$$\frac{\Omega_m}{\Omega_{m,0}} = \frac{\rho_m \rho_{c,0}}{\rho_{m,0}\rho_c} = \frac{\rho_m}{\rho_{m,0}}\frac{H_0^2}{H^2} = \frac{1}{R_U^3}\frac{H_0^2}{H^2}$$

$$\frac{\Omega_\gamma}{\Omega_{\gamma,0}} = \frac{1}{R_U^4}\frac{H_0^2}{H^2} \quad \text{and} \quad \frac{\Omega_\Lambda}{\Omega_{\Lambda,0}} = \frac{H_0^2}{H^2}$$

- Write down the **Friedmann Equation** with the **boundary condition**:

$$H^2\ [1-(\Omega_m+\Omega_\gamma+\Omega_\Lambda)]\ R_U^2 = -kc^2 = H_0^2(1-\Omega_0)$$

  then plug in the **density parameter relations** and rearrange:

$$H^2 = H_0^2[(1-\Omega_0)R_U^{-2} + \Omega_{m,0}R_U^{-3} + \Omega_{\gamma,0}R_U^{-4} + \Omega_{\Lambda,0}]$$

- Examples:
  - For **an empty universe**:

$$\Omega_0 = \Omega_{m,0} = \Omega_{\gamma,0} = \Omega_{\Lambda,0} = 0 \Rightarrow H = H_0/R_U = H_0(1+z)$$

  - For **a matter-only flat universe (Einstein-de Sitter universe)**:

$$\Omega_0 = \Omega_{m,0} = 1, \Omega_{\gamma,0} = \Omega_{\Lambda,0} = 0 \Rightarrow H = H_0/R_U^{3/2} = H_0(1+z)^{3/2}$$

How to Solve the Complete Friedmann Equation? t(z) or $R_u(t)$ solution

- Write down the Friedmann Equation with the boundary condition and replace Hubble parameter with scale factor, $H(t) \equiv \dot{R}_U/R_U$, we have

$$H^2 = \left(\frac{1}{R_U}\frac{dR_U}{dt}\right)^2 = H_0^2[(1-\Omega_0)R_U^{-2} + \Omega_{m,0}R_U^{-3} + \Omega_{\gamma,0}R_U^{-4} + \Omega_{\Lambda,0}]$$

- Separate time and scale factor into two sides of the equation:

$$dt = \frac{1}{H_0}\frac{R_U^{-1}dR_U}{\sqrt{(1-\Omega_0)R_U^{-2} + \Omega_{m,0}R_U^{-3} + \Omega_{\gamma,0}R_U^{-4} + \Omega_{\Lambda,0}}}$$

- Integrating it from $R_U=0$ (i.e., t=0) to $R_U = 1/(1+z)$ [i.e., t(z)], we can solve for the t(z) relation for *any given values of the density parameters*.
- For example, for **a matter-only critical/flat universe (a.k.a. the Einstein-de Sitter universe),** we have solved for both H(z) and t(z):

$$\Omega_0 = \Omega_{m,0} = 1,\ \Omega_{\gamma,0} = \Omega_{\Lambda,0} = 0$$

$$\Rightarrow H = H_0/R_U^{3/2} = H_0(1+z)^{3/2}$$

$$\Rightarrow t(z) = \frac{2}{3}t_H(1+z)^{-3/2}$$

Lookback Time and Distance

- Start from the **Friedmann Equation** with the boundary condition:

$$H^2 = H_0^2[(1-\Omega_0)R_U^{-2} + \Omega_{m,0}R_U^{-3} + \Omega_{\gamma,0}R_U^{-4} + \Omega_{\Lambda,0}]$$

- We can define the **dimensionless Hubble parameter**:

$$E(R_U) \equiv \frac{H(R_U)}{H_0} = \sqrt{(1-\Omega_0)R_U^{-2} + \Omega_{m,0}R_U^{-3} + \Omega_{\gamma,0}R_U^{-4} + \Omega_{\Lambda,0}}$$

- Given the definition $H \equiv \frac{\dot{R}_U}{R_U}$, one obtains: $dt = \frac{dR_U}{R_U H(R_U)} = \frac{dR_U}{H_0 R_U E(R_U)}$

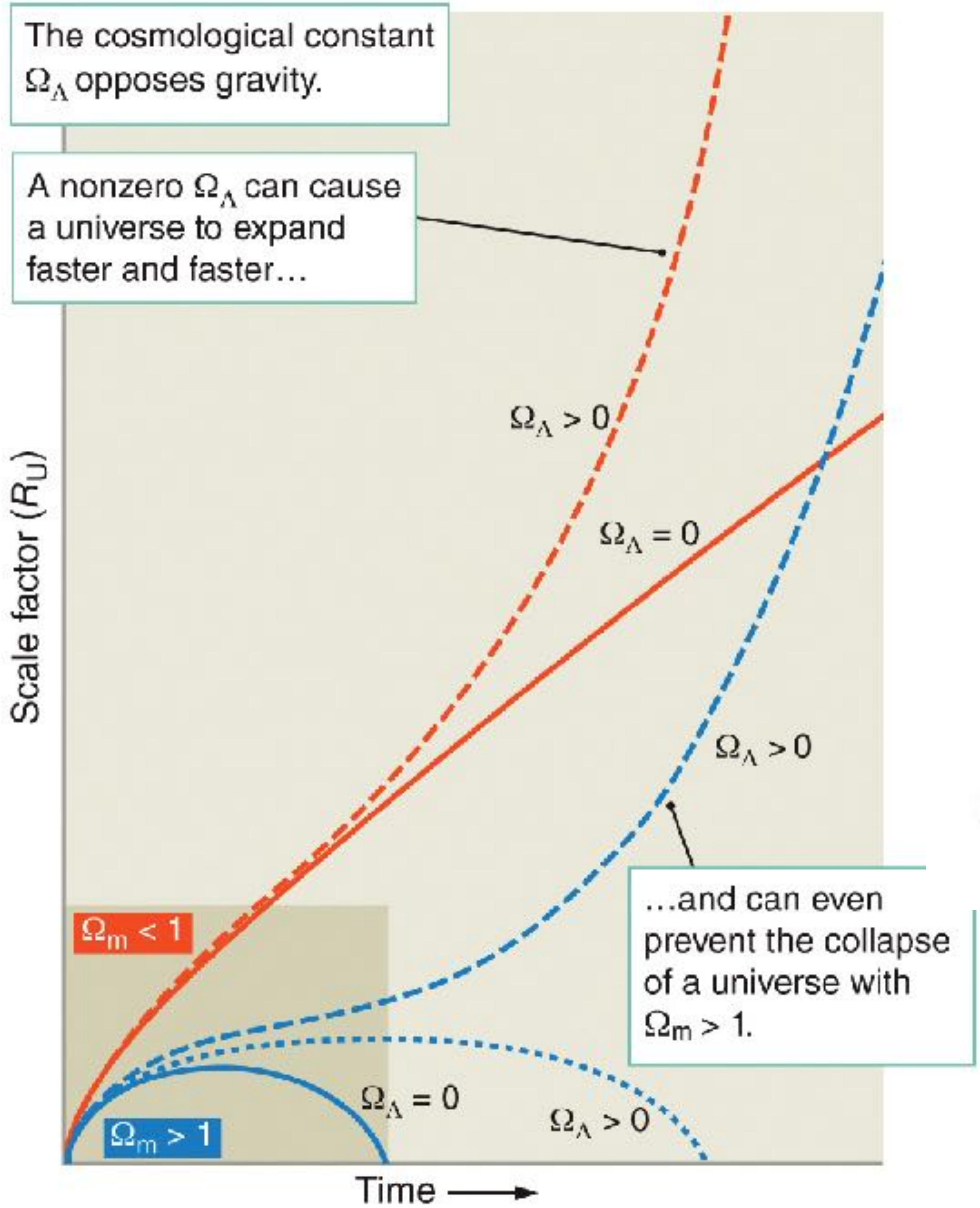
- Integrate the above relation, one obtains the **lookback time:**

$$\int_{t_{emit}}^{t_0} dt = \frac{1}{H_0}\int_{R_U(t_{emit})}^{1} \frac{dR_U}{R_U E(R_U)}$$

- **Distance** is simply **speed of light** multiplied by the **lookback time:**

$$D = c\int_{t_{emit}}^{t_0} dt = \frac{c}{H_0}\int_{R_U(t_{emit})}^{1} \frac{dR_U}{R_U E(R_U)}$$

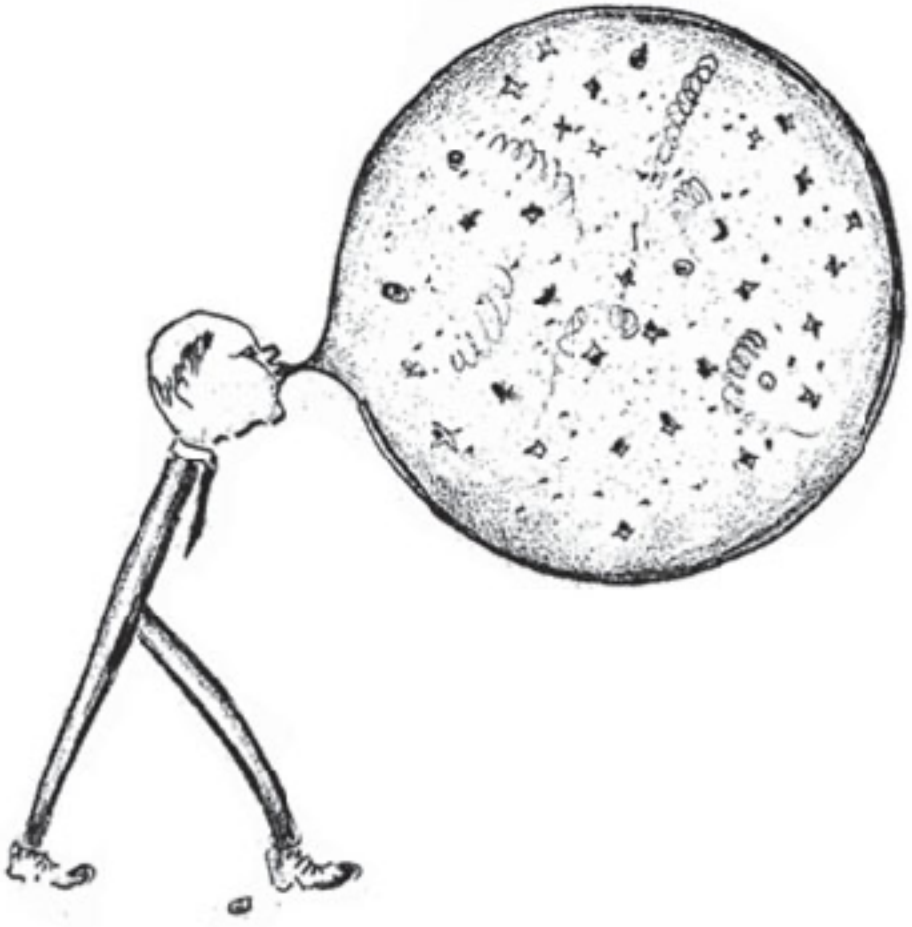
Various Possible Expansion Histories — All begin at t=0



The cosmological constant Ω_Λ opposes gravity.

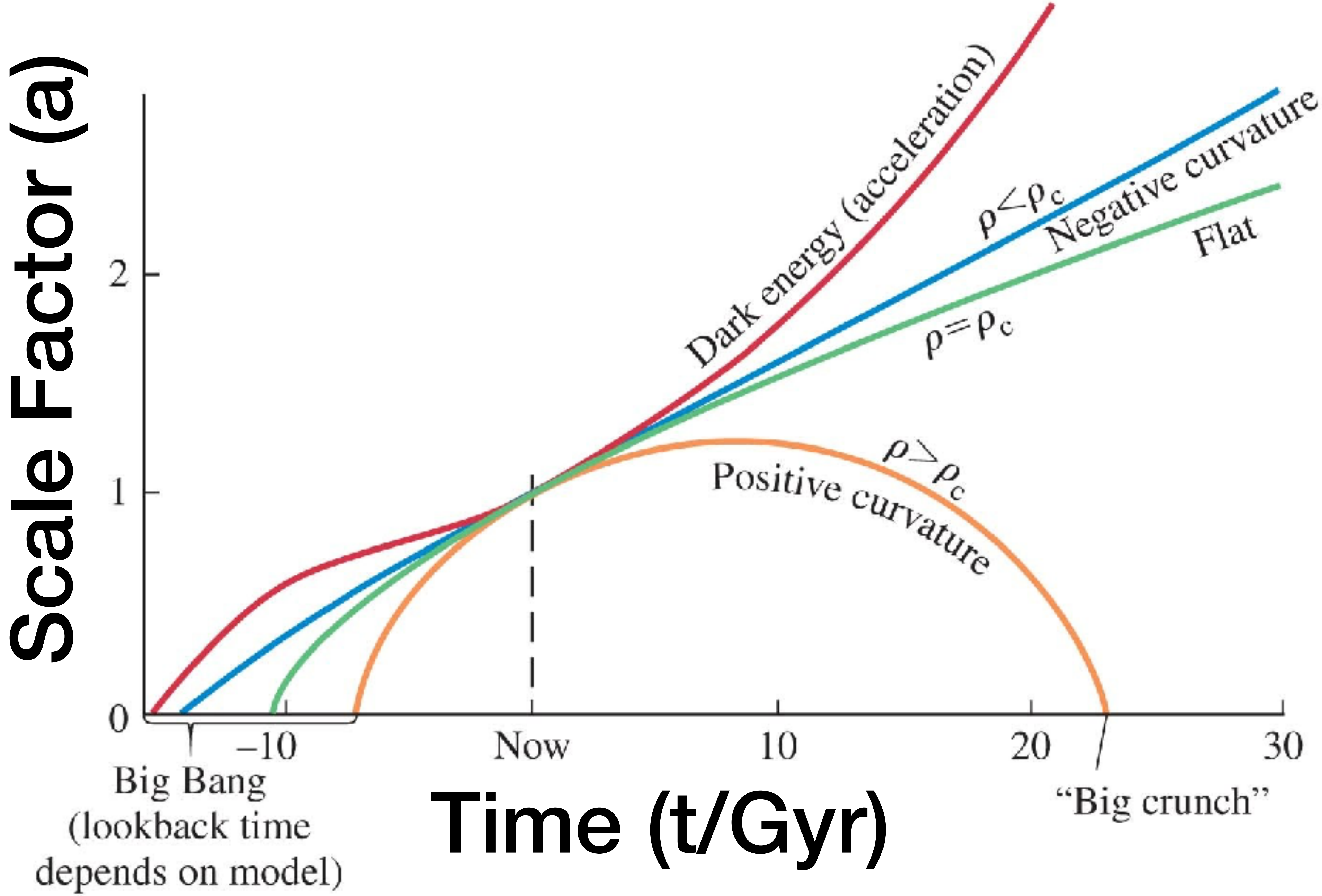
A nonzero Ω_Λ can cause a universe to expand faster and faster…

…and can even prevent the collapse of a universe with $\Omega_m > 1$.



Λ as the source of cosmic expansion: De Sitter in 1930

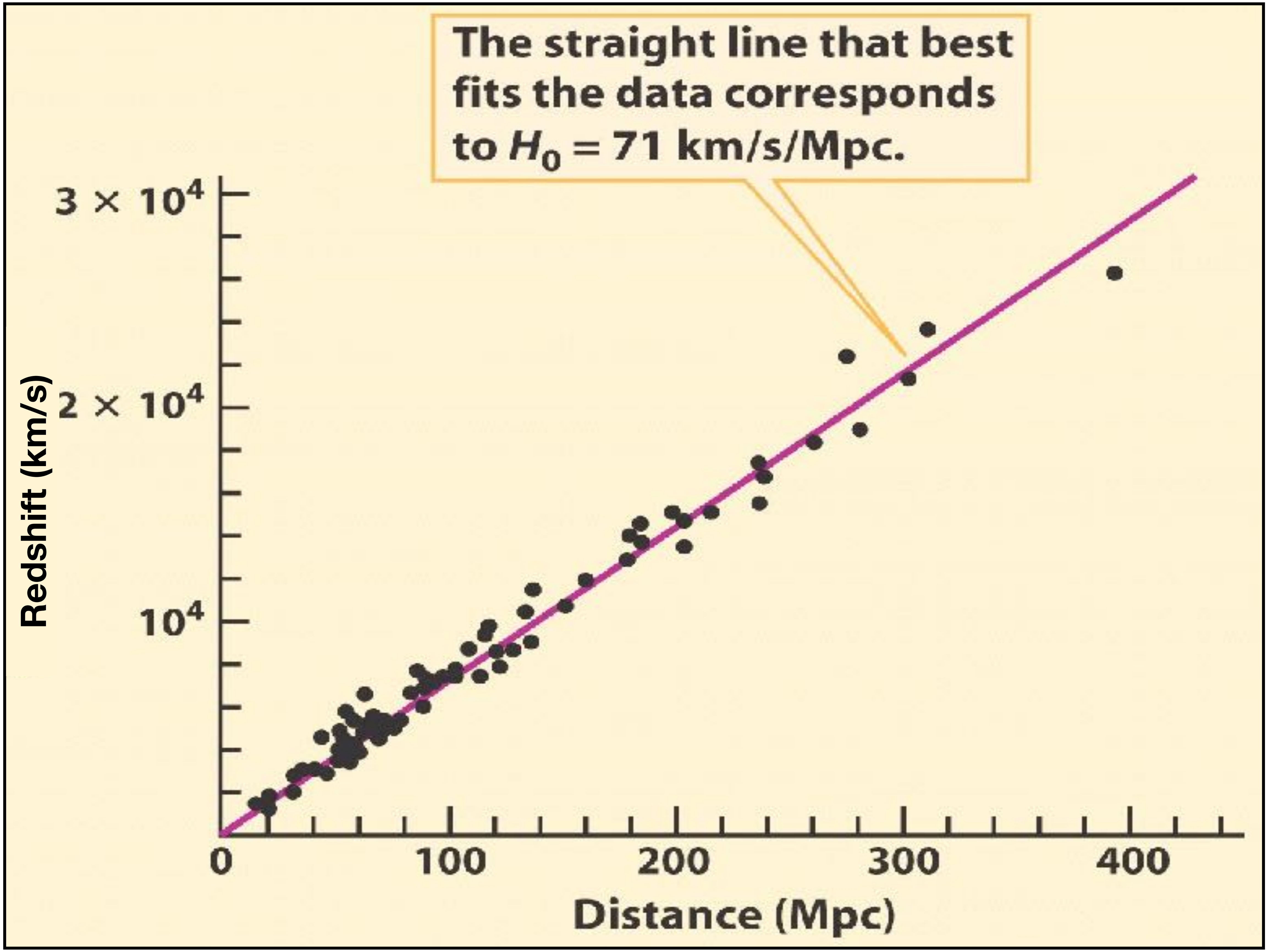
Another Way to Present the Various Possible Expansion Histories — Normalization at Present Time



Constraints on Cosmological Parameters:

distance-redshift relation up to $z \sim 1$

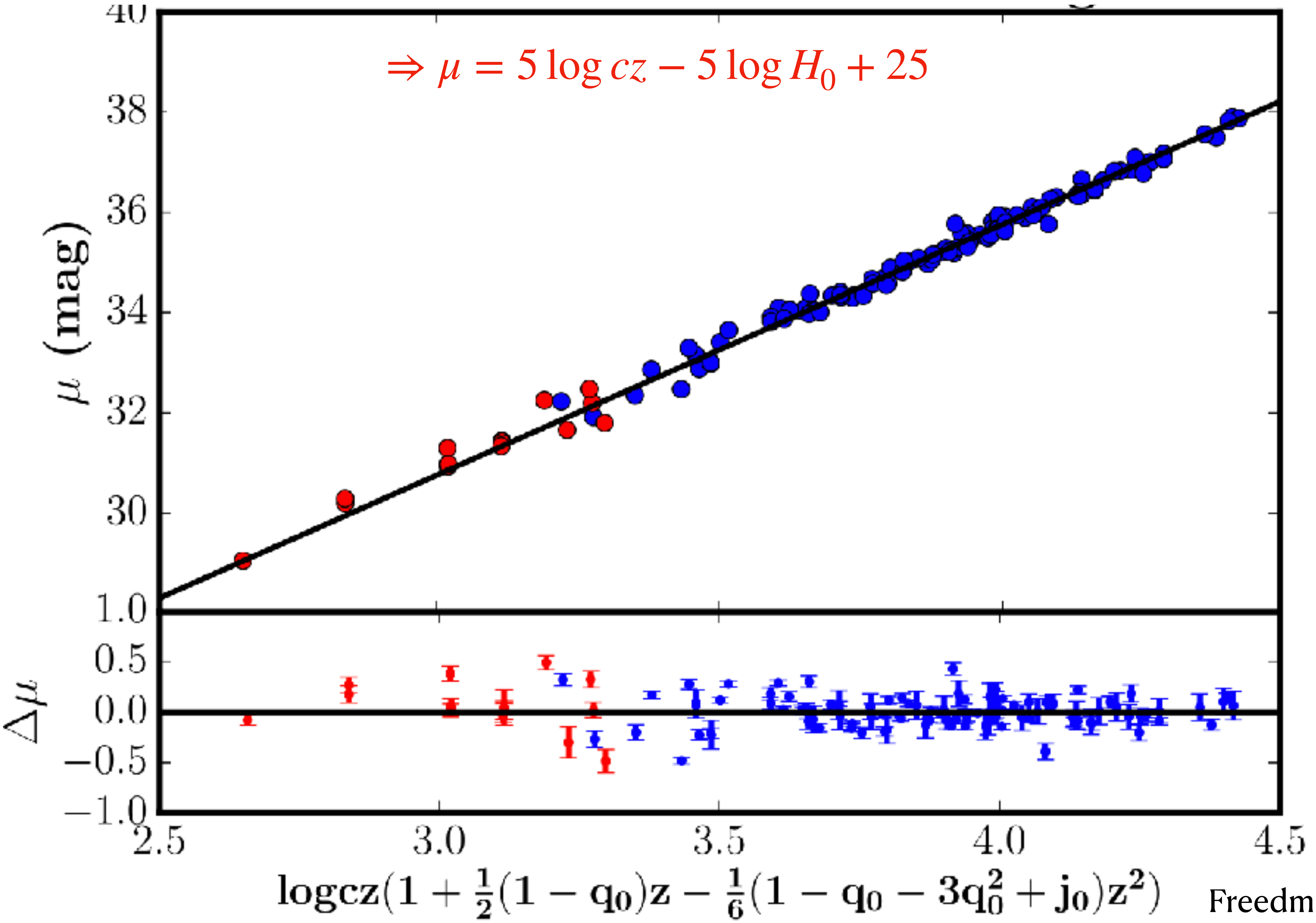
Simple Distance Predictions from Hubble's Law: $cz = H_0 D \rightarrow D = cz/H_0$, this approximation is valid at 0.02 < z < 0.2 (think about why there is a lower limit?)



Modern Hubble Diagram: Distance Modulus vs. redshift

$$\mu = m - M = 5\log(D/\mathrm{Mpc}) + 25 = 5\log(cz/H_0) + 25$$

$$\Rightarrow \mu = 5\log cz - 5\log H_0 + 25$$



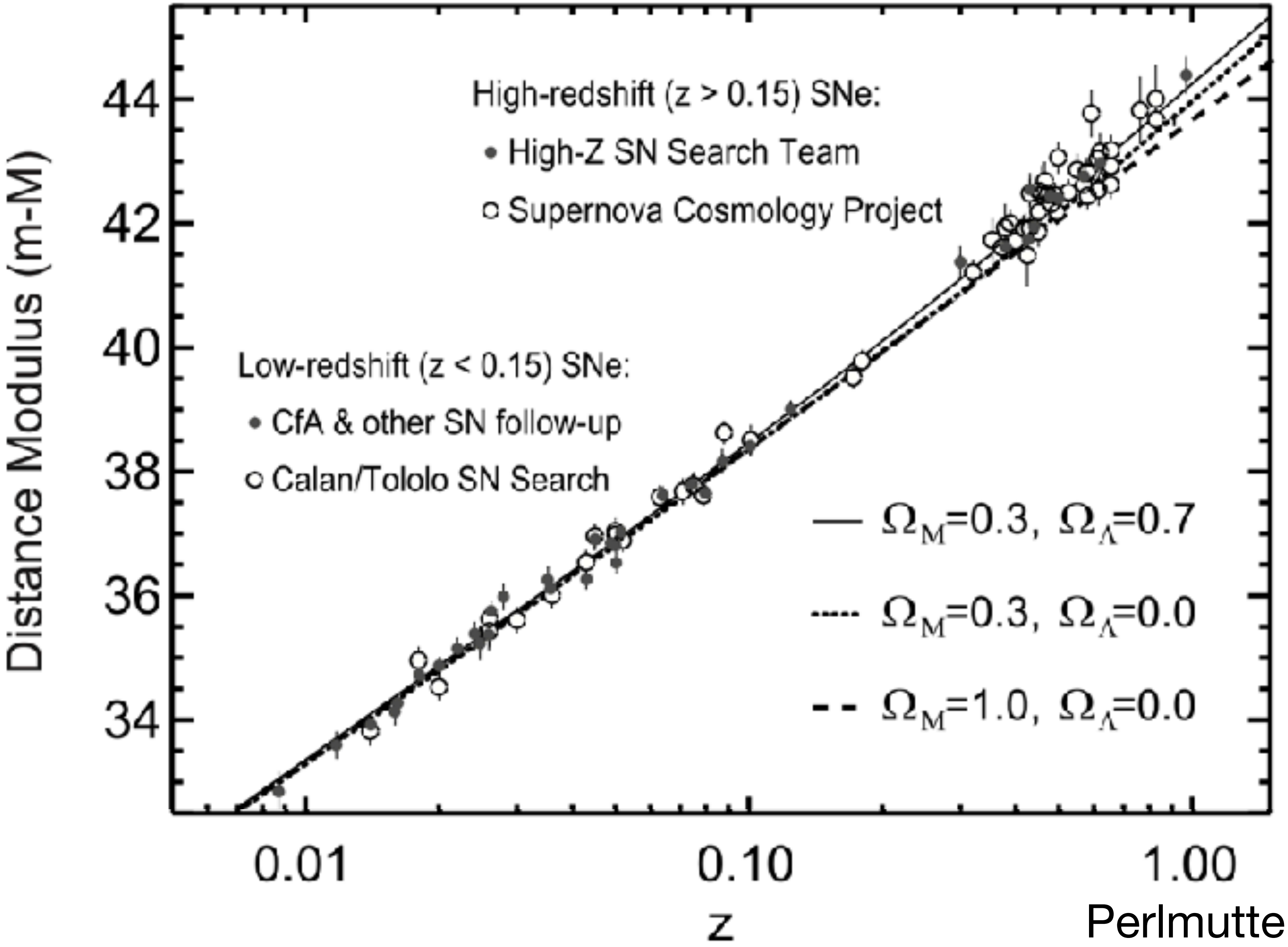
Freedman+2019

Hubble Diagram: Distance Modulus vs. Cosmological Redshifts

Cosmological parameters can be constrained by **comparing DM measurements (data points) with model predictions (curves)** for a range of redshifts. For standard model, one uses these Eqs:

$$D_L = (1+z)\int_0^z \frac{c\,dz'}{H(z')} \qquad H^2 = \frac{H_0^2}{R_U^2}\left[(1-\Omega_0) + \Omega_{m,0}/R_U + \Omega_{\gamma,0}/R_U^2 + \Omega_{\Lambda,0}R_U^2\right]$$

$$\mu = m - M = 5\log D_L^{Mpc}(z; H_0, \Omega_M, \Omega_\Lambda) + 25$$



Perlmutter & Schmidt 2003

2011 Nobel Prize in Physics

The Supernova Cosmology Project **The High-z Supernova Search Project**

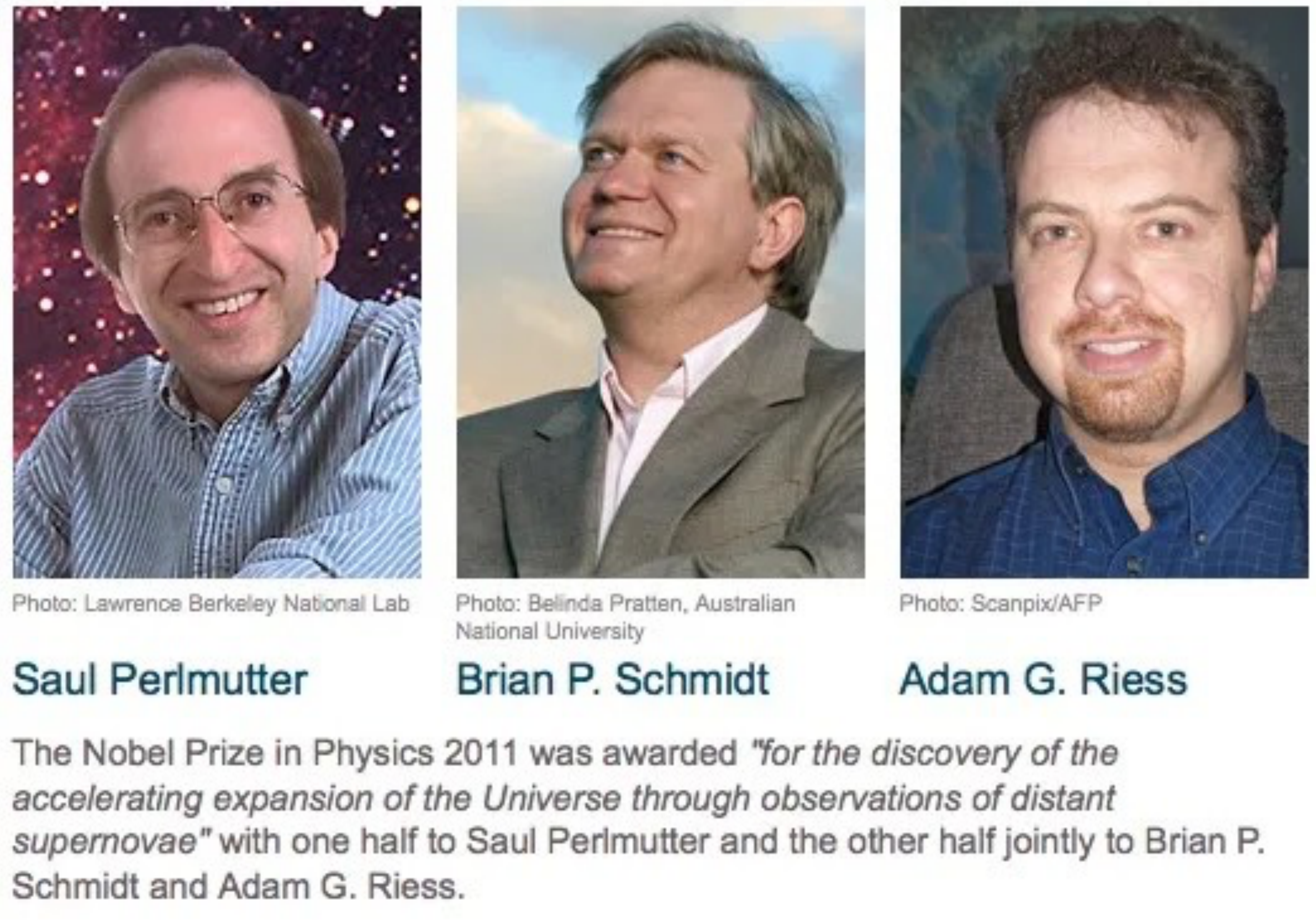


Photo: Lawrence Berkeley National Lab

Photo: Belinda Pratten, Australian National University

Photo: Scanpix/AFP

Saul Perlmutter

Brian P. Schmidt

Adam G. Riess

The Nobel Prize in Physics 2011 was awarded *"for the discovery of the accelerating expansion of the Universe through observations of distant supernovae"* with one half to Saul Perlmutter and the other half jointly to Brian P. Schmidt and Adam G. Riess.

The Density Parameters of the Universe Today

How do they evolve over time?
What’s the predicted future?

Density Parameters Today

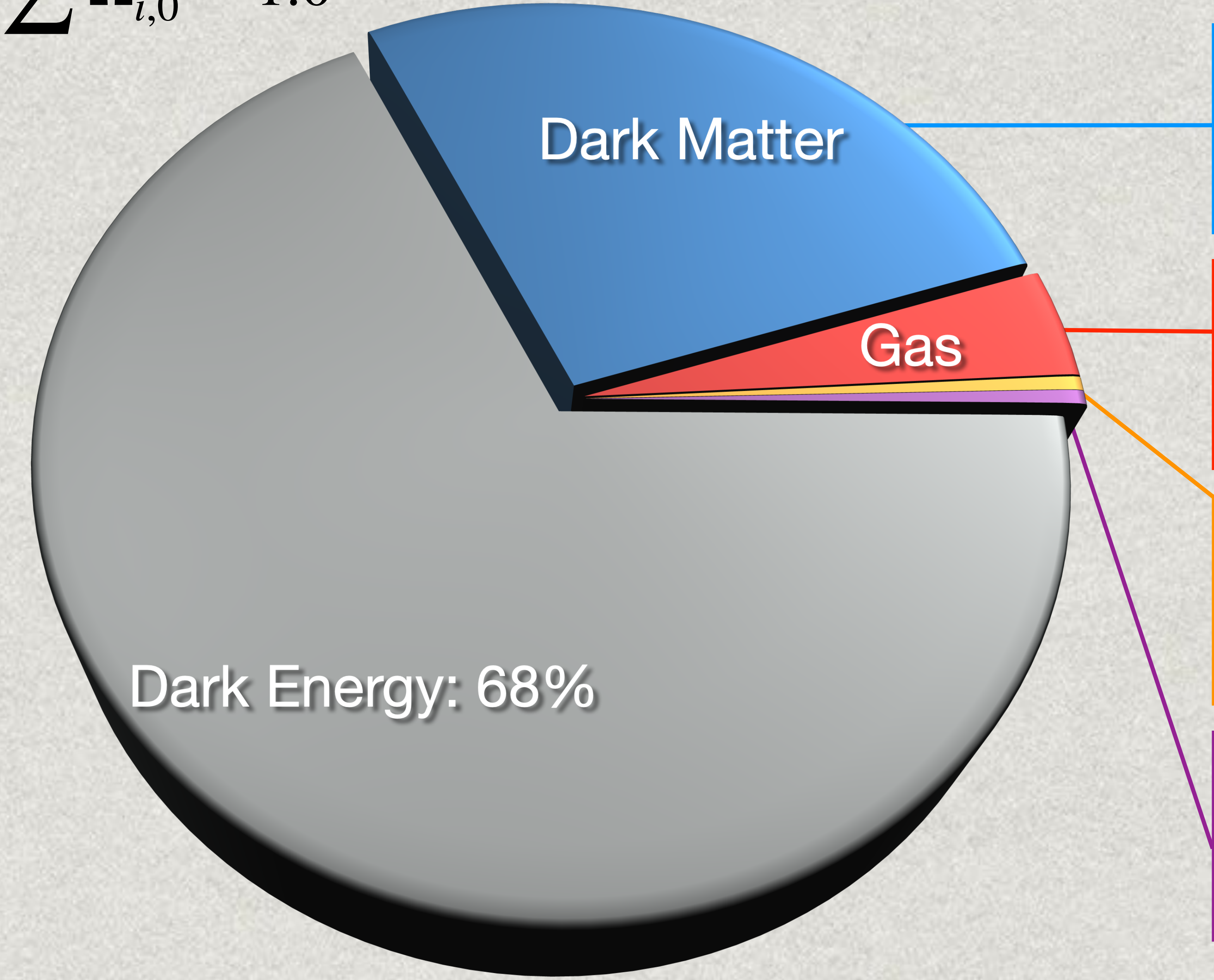
$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}$$

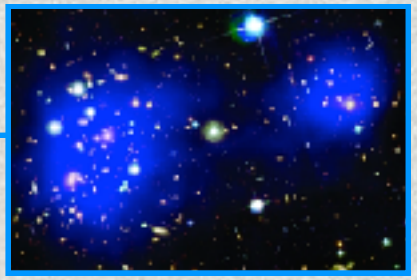
Critical density as a function of time. Value below is present value, based on present value of the Hubble parameter H

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = 9.47x10^{-27}\,kg/m^3$$

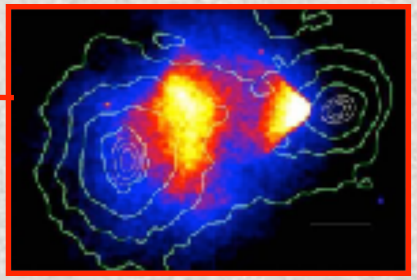
Planck Collaboration (2013)

$$\sum \Omega_{i,0} = 1.0$$



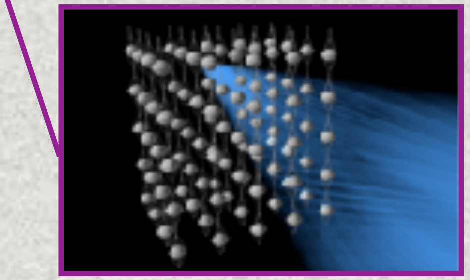


Dark matter: 27%



Gas: 4%

Galaxies: 0.5%



Neutrinos: <0.5%

The Evolution of Dimensionless Density Parameters

- Given density parameters-scale factor relations from equations of state:

$$\frac{\Omega_m}{\Omega_{m,0}} = \frac{\rho_m}{\rho_c}\frac{\rho_{c,0}}{\rho_{m,0}} = \frac{\rho_m}{\rho_{m,0}}\frac{H_0^2}{H^2} = \frac{1}{R_U^3}\frac{H_0^2}{H^2}$$

$$\frac{\Omega_\gamma}{\Omega_{\gamma,0}} = \frac{1}{R_U^4}\frac{H_0^2}{H^2} \quad \& \quad \frac{\Omega_\Lambda}{\Omega_{\Lambda,0}} = \frac{H_0^2}{H^2}$$

- And the FE1 with the boundary condition:

$$H^2 = H_0^2[(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}]$$

- It's easy to see that:

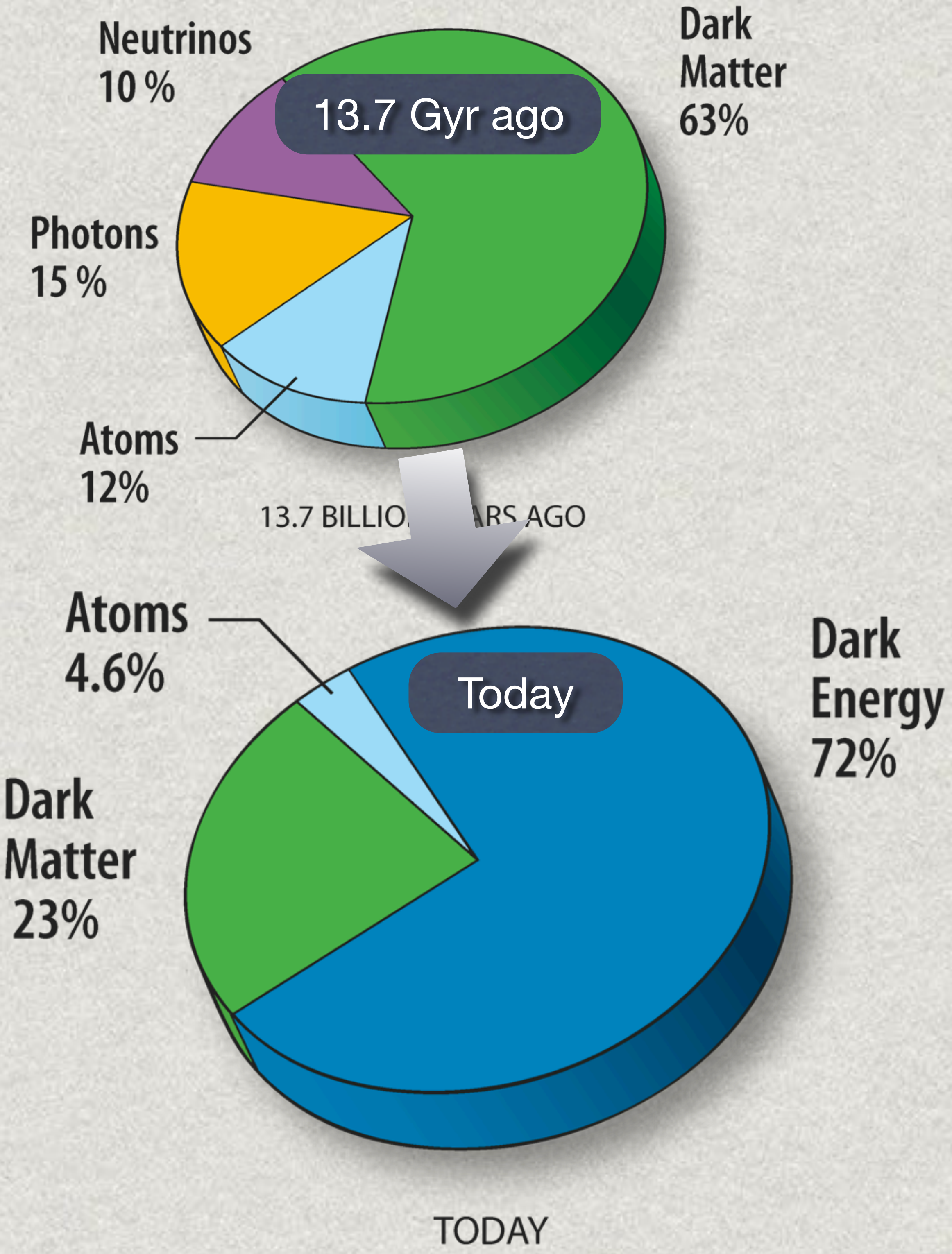
$$\frac{\Omega_m}{\Omega_{m,0}} = \frac{1}{R_U^3}\frac{H_0^2}{H^2} = \frac{1}{R_U^3[(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}]}$$

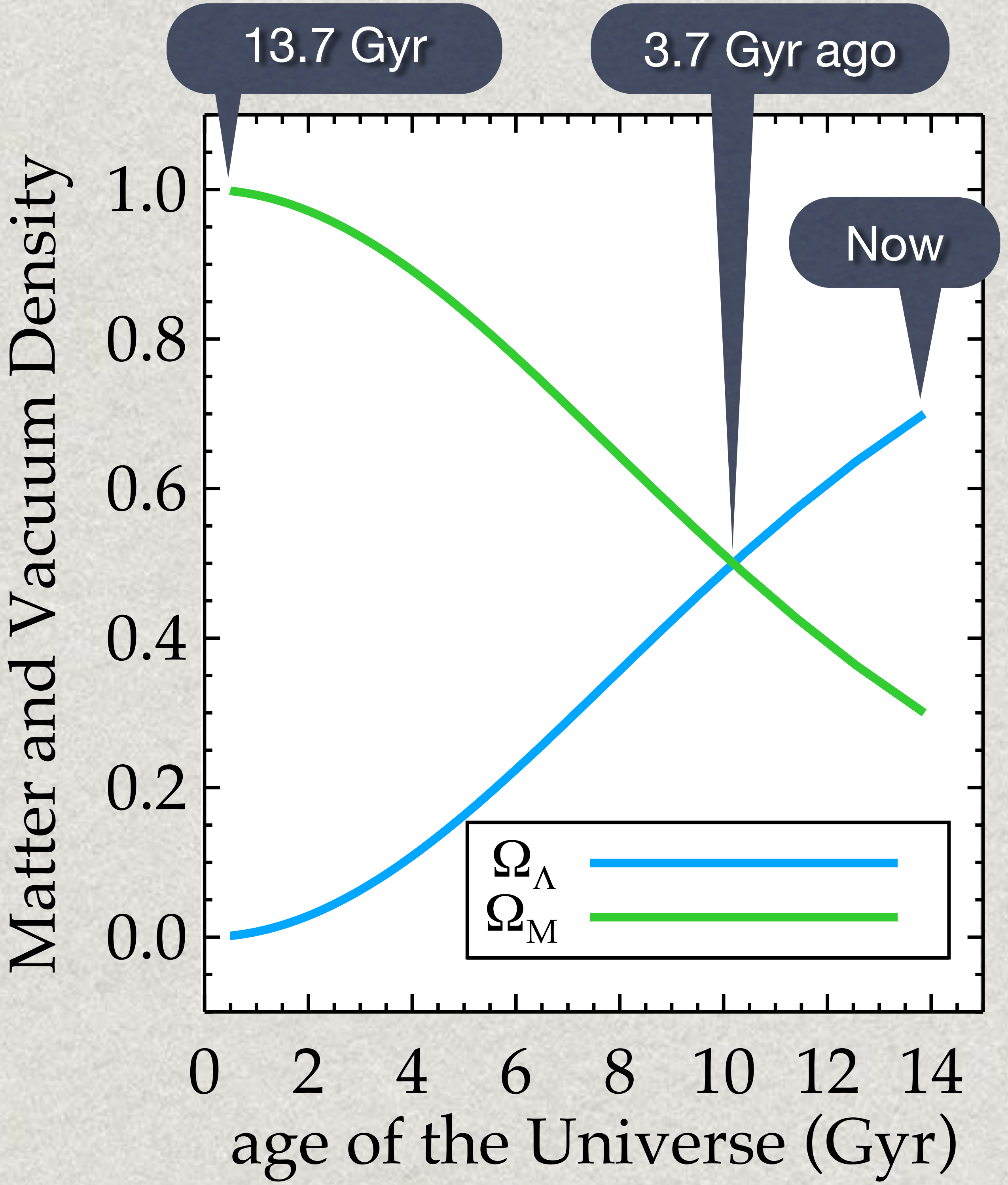
$$\frac{\Omega_\gamma}{\Omega_{\gamma,0}} = \frac{1}{R_U^4}\frac{H_0^2}{H^2} = \frac{1}{R_U^4[(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}]}$$

$$\frac{\Omega_\Lambda}{\Omega_{\Lambda,0}} = \frac{H_0^2}{H^2} = \frac{1}{(1-\Omega_0)/R_U^2 + \Omega_{m,0}/R_U^3 + \Omega_{\gamma,0}/R_U^4 + \Omega_{\Lambda,0}}$$

Density Parameters vs. Time

- Dark Matter-dominated in the first 10 Gyrs, then Dark Energy dominated





Predicted Evolution of Density Parameters

- Dark Matter-dominated in the first 10 Gyrs, then Dark Energy dominated

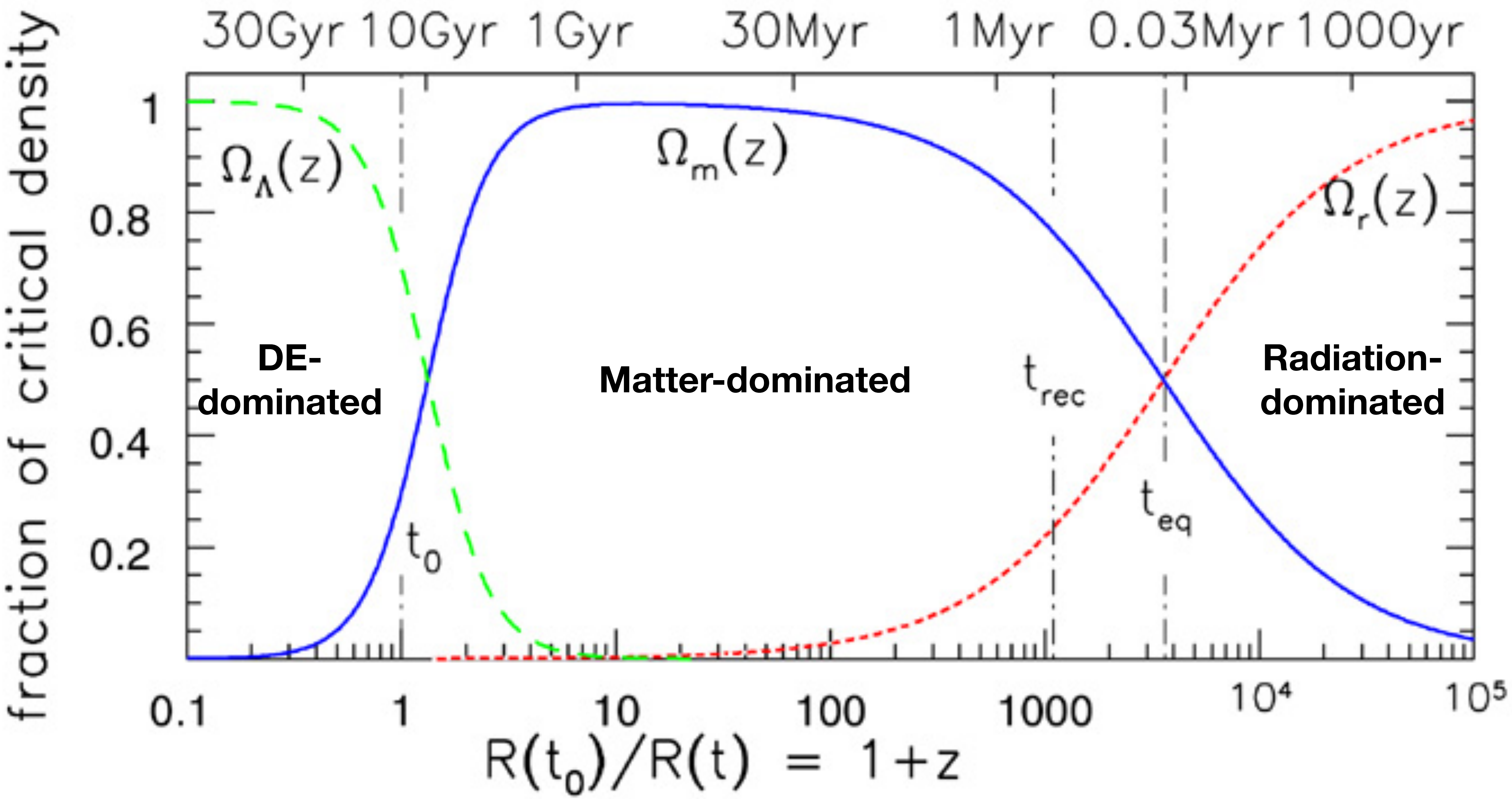
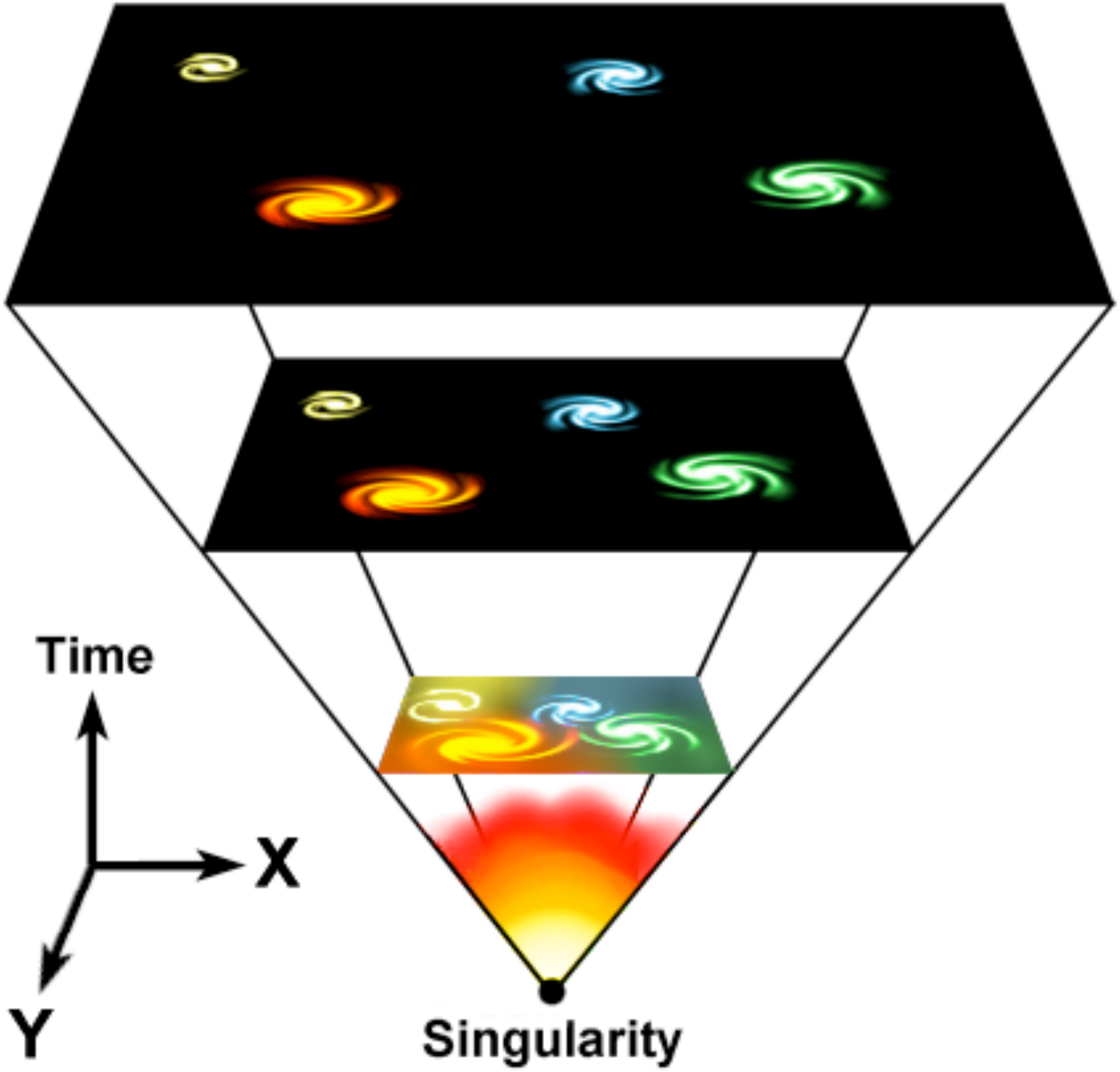
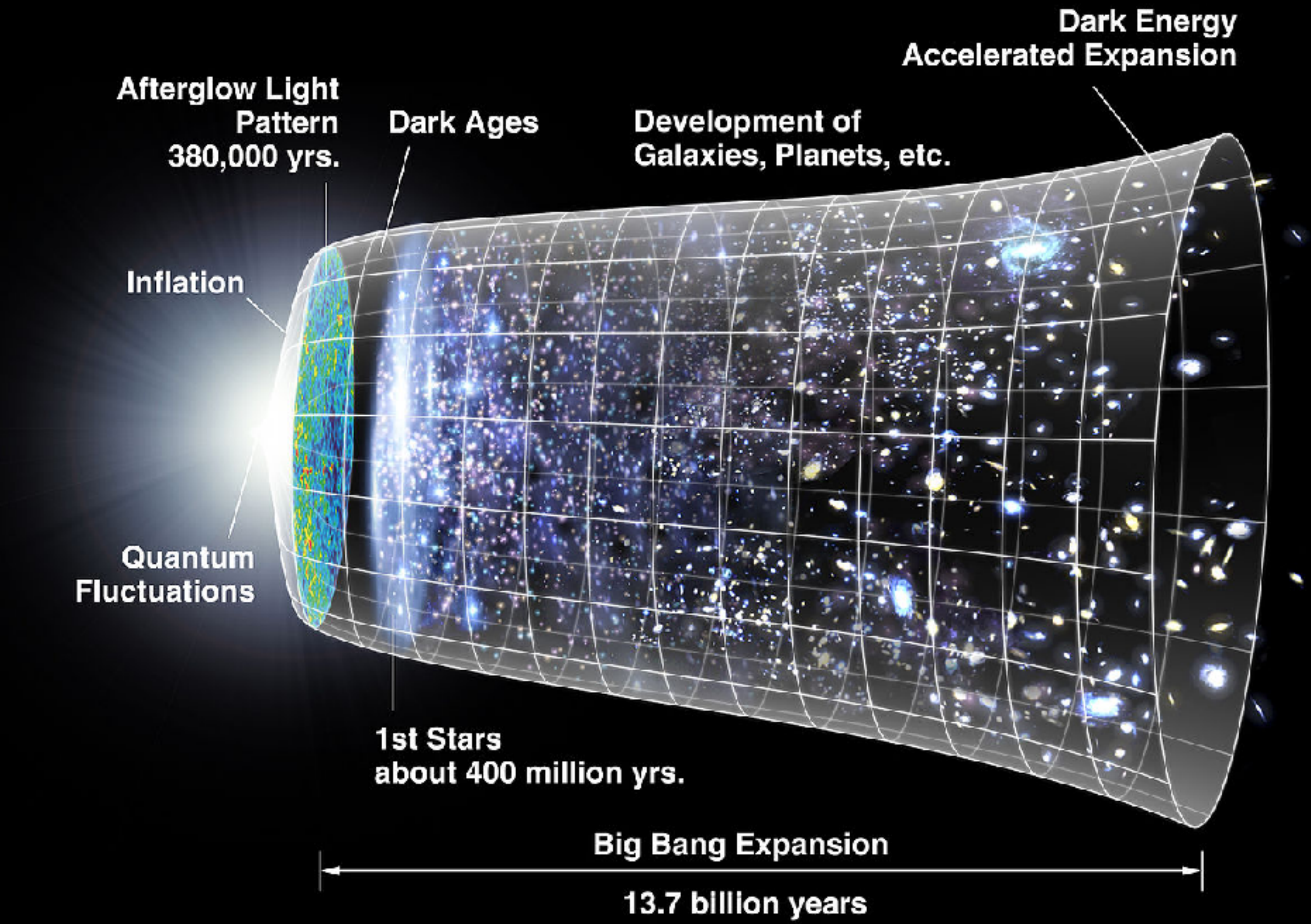


Fig 8.7 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

The Expanding Universe Had a Hot Beginning



Big Bang - Particles created - Ionized universe (opaque) - Recombination (z ~ 1100) - Dark Ages - Reionization (z ~ 20-7)

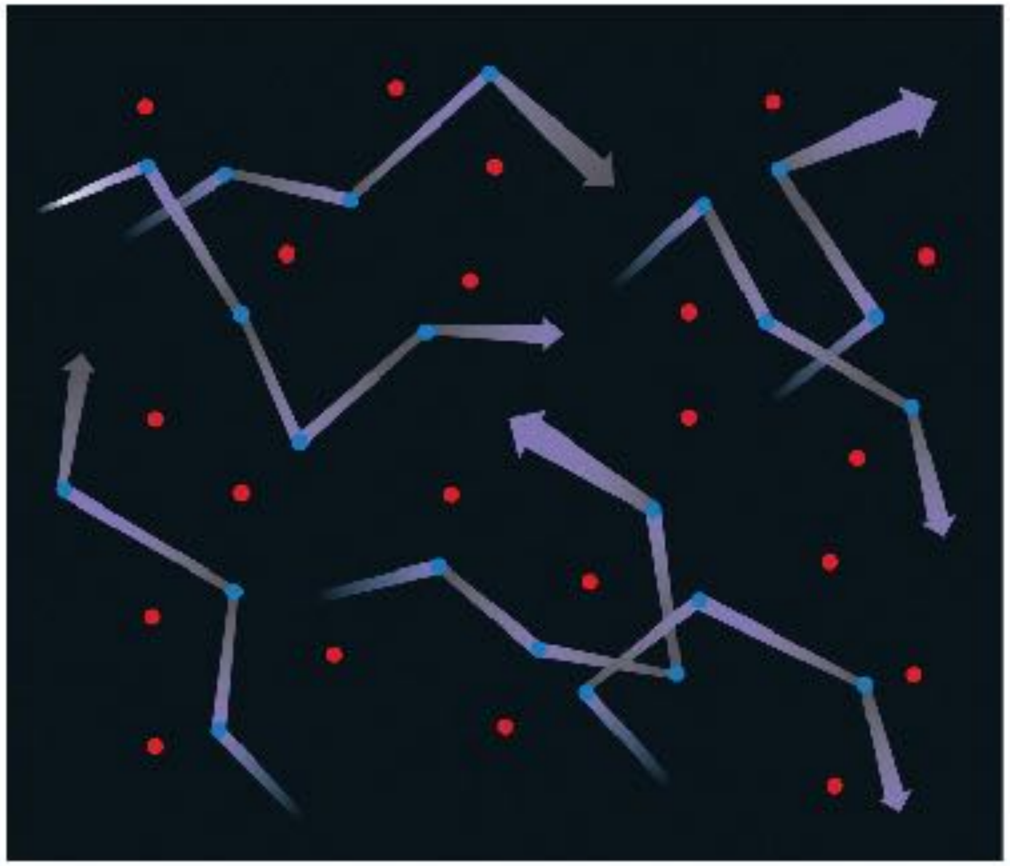


The epoch of hydrogen recombination and the emergence of the EM radiation background

How did the CMB emerge?
First, EM waves are trapped in an ionized universe

- When the universe was hot and the gas was ionized, photons were trapped with matter.
  - Free electrons interact strongly with photons (***Thomson scattering***).
  - We cannot observe anything during this era. It’s as if the universe is filled with a dense fog.

In the ionized early universe, light was trapped by free electrons. Radiation had a blackbody spectrum.



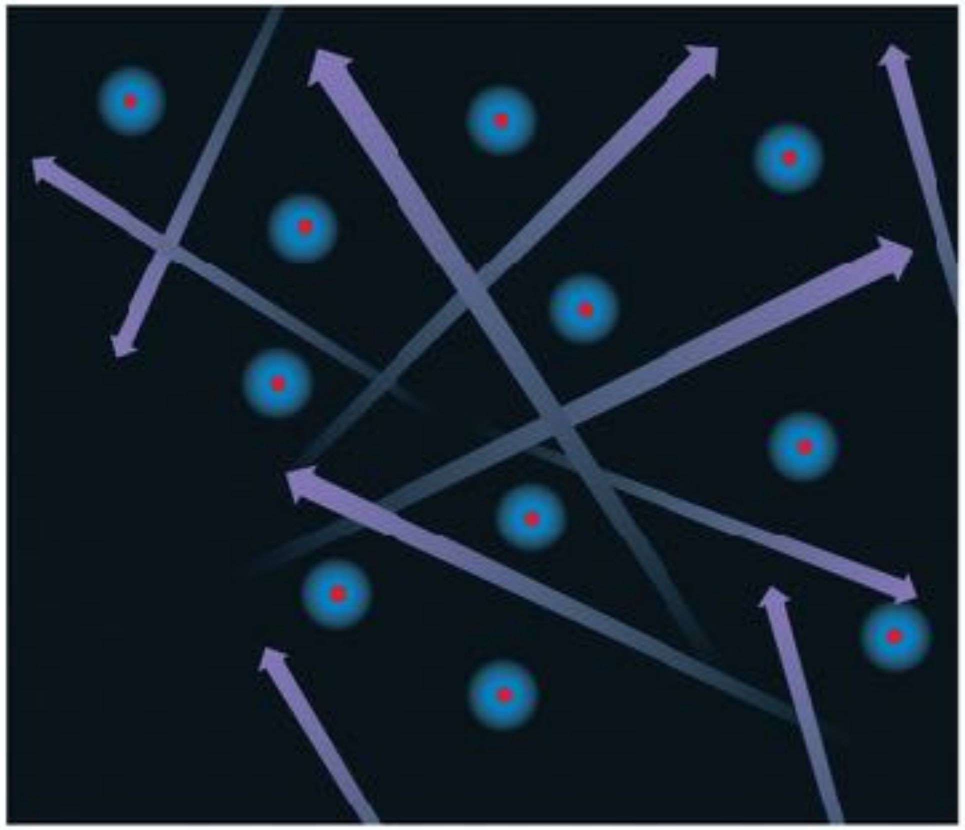
At that time, it was as though the universe was filled with a thick fog.

Emil Oprisa/Alamy Stock Photo

KEY ● Proton ● Electron ➚ Path of photon

How did the CMB emerge?
Then, protons and electrons recombined to form hydrogen

- Eventually, the expansion causes the temperature to cool enough that **protons** and **electrons** could form **neutral H atoms**: this phase-transition of the Universe is called **the epoch of recombination (EoR)**.
- At that time, light was no longer blocked from its travel by free electrons.
- **EoR** marks the earliest point in the universe that we can observe.



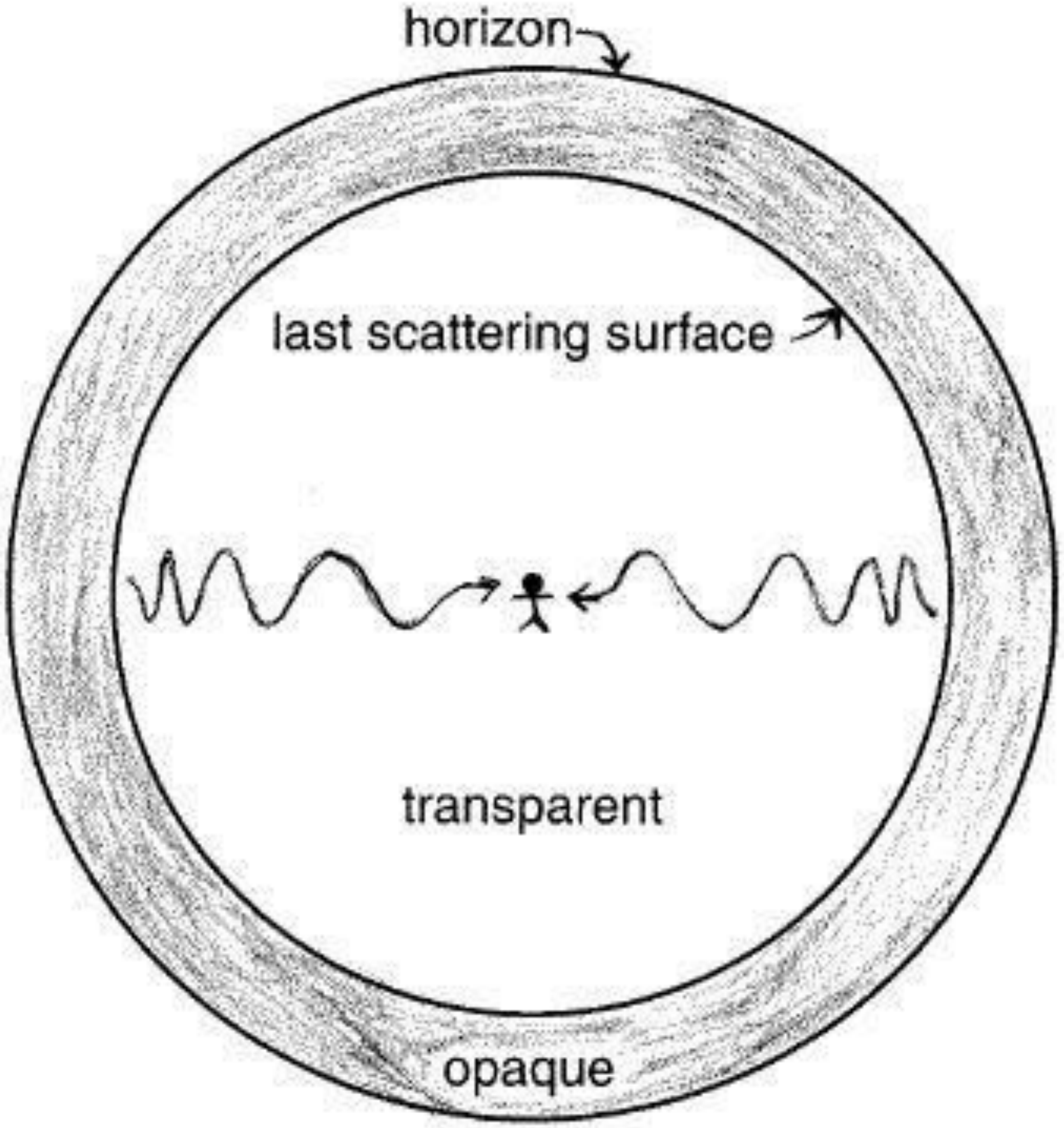
At recombination, the universe became transparent, and the blackbody radiation traveled freely through the universe.

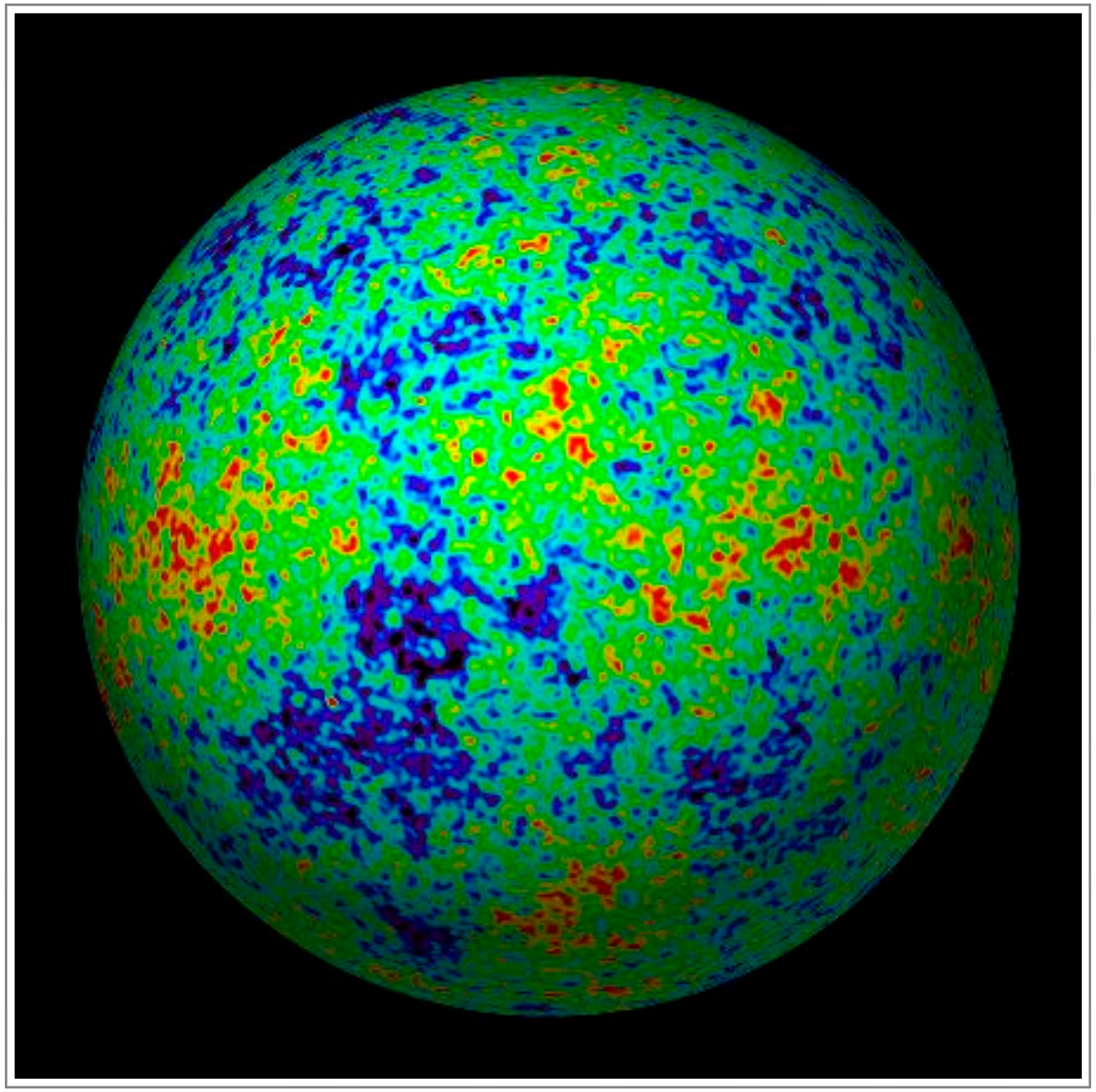
Sasho Bogoev/Alamy Stock Photo

Recombination was like the fog suddenly clearing.

CMB Photons travel straight to us from the last scattering surface

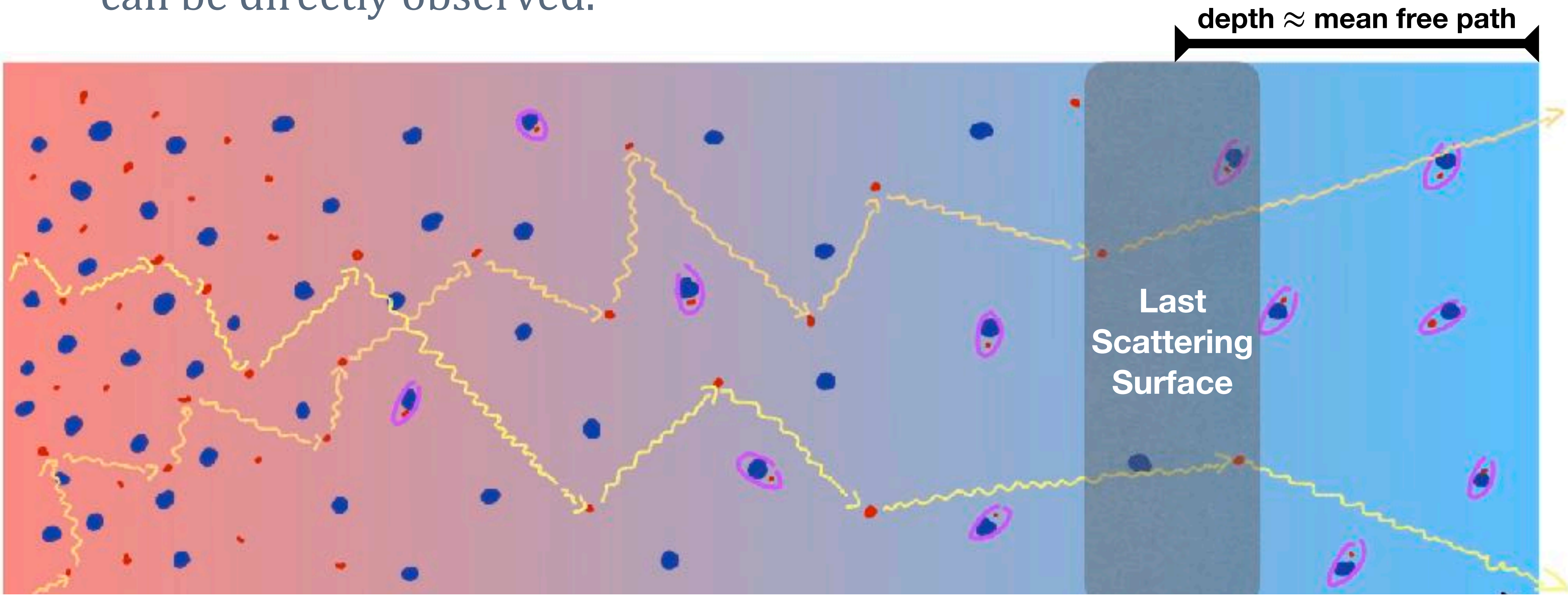
- Analogous to the *last scattering surface* that marks the surface of the Solar photosphere





Last Scattering Surface of the Solar Photosphere

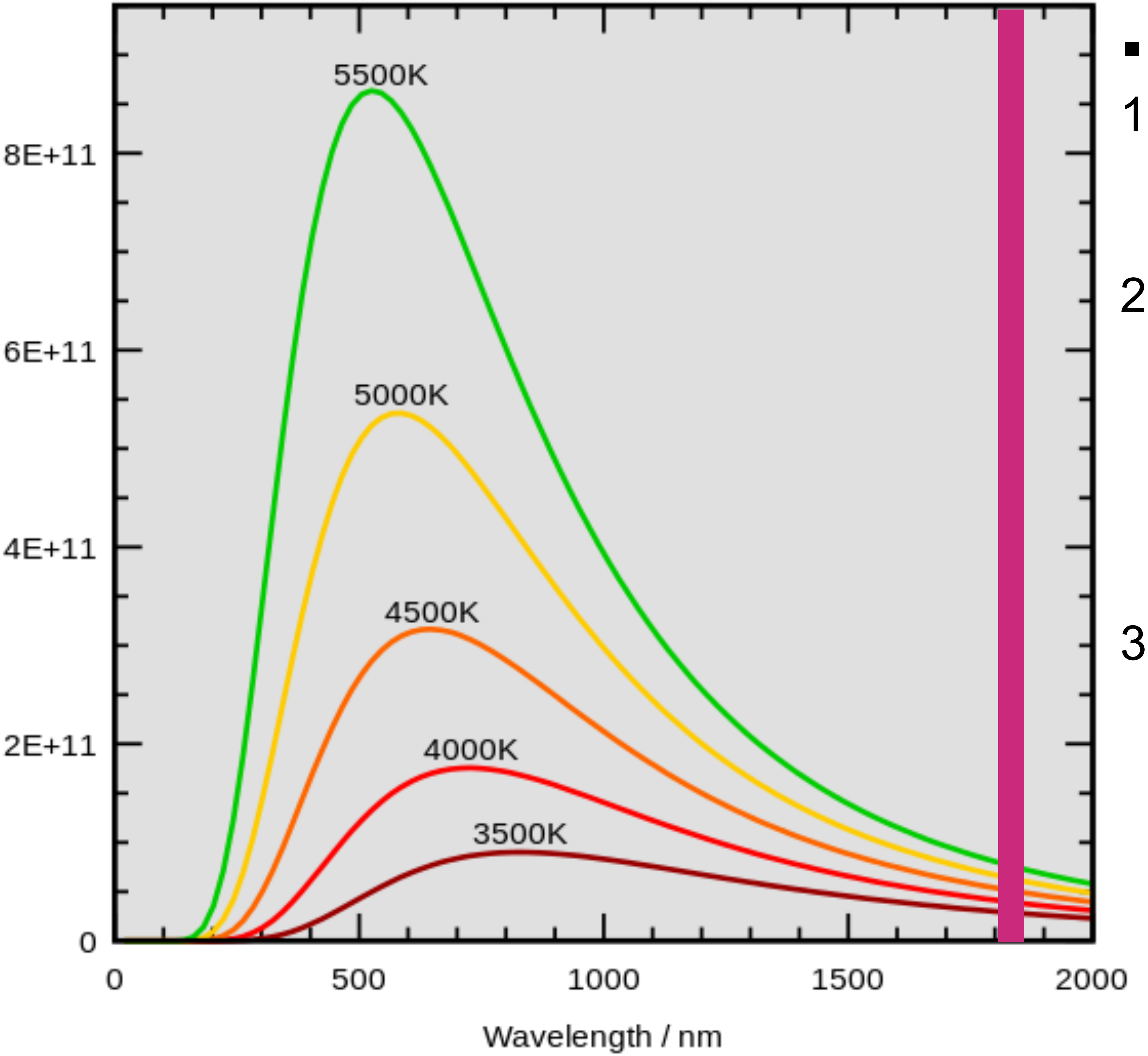
- The Sun has no solid surface, but the apparent surface of the Sun is the surface at which light can directly escape into space.
- Let's call this surface the **last scattering surface** (a concept also used in **cosmology**). Note that its depth depends on **(1) the angle we look into the Sun and (2) the wavelength of the photons**
- The layers above this point are known as the **atmosphere**, which can be directly observed.



$$B_\lambda(T) \equiv \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc/\lambda}{kT}}-1} \approx \frac{2hc^2}{\lambda^5}\frac{kT\lambda}{hc} = \frac{2ckT}{\lambda^4}$$

Rayleigh-Jeans Tail

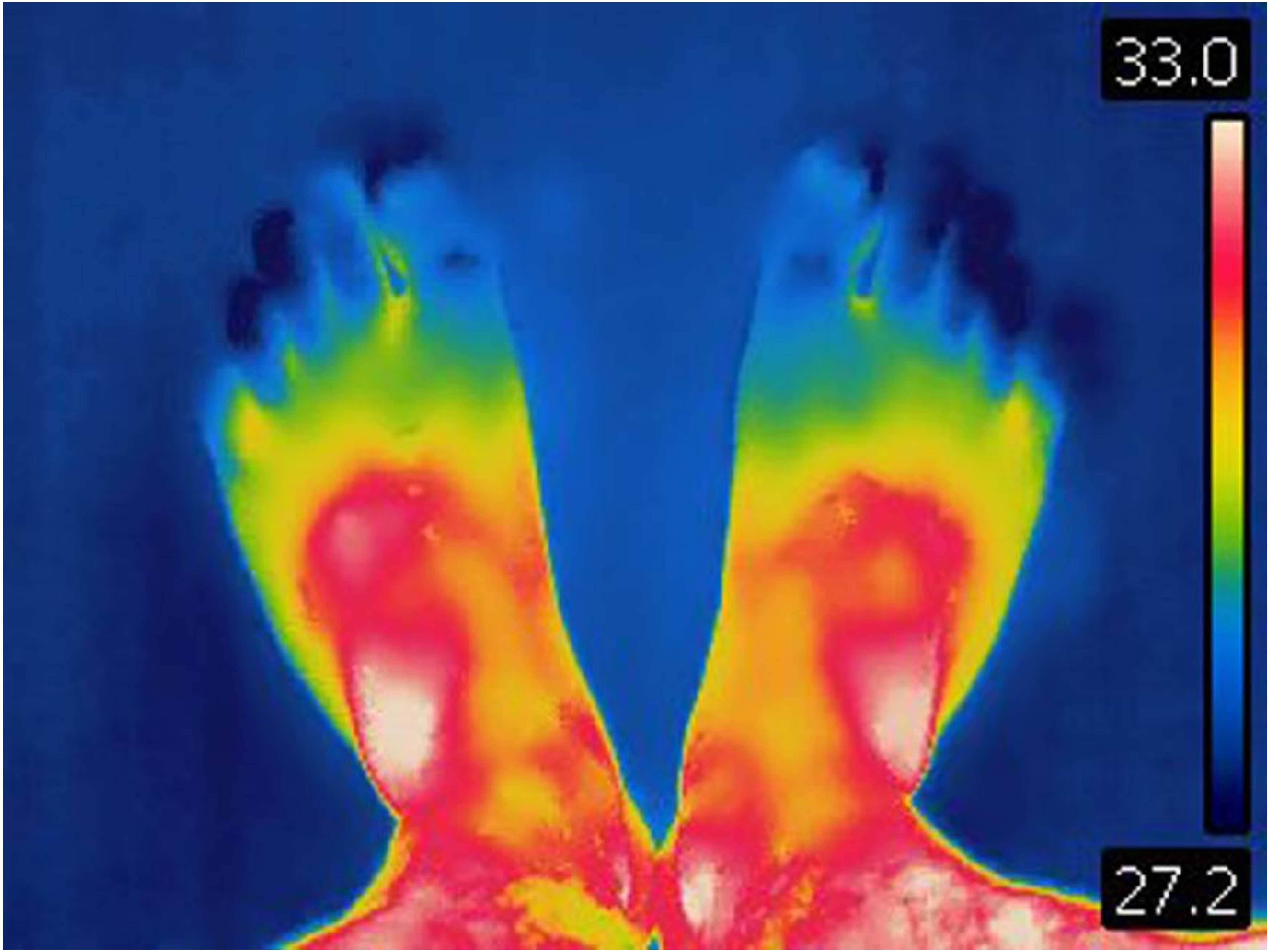
$hc/\lambda \ll kT$



- **When *T* increases:**

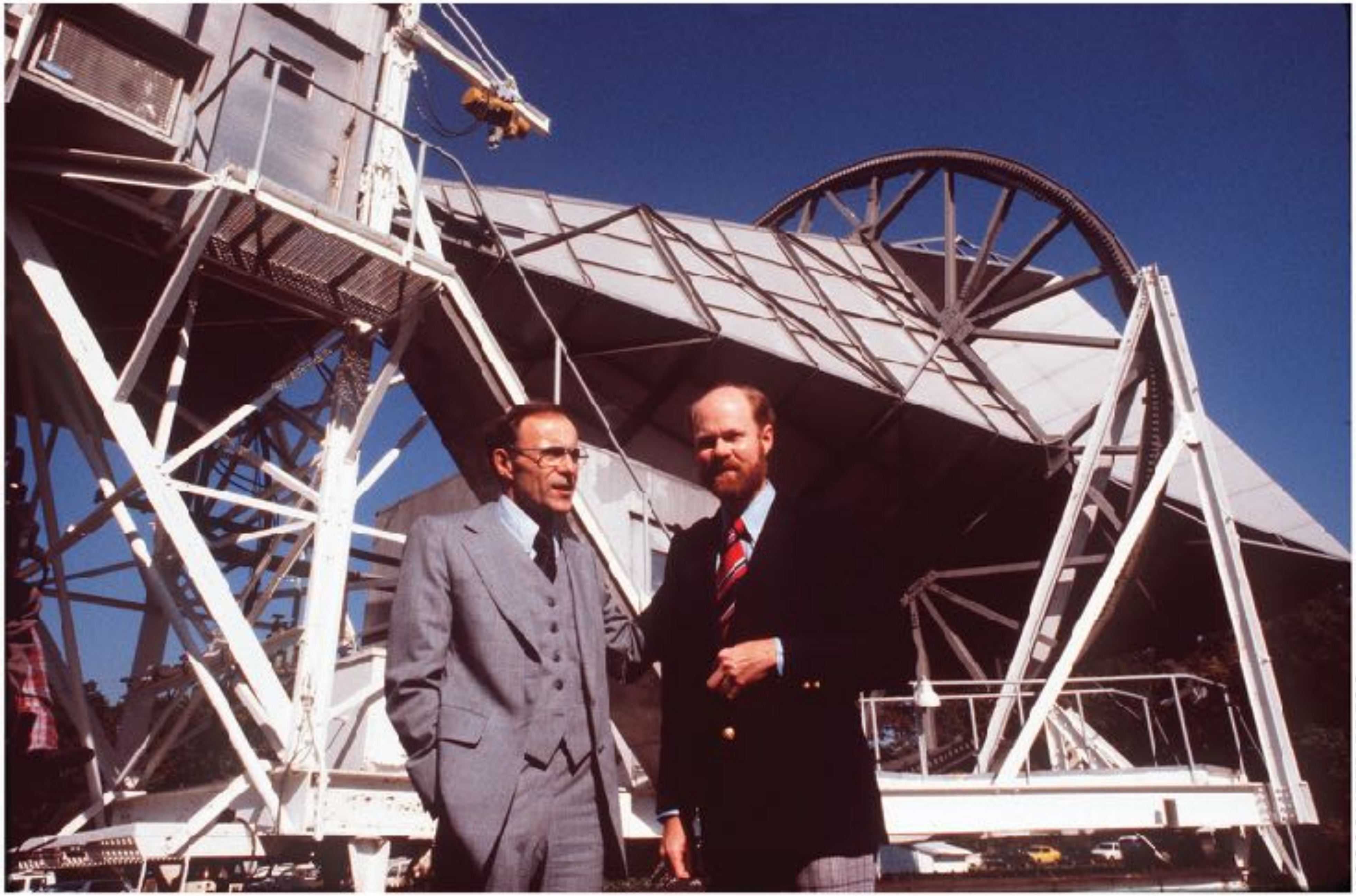
1. **Peak** shifts to shorter wavelength - **Wien's Displacement Law**
2. **Surface Flux,** the total area under each Planck curve, increases w/ ***T***^{4} - **Stefan-Boltzmann Law**
3. **Surface brightness (intensity)** increases at all wavelengths - **brightness temperature (IR/ radio thermometer)**

Temperature Map from Single-Wavelength Intensity Measurement



Measuring the Cosmic Microwave Background

- This predicted spectrum was *accidentally* discovered in **1963** by **Arno Penzias** and **Robert Wilson**. They measured **4.08 GHz (7.4 cm)** emission in all directions and the **brightness temperature** of the emission was about **3 K**.
- The **cosmic microwave background (CMB)** was the first clear evidence of **the Big Bang,** earning the duo the **1978 Nobel Prize in Physics.**



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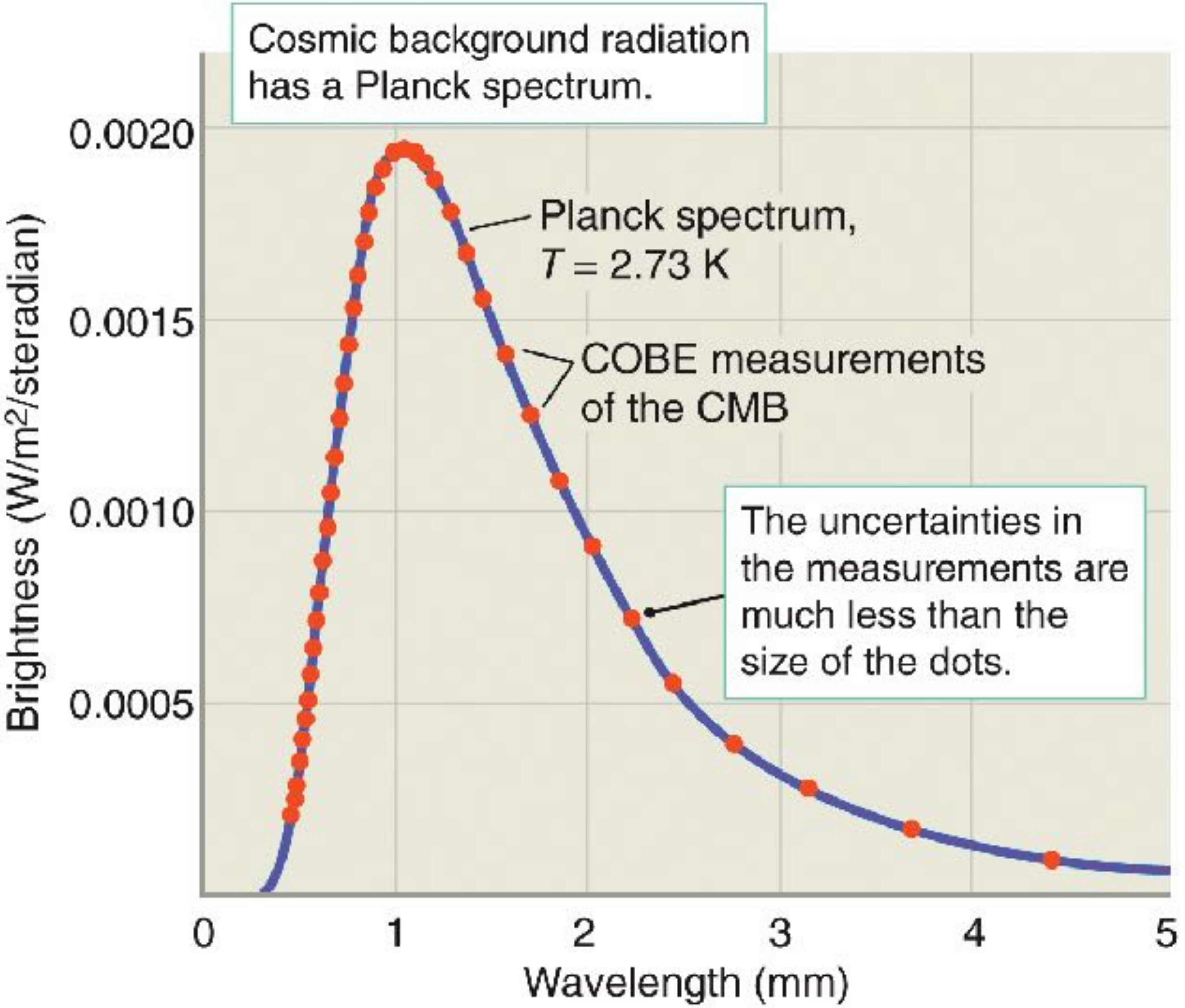
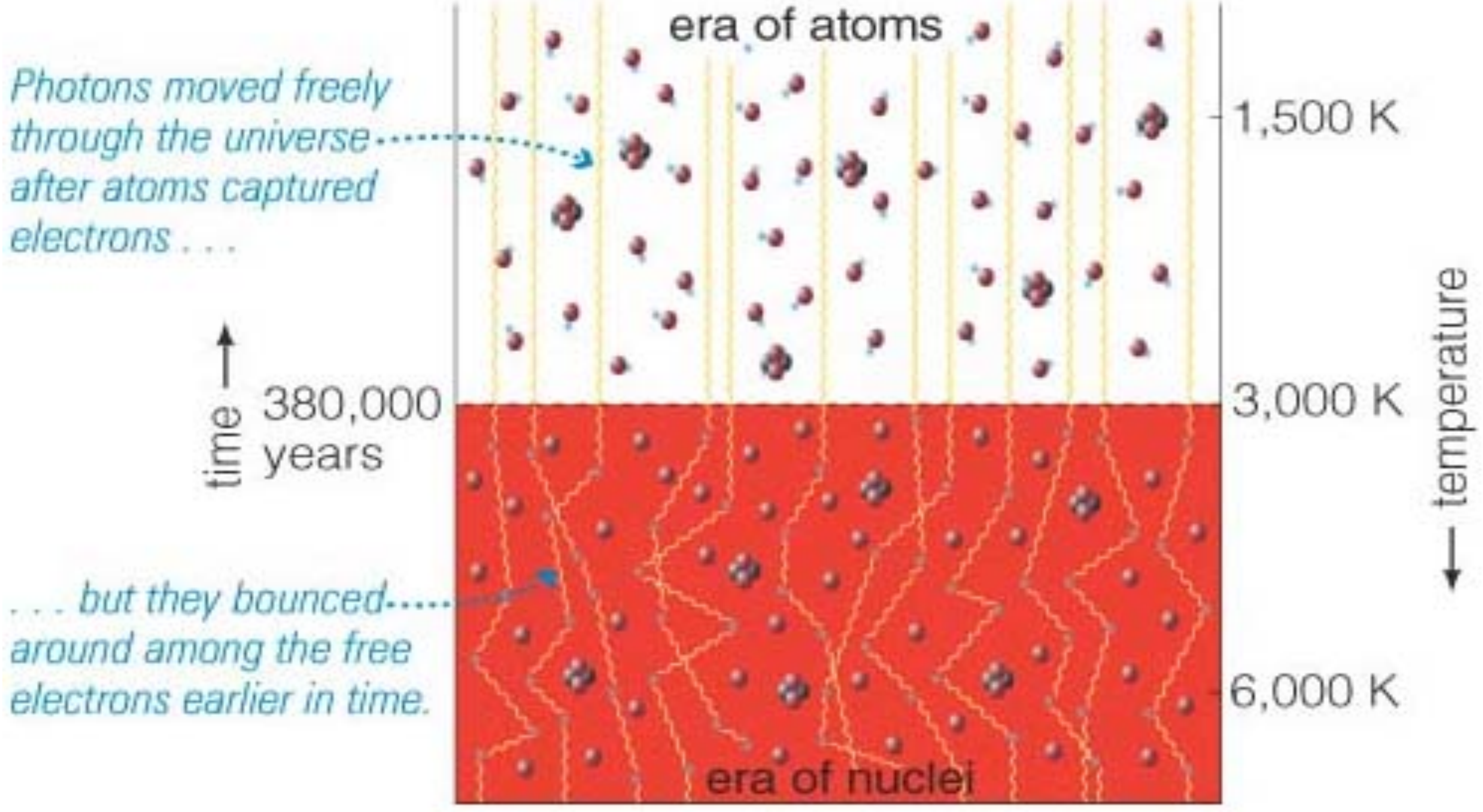
The CMB Temperature from the Full Blackbody Spectrum

- **The COBE satellite (launched in 1989)** was the first instrument to provide very accurate measurements of **the CMB spectrum**.
- CMB spectrum peaks at **1.1 mm (283 GHz) in** B_λ and at 160 GHz (1.9 mm) in B_ν, the best-fit Planck function determined a temperature of **2.73 K.**



The EM radiation background emerges when recombination completed

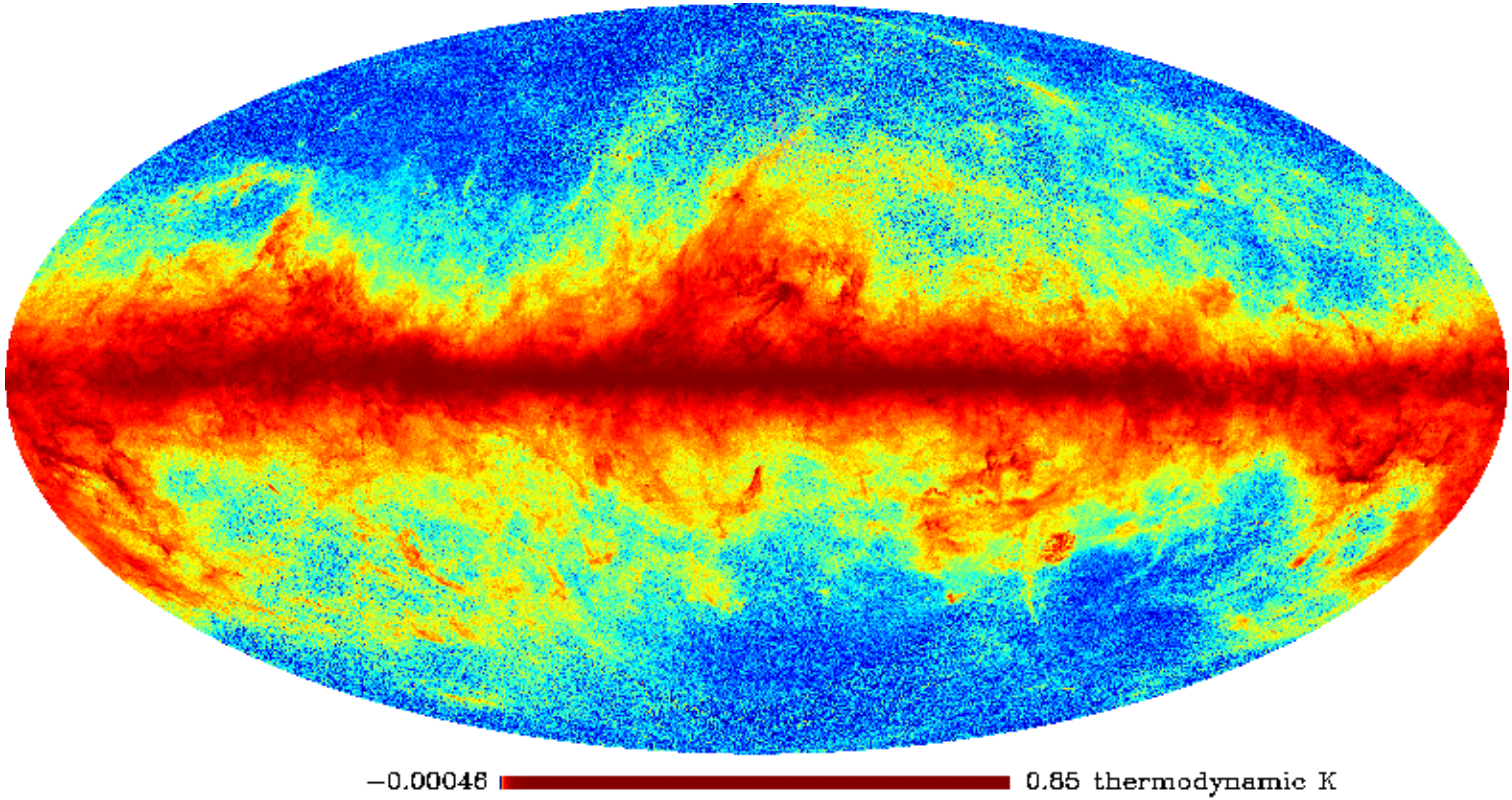
- Before **recombination**, photons cannot travel far before it is scattered by e-; after **recombination**, photons can freely travel and eventually reach us.
- Given the baryon density of the Universe, it can be shown that **Hydrogen recombination** completed when the universe was ~**3000 K.** This is **the original temperature of the cosmic EM radiation field**.
- Because of redshift, $T_{\mathrm{observed}} = T_{\mathrm{emitted}}/(1+z)$, and we know the CMB has a temperature is 2.7 K today, the redshift at which the CMB emerged must be around 1100: $1+z = T_{\mathrm{emitted}}/T_{\mathrm{observed}} = 3000\ \mathrm{K}/\ 2.7\ \mathrm{K} \sim 1100$



If the entire sky glows in microwave radiation, why not get an all-sky map of the CMB?

The discovery of CMB anisotropies

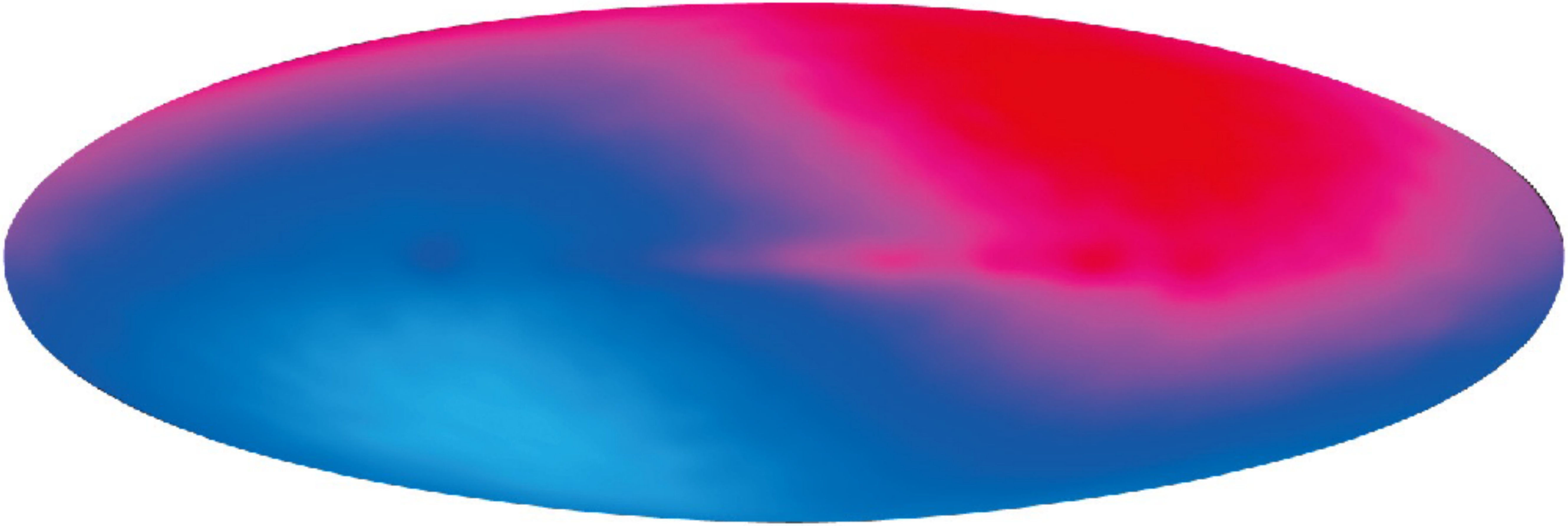
The All-sky Temperature Map of the CMB in Mollweide (equal-area) projection



After subtracting the Milky Way, there is a strong dipole signal in the CMB ($\langle T \rangle = 2.7$ K)

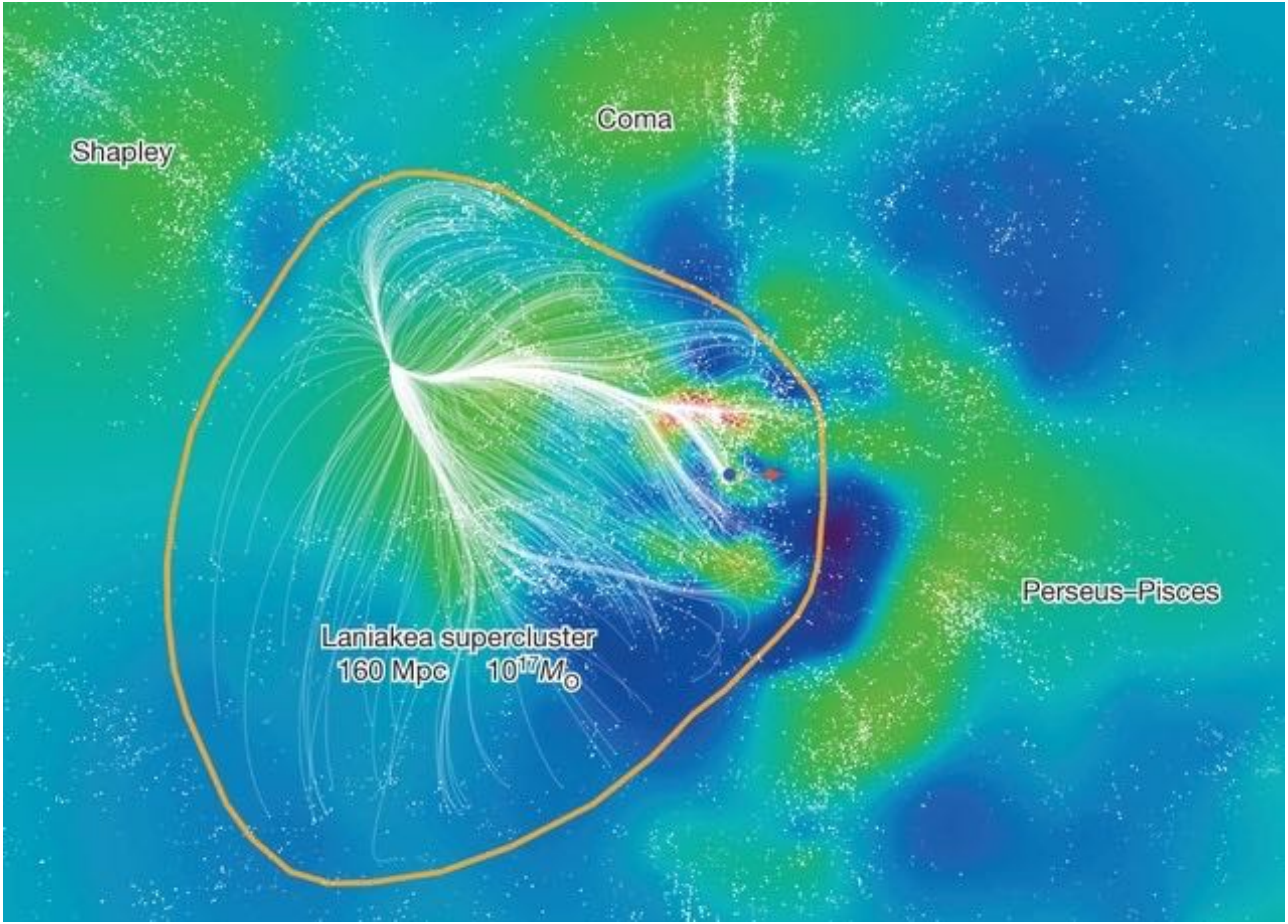
Dipole Maximum Direction: $\mathrm{RA} = 167\overset{\circ}{.}942 \pm 0\overset{\circ}{.}007$, $\mathrm{Dec} = -6\overset{\circ}{.}944 \pm 0\overset{\circ}{.}007$ (J2000).

Dipole Maximum Amplitude: 3.362 mK

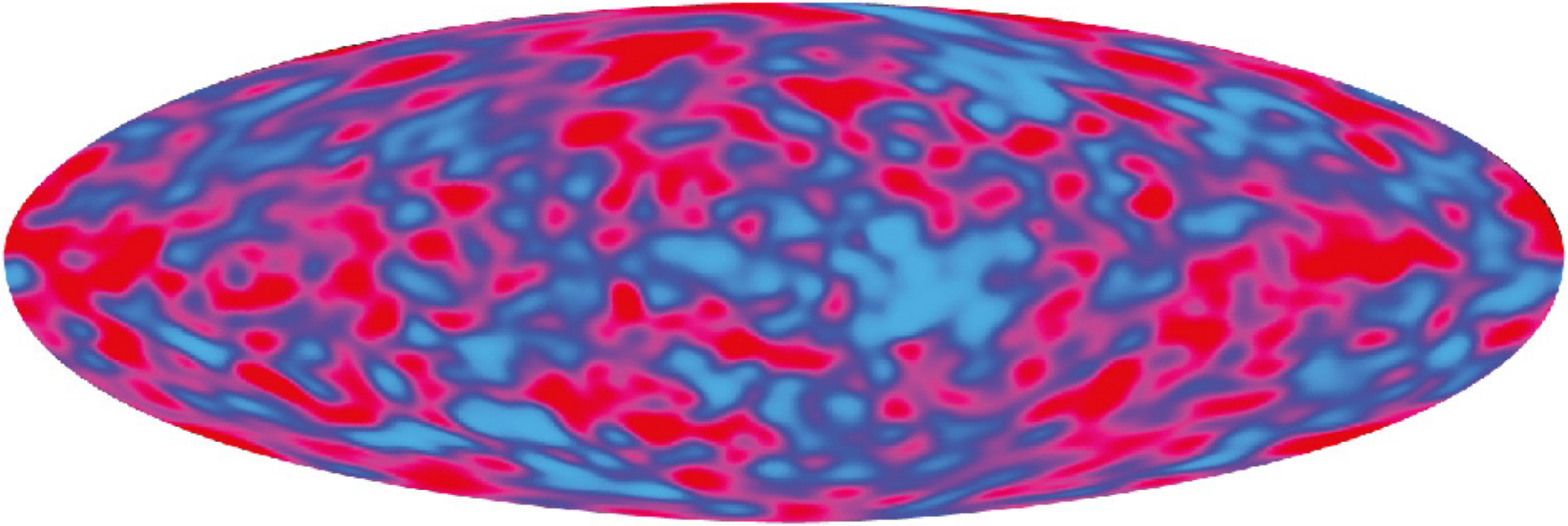


What is the relative velocity between the Solar System and the CMB rest frame?

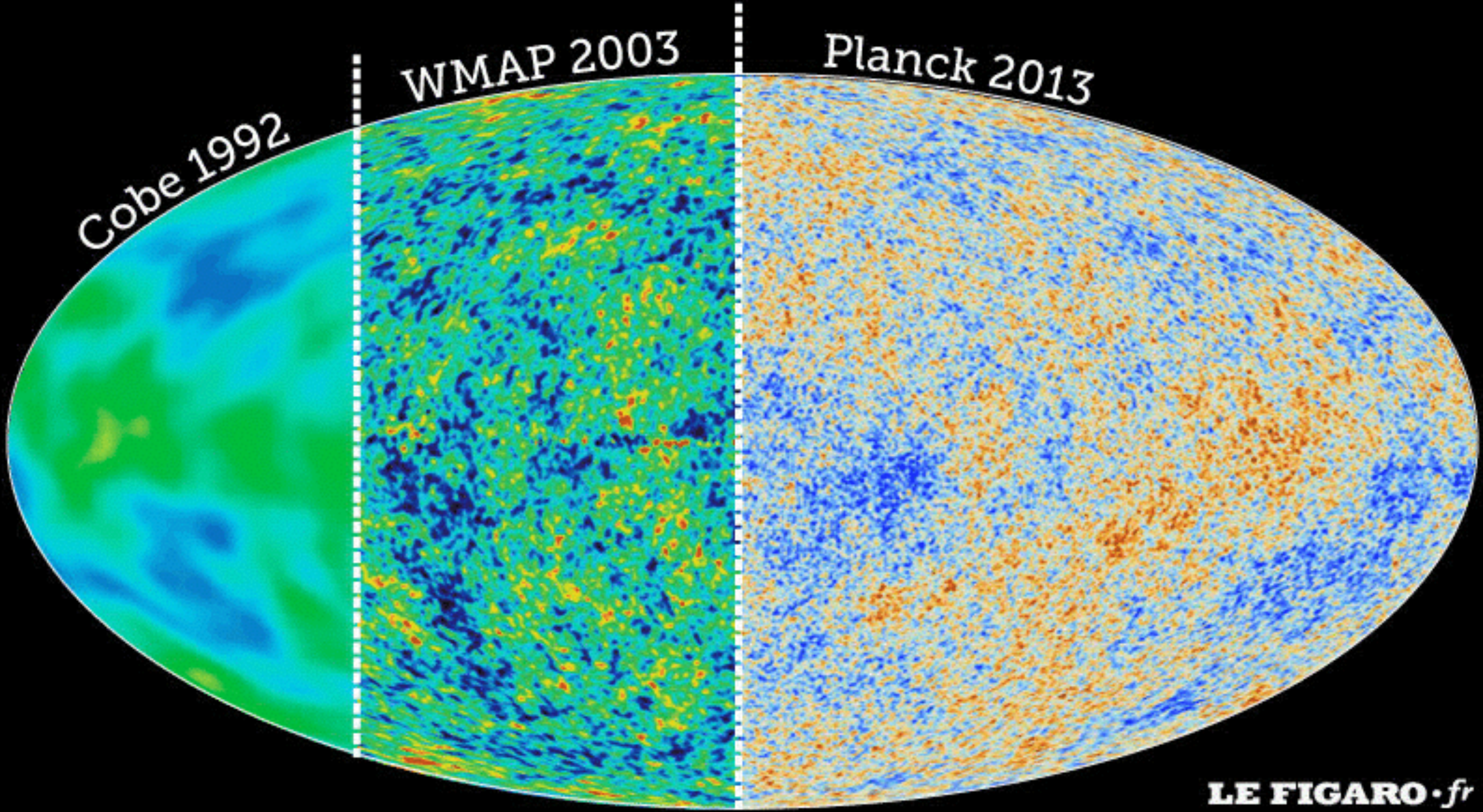
The CMB Solar Dipole shows the Combined Motion of (1) the Solar System in the Milky Way and (2) the Milky Way Galaxy in the Local Supercluster



Subtraction of the dipole reveals smaller scale fluctuations in the CMB (the anisotropies)



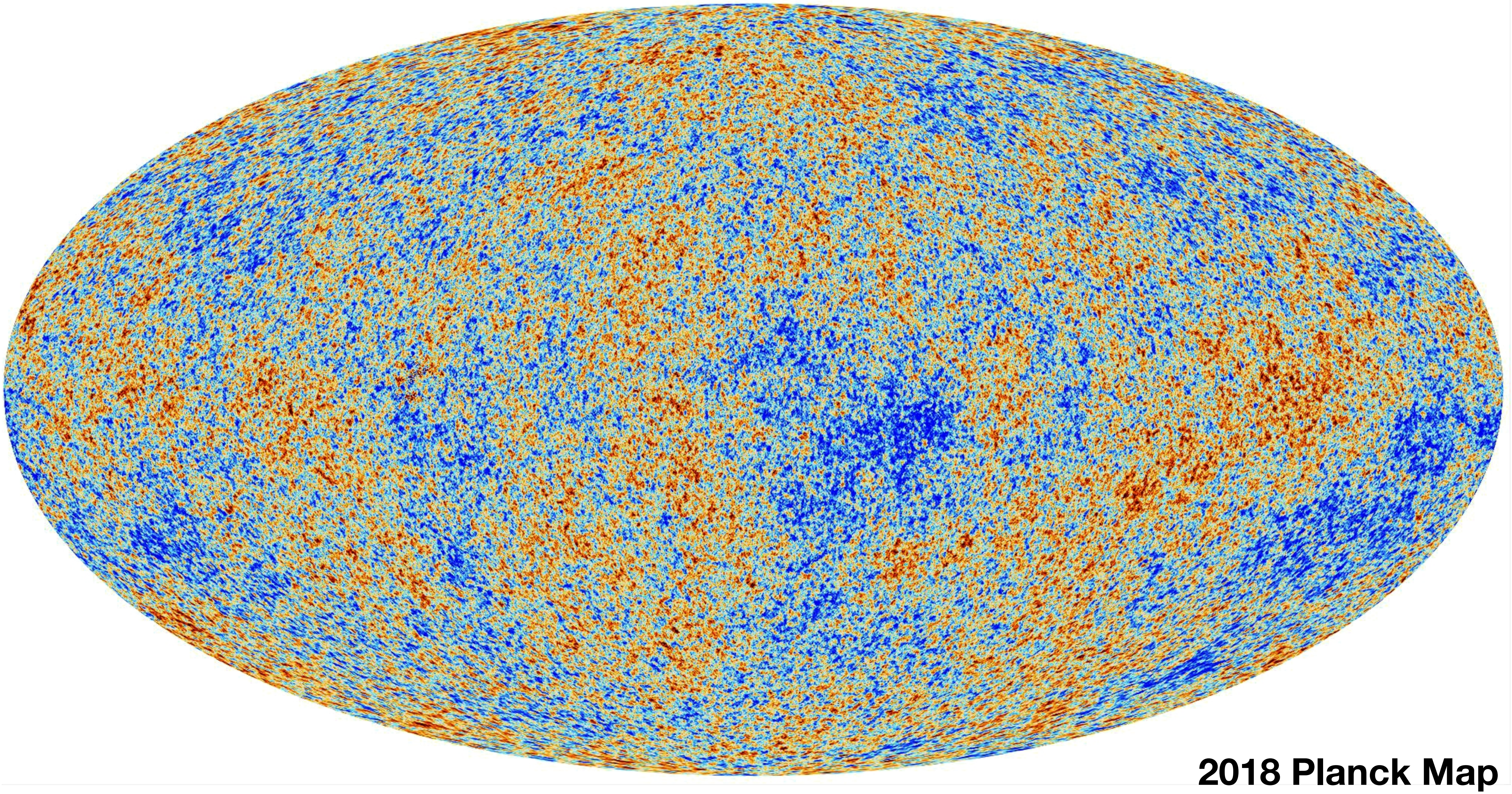
Improved angular resolutions over three generations of satellites



LE FIGARO·fr

The CMB anisotropy map shows temperature fluctuations of $\delta T/T \approx 10^{-5}$

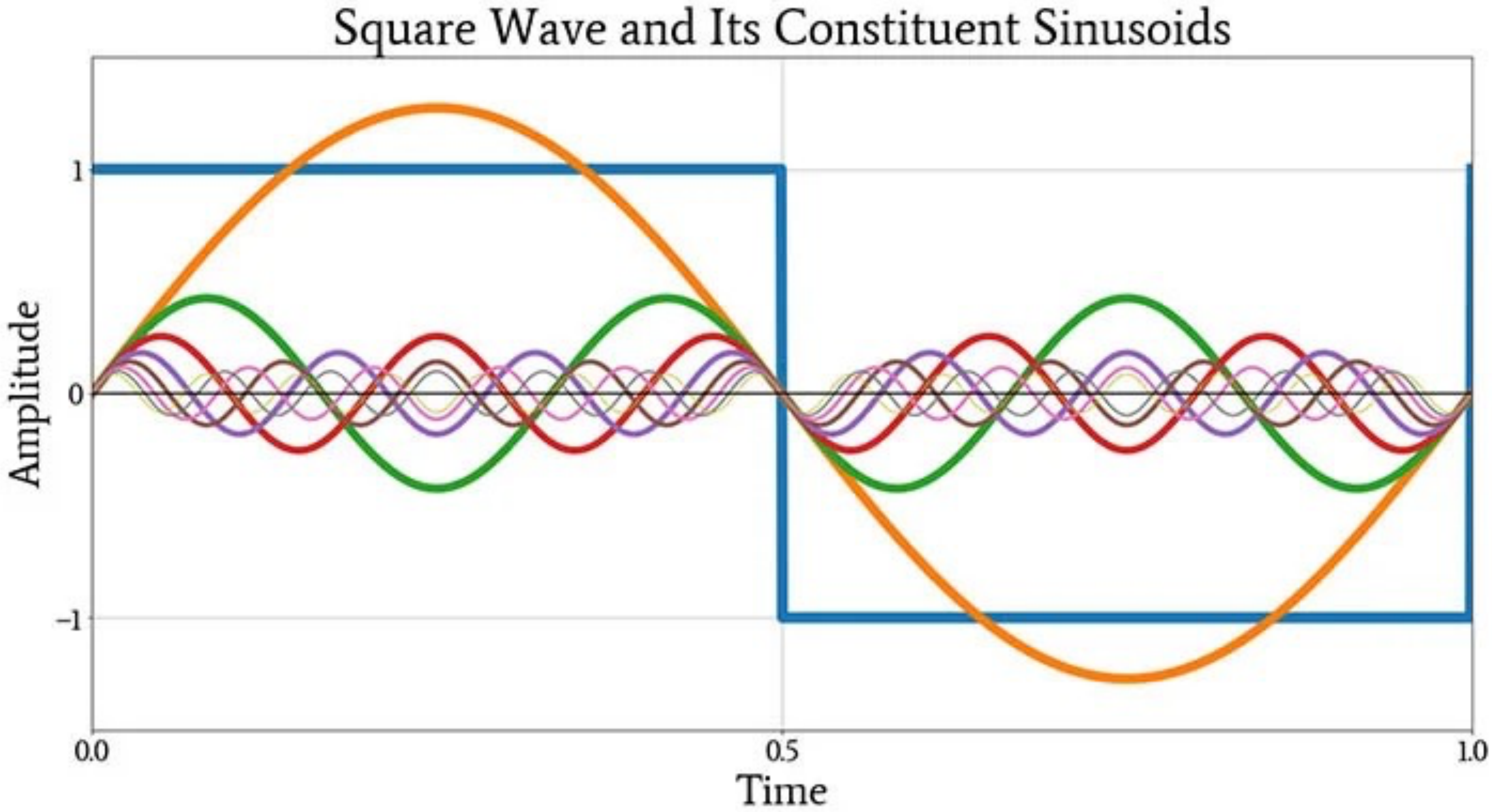
Hotter regions have *higher density (seeds of DM halos)*,
CMB anisotropy shows density fluctuations of $3\text{x}10^{-5}$ at z ~ 1000



2018 Planck Map

Quantifying CMB anisotropies w/ **angular power spectrum**

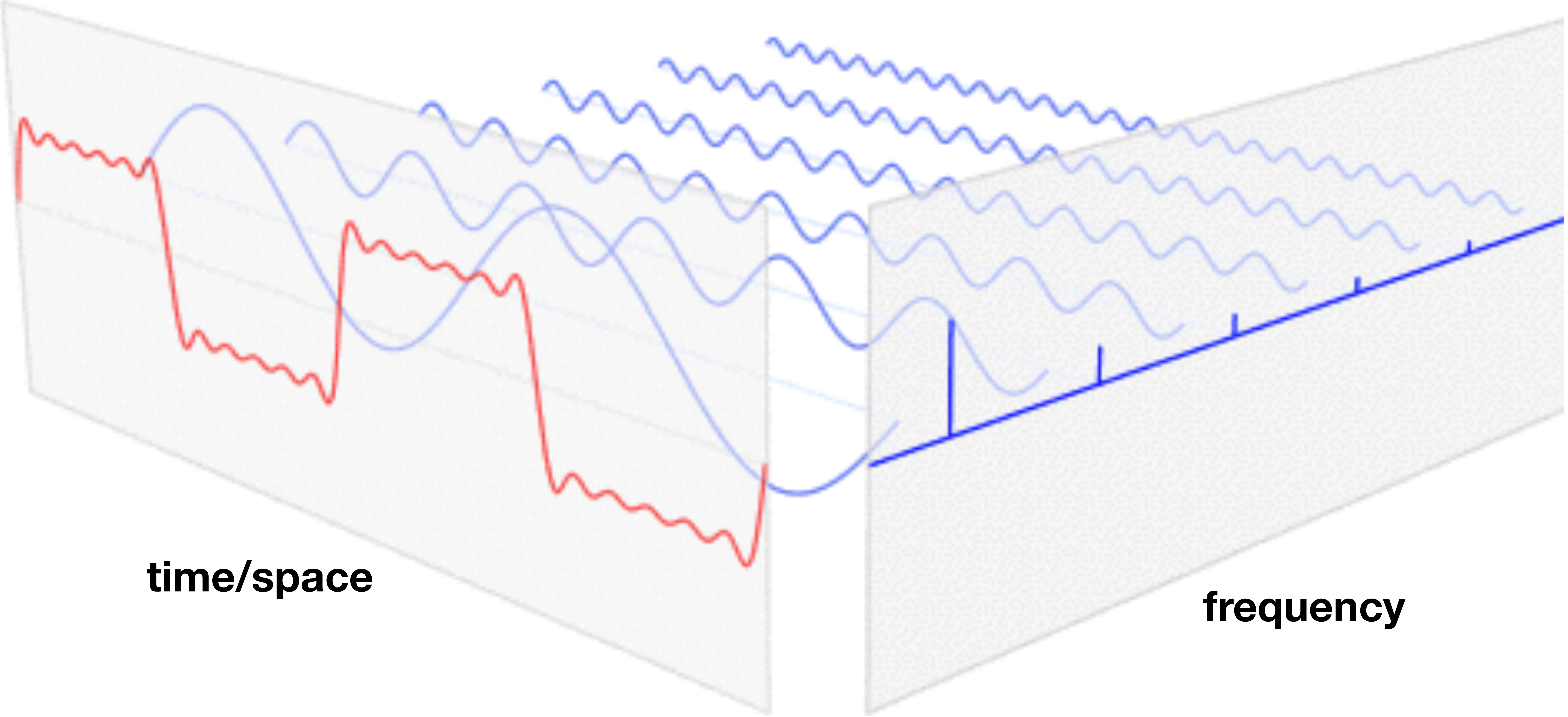
Fourier (1822): the standard tool to analyze periodic signals



Decomposing a *periodic* time or space signal into series of sine & cosine functions

$$\widehat{f}(\xi) = \underbrace{Ae^{i\theta}}_{\text{polar coordinate form}} = \underbrace{A\cos(\theta) + iA\sin(\theta)}_{\text{rectangular coordinate form}}.$$

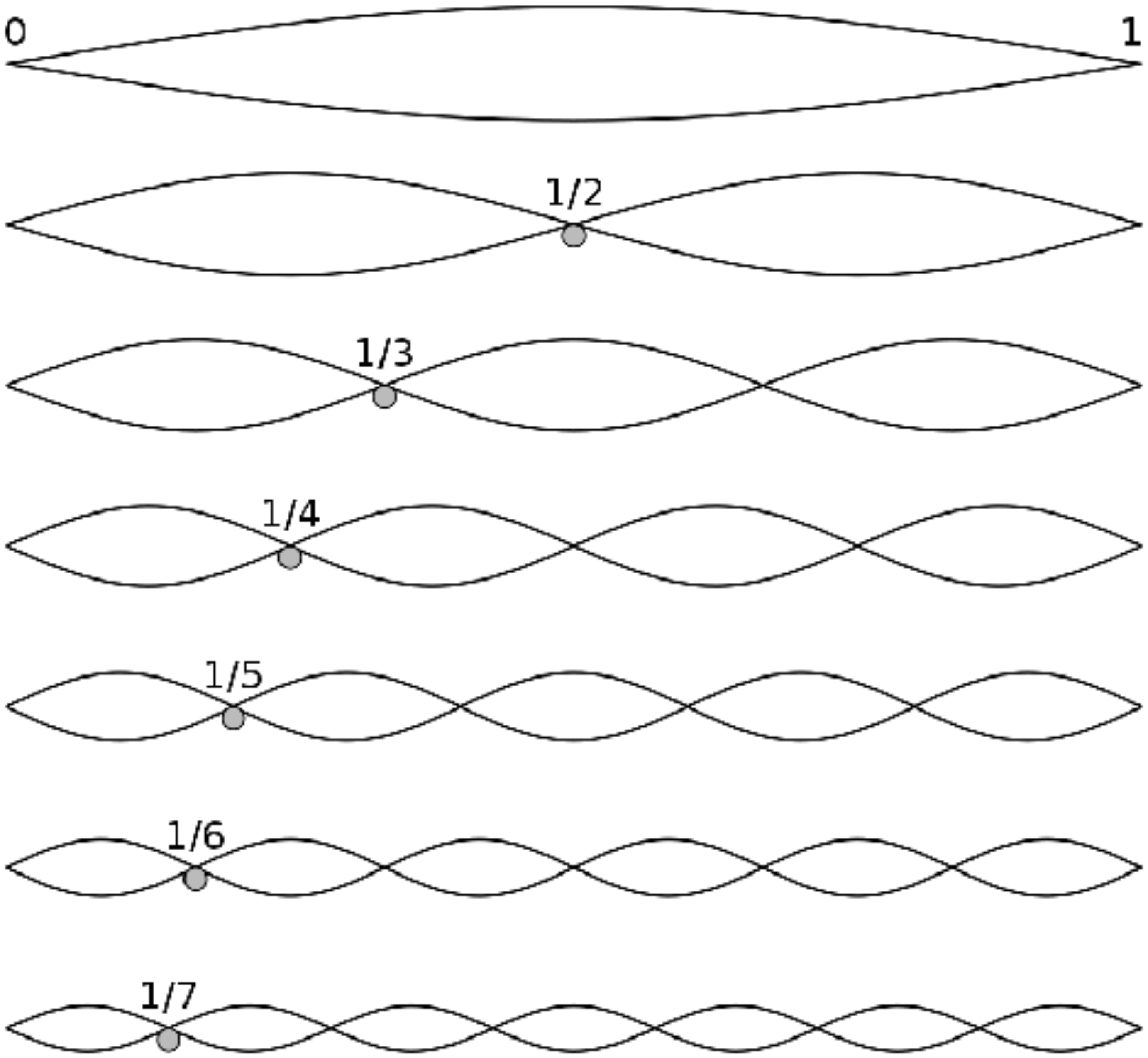
Fourier Transform: time/space domain -> frequency domain



Decomposing a *periodic* time or space signal into series of sine & cosine functions

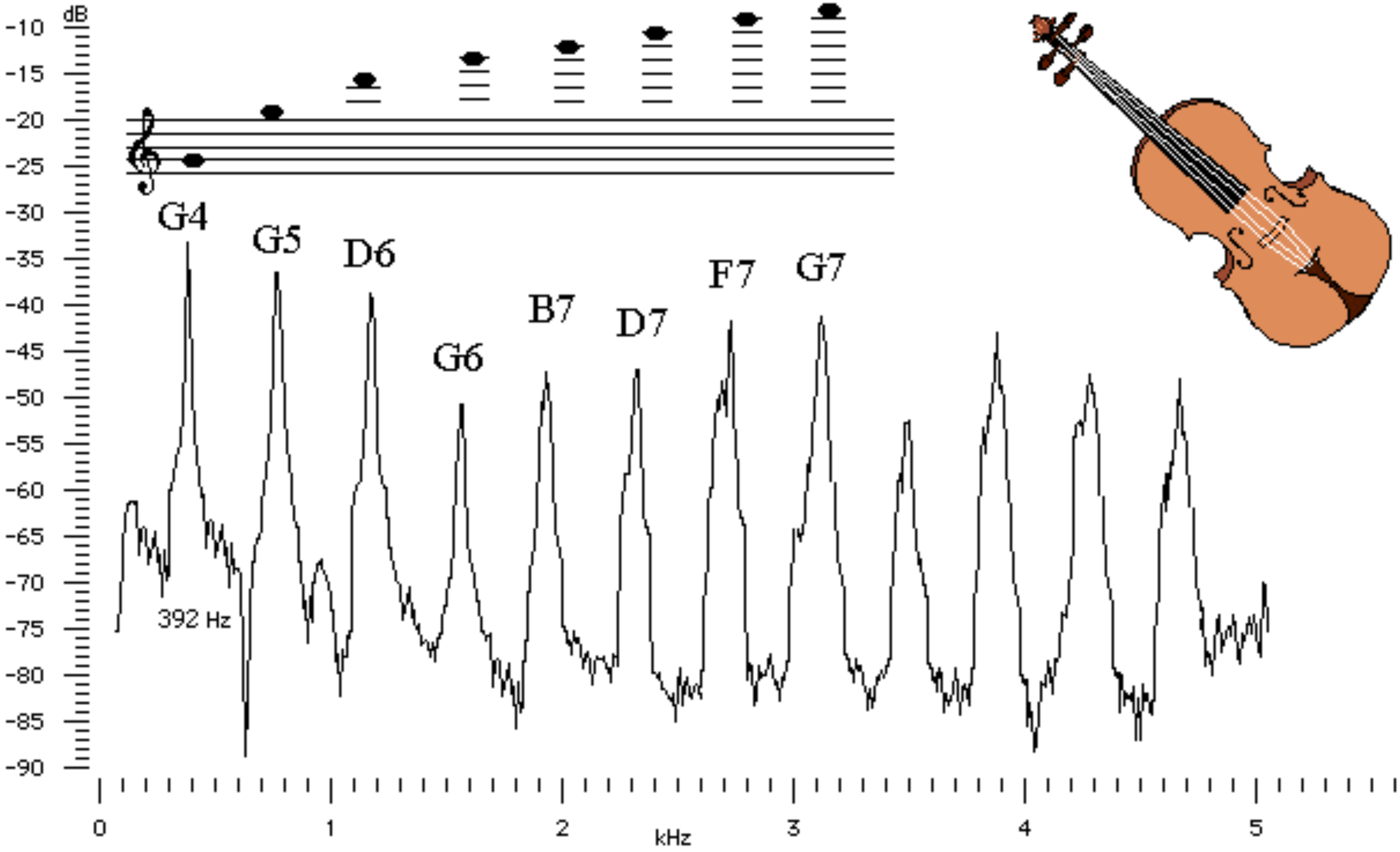
$$\widehat{f}(\xi) = \underbrace{Ae^{i\theta}}_{\text{polar coordinate form}} = \underbrace{A\cos(\theta) + iA\sin(\theta)}_{\text{rectangular coordinate form}}.$$

Harmonics of a string showing the periods of the pure-tone harmonics



acoustic resonators based on strings, which some call “music instruments”

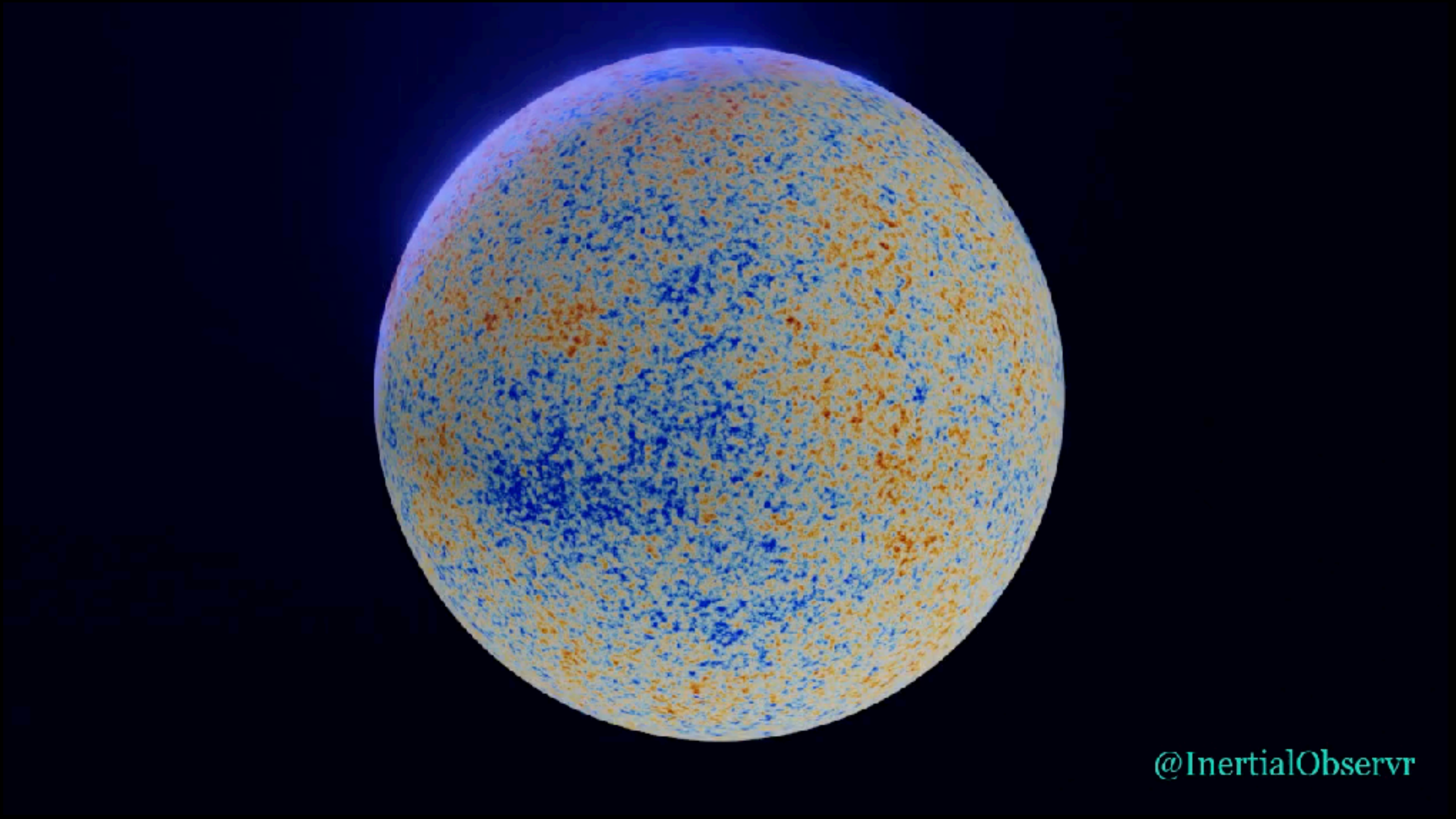
The harmonic spectrum of a Violin



https://www.intmath.com/fourier-series/6-line-spectrum.php

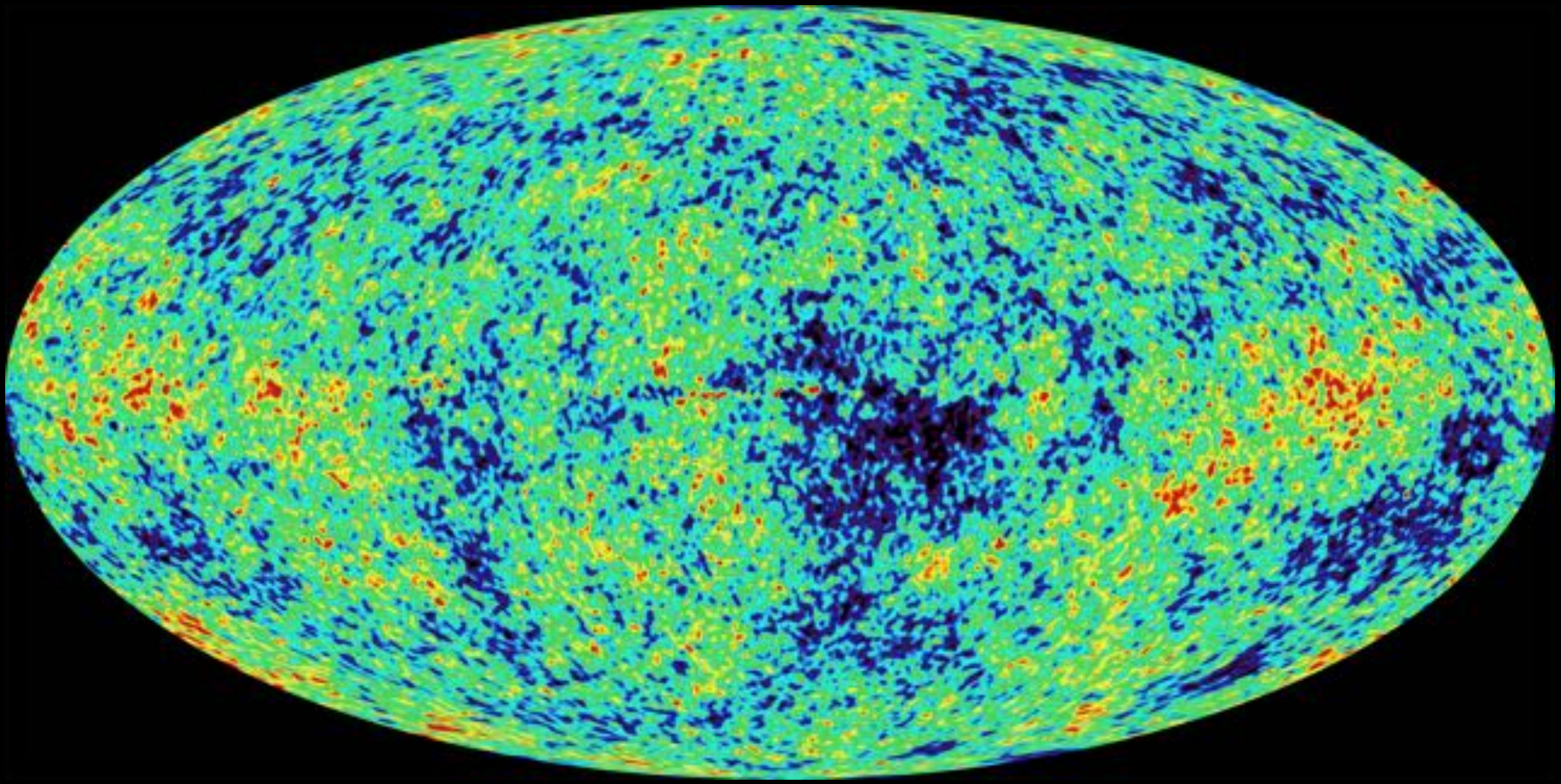
The cosmic harmonics frozen in time

"What makes the music of heaven?" - Chuang Tzu (300 BC)



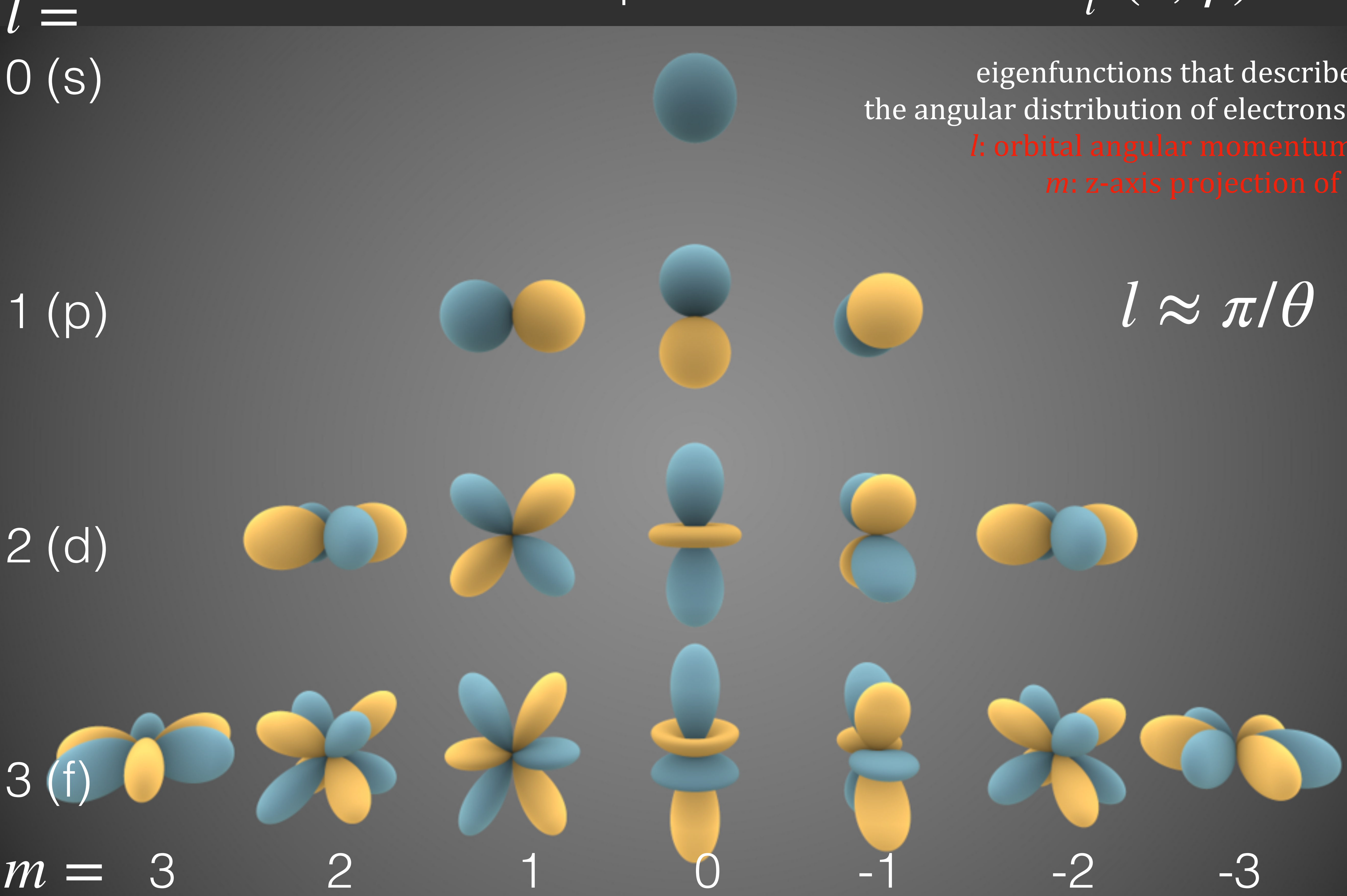
@InertialObservr

CMB Anisotropy in Mollweide (equal-area) projection



$\delta T/T \approx 0.00001$

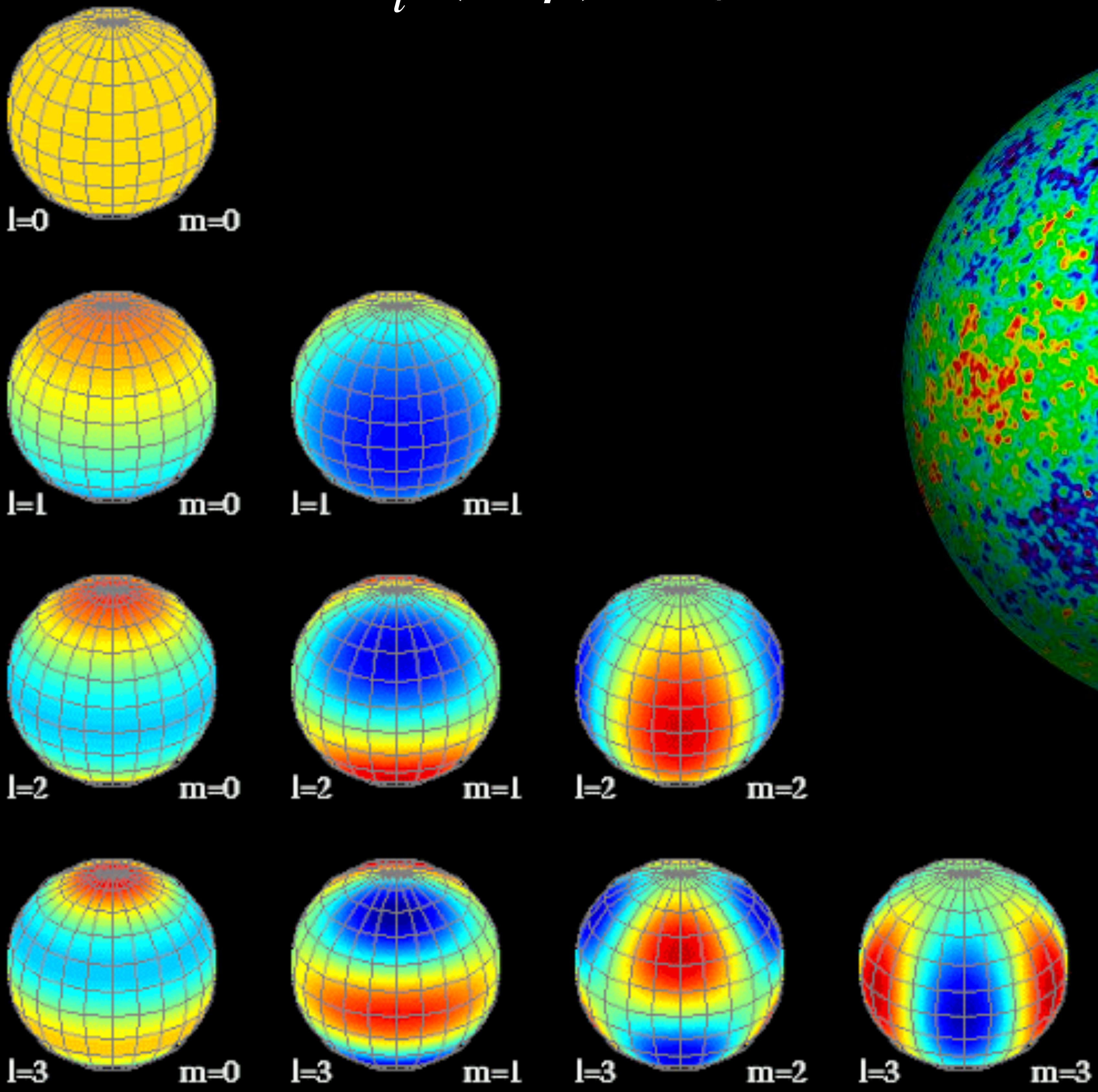
Quantum Mechanics: spherical harmonics $Y_l^m(\theta, \phi)$

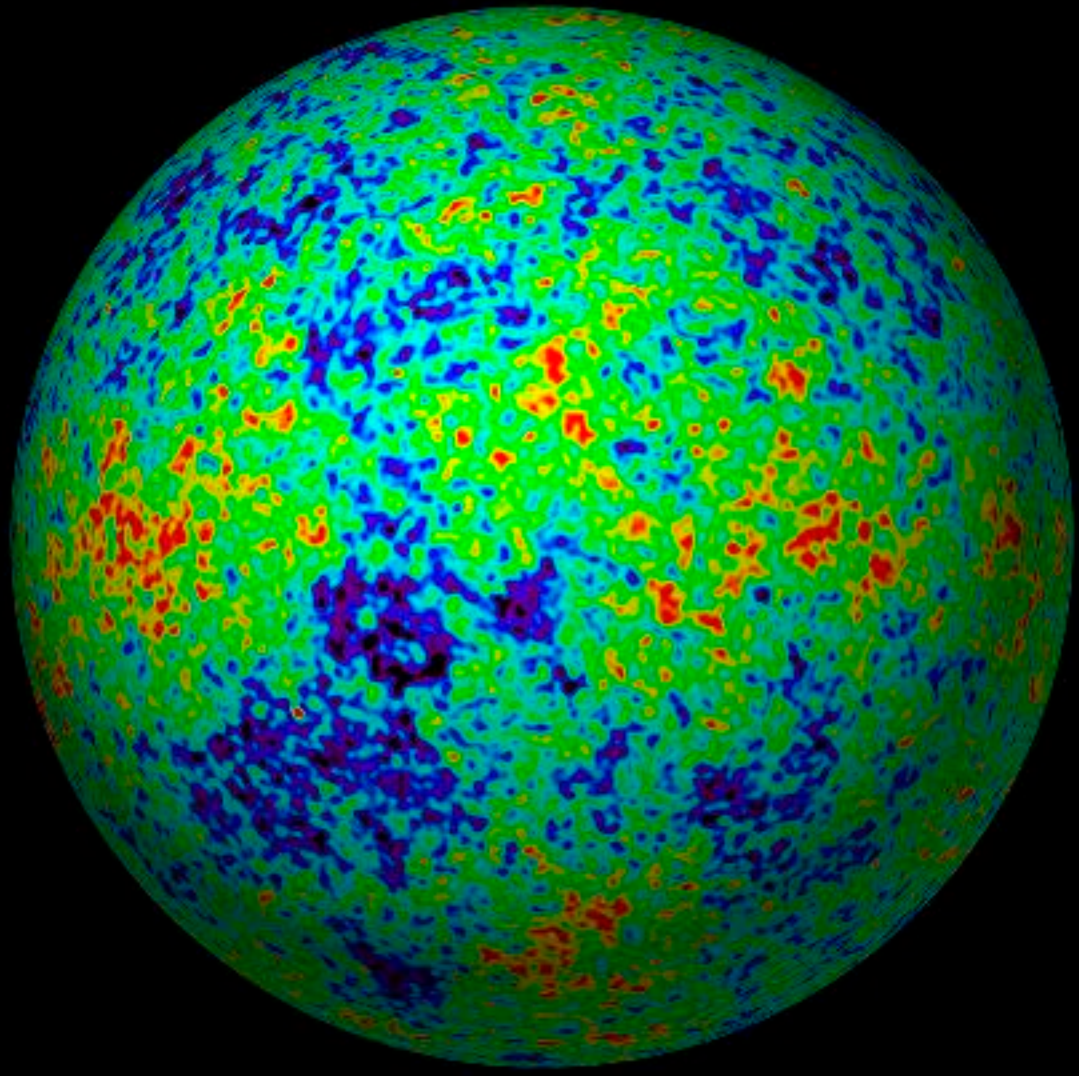


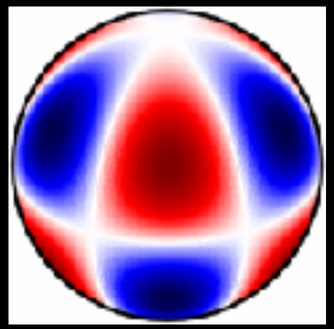
eigenfunctions that describe
the angular distribution of electrons:
l: orbital angular momentum
m: z-axis projection of *l*

$$l \approx \pi/\theta$$

Representing CMB anisotropies as a sum of spherical harmonics $Y_l^m(\theta, \phi)$ [Laplace 1782]

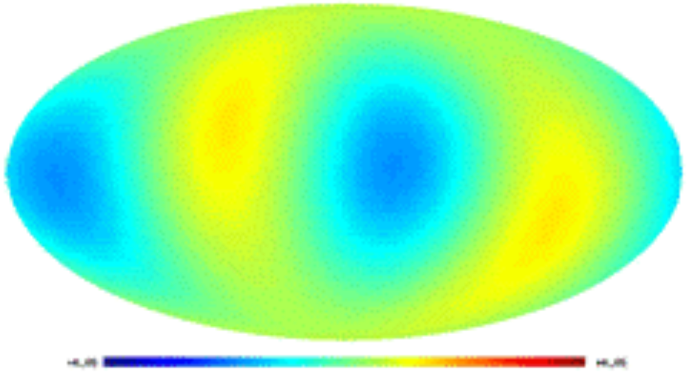




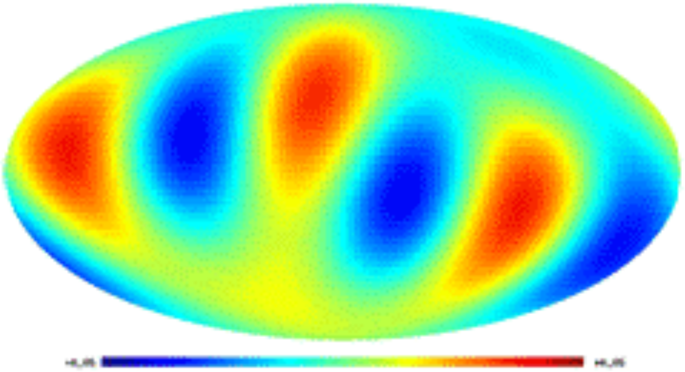


Spherical harmonics in Mollweide projection

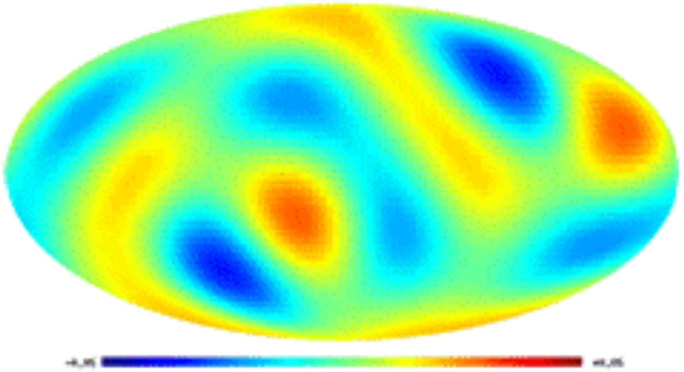
$m = l$



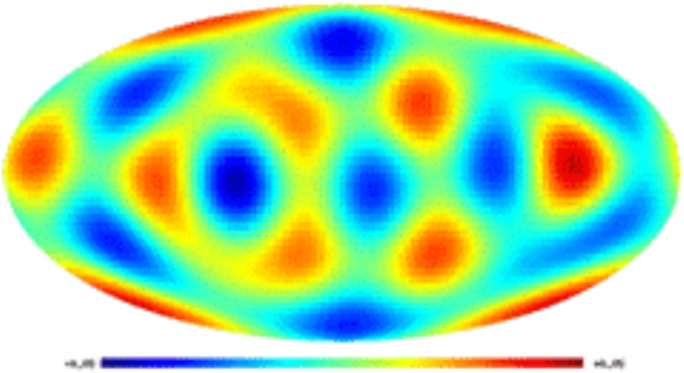
$\ell = 2$



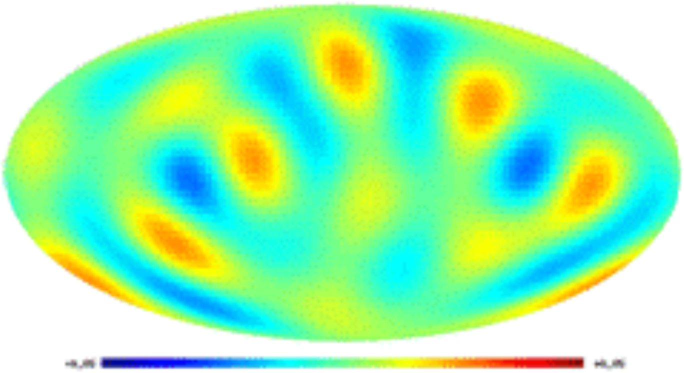
$\ell = 3$



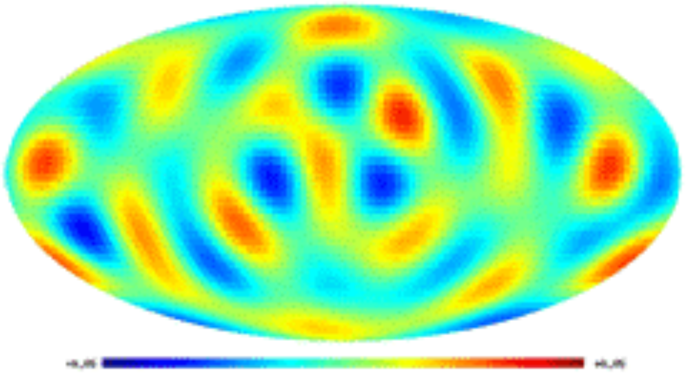
$\ell = 4$



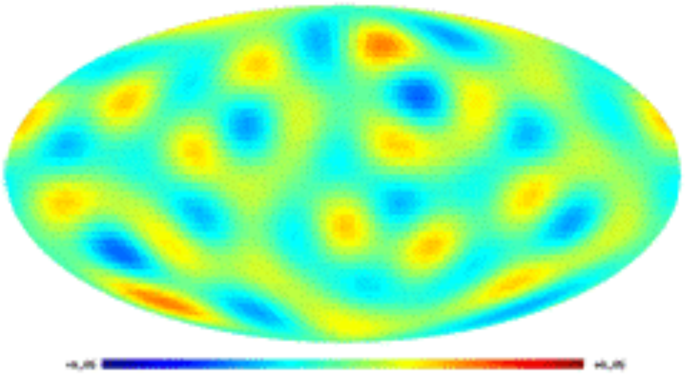
$\ell = 5$



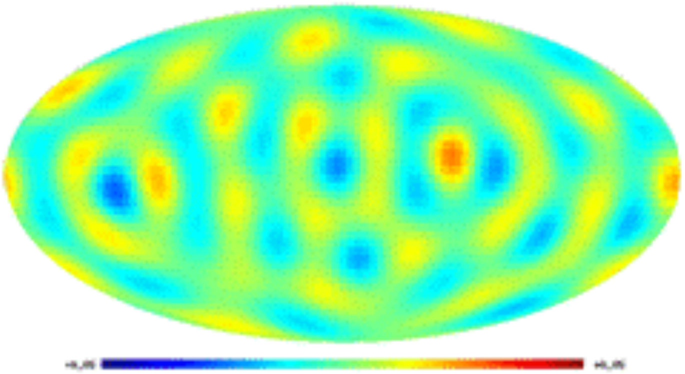
$\ell = 6$



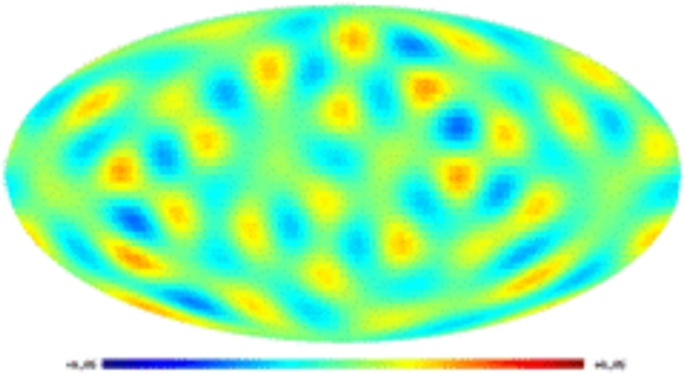
$\ell = 7$



$\ell = 8$

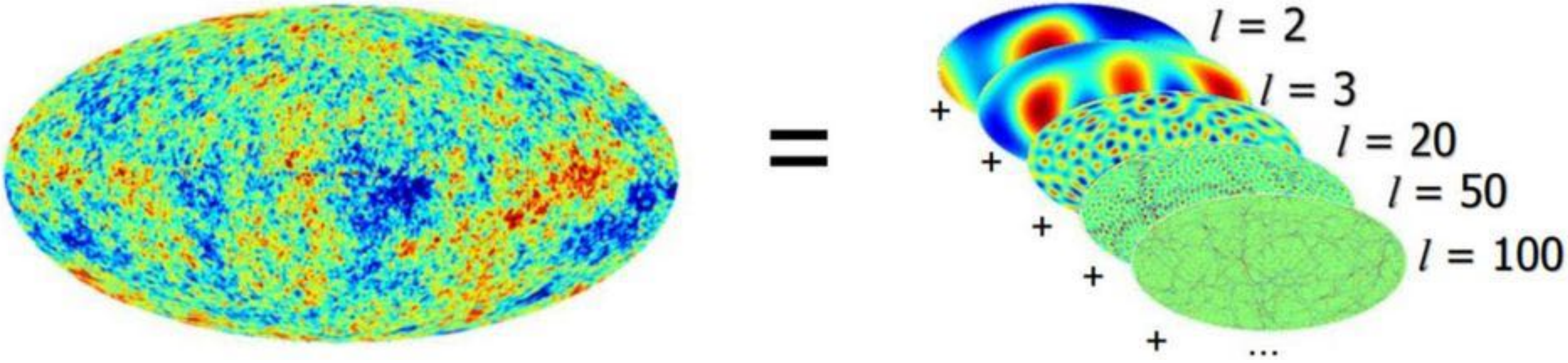


$\ell = 9$



$\ell = 10$

Expressing anisotropies as sum of spherical harmonics



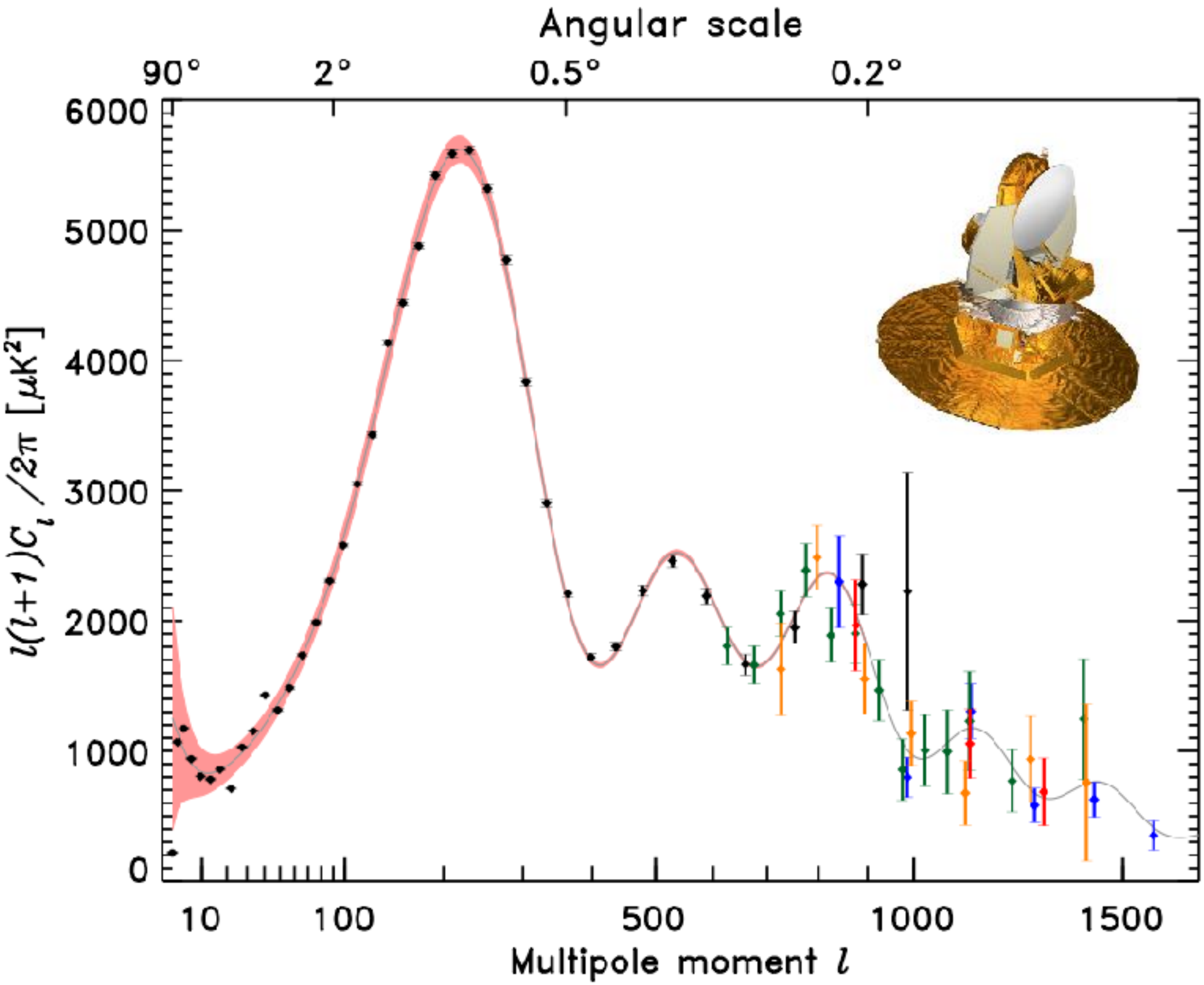
Harmonic Decomposition:

$$\delta_T(\theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{l,m} Y_l^m(\theta, \phi)$$

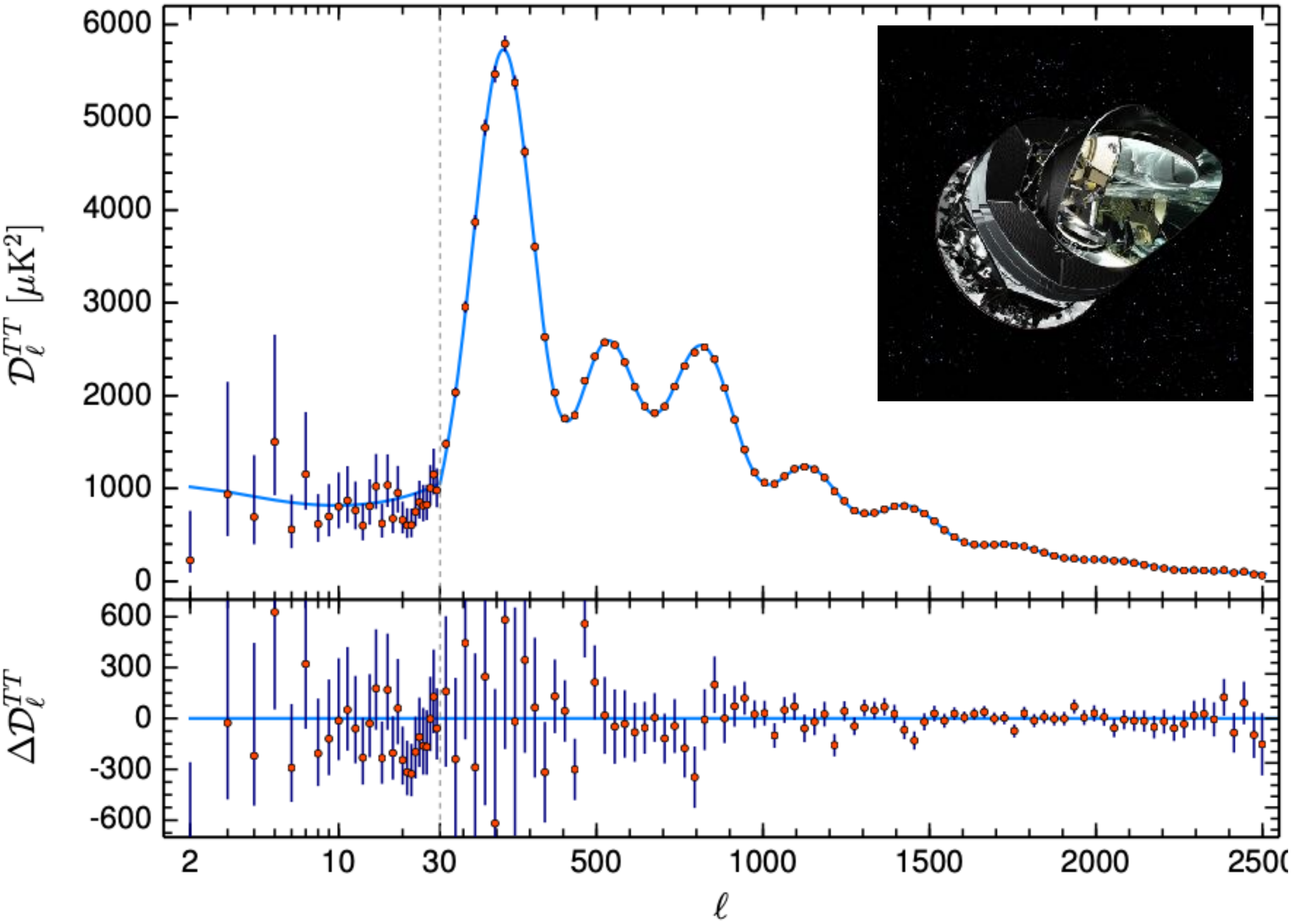
Temp. Power Spectrum:

$$P(l) = \frac{l(l+1)}{2\pi} C_l = \frac{l(l+1)}{2\pi} \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{l,m}|^2$$

Power spectrum of CMB anisotropy (WMAP: launched in 2001)



Power spectrum of CMB anisotropy (Planck: launched in 2009)

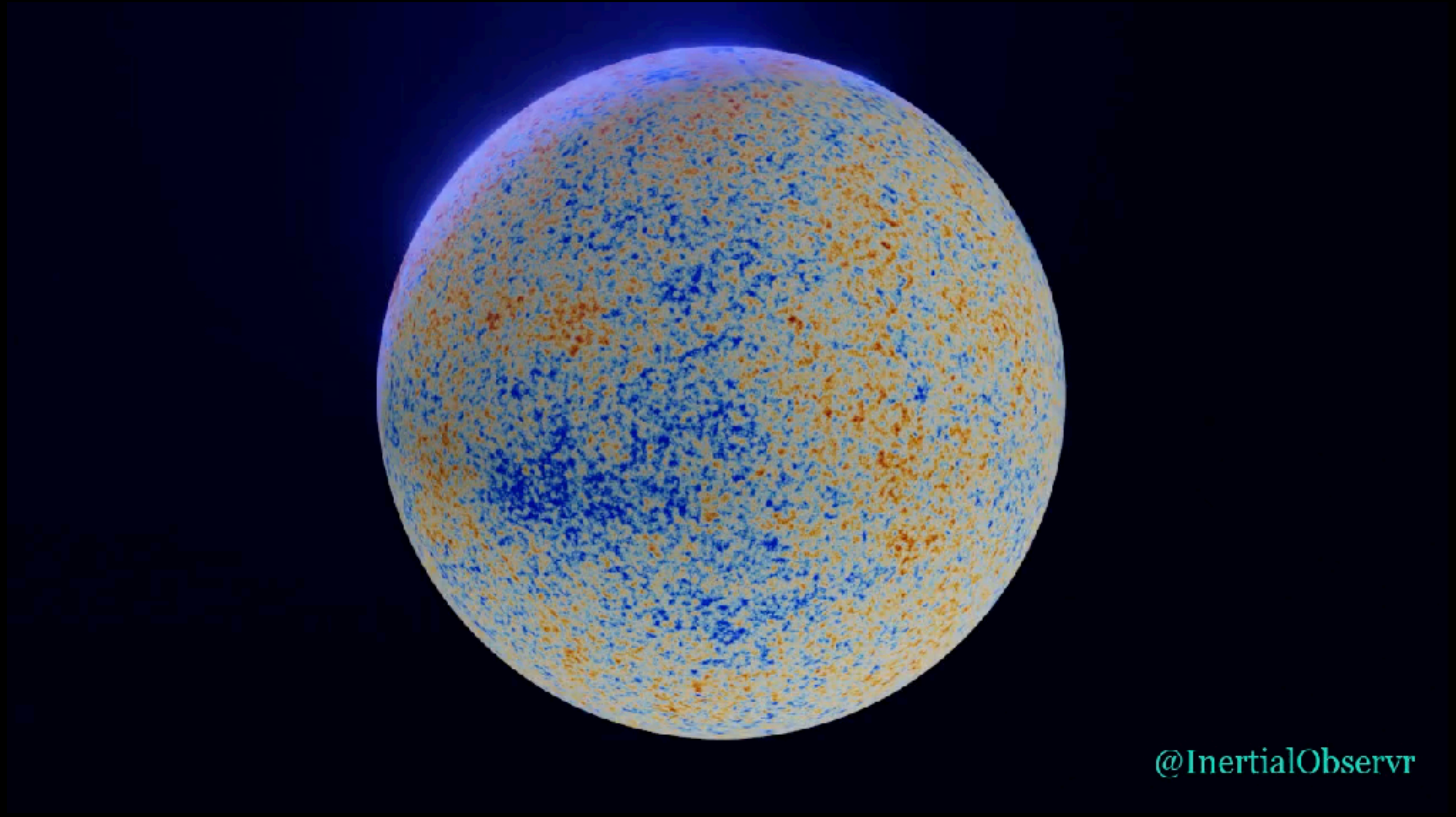


Constraints on Cosmological Parameters:

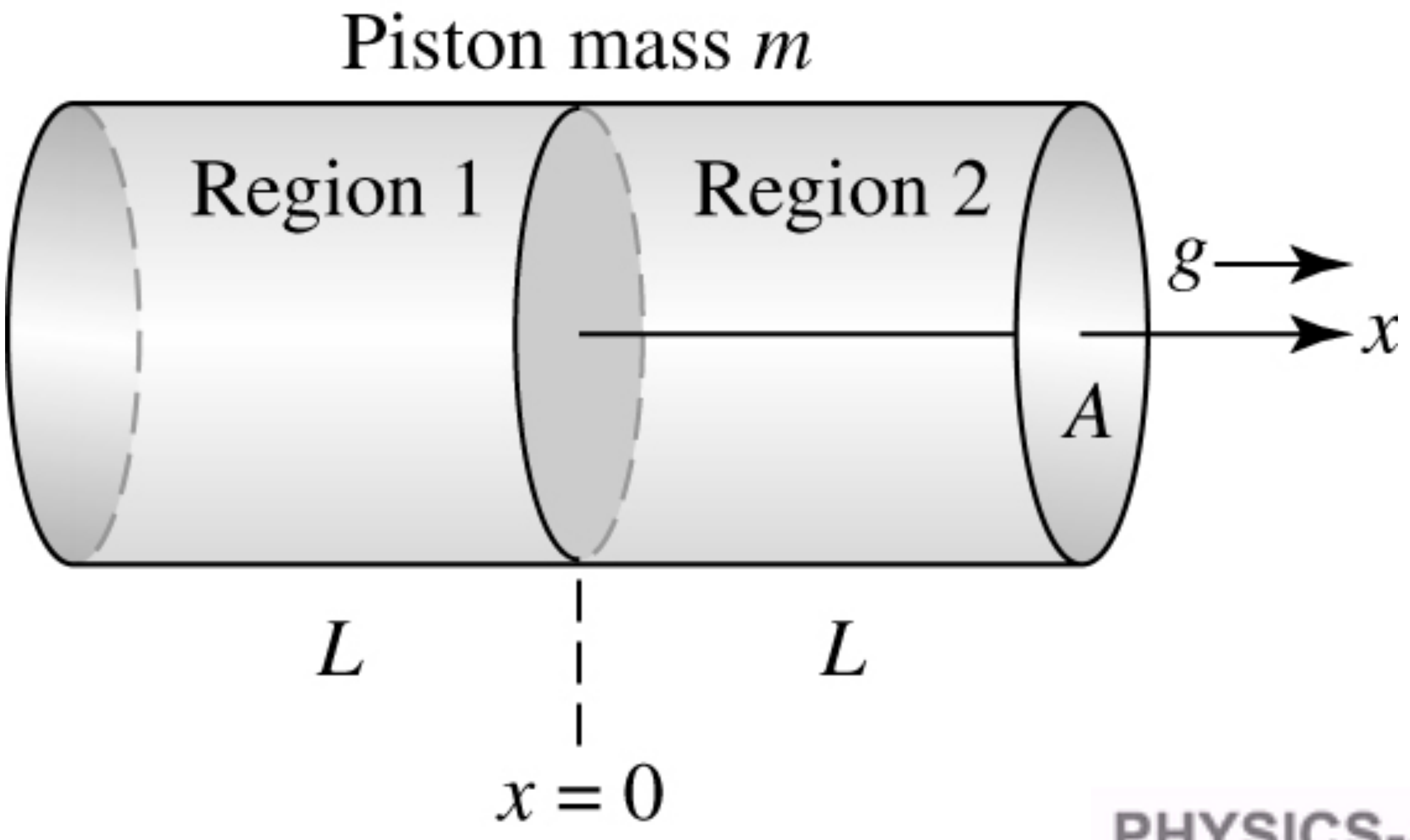
CMB Anisotropies

The cosmic harmonics frozen in time

“What makes the music of heaven?” - Chuang Tzu (300 BC)



Because overdensities of the baryon+photon fluid cannot collapse (Jeans length > Horizon size), they undergo acoustic oscillations



Ideal Gas Solution

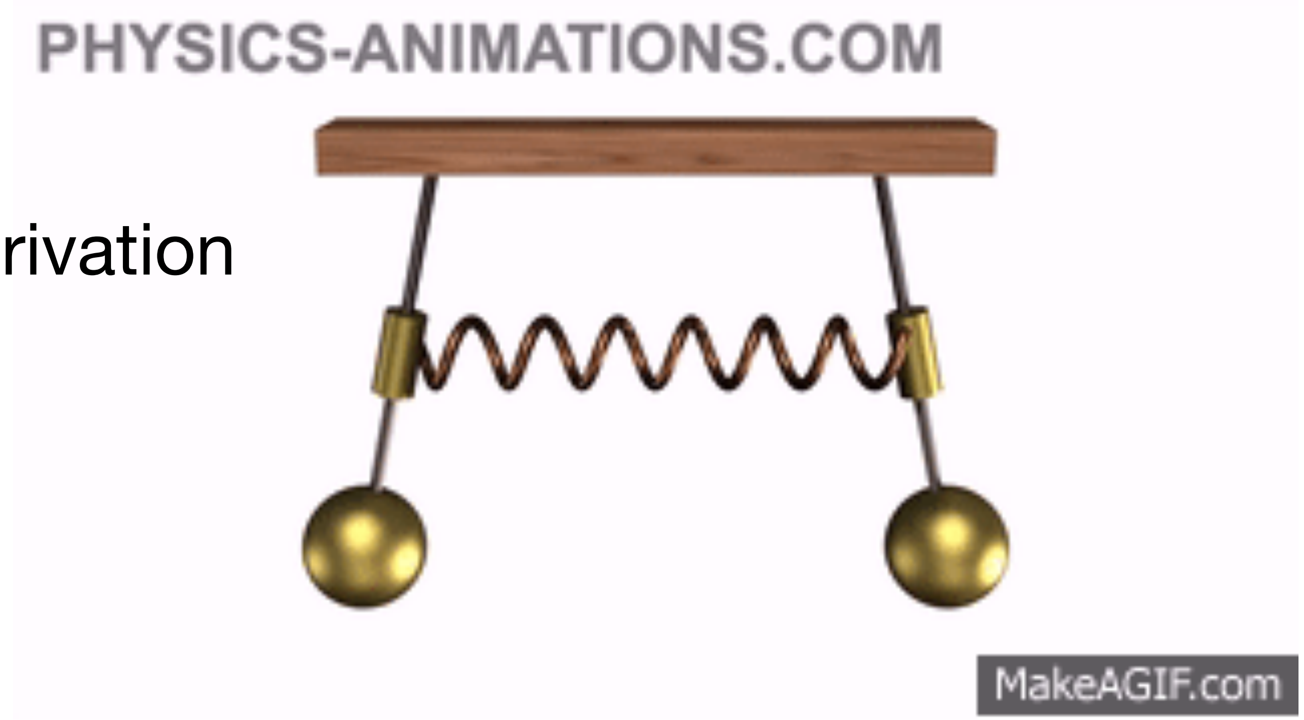
$$x(t) = x_0 \sin(c_s t/L)$$

$$\tau = 2\pi L/c_s$$

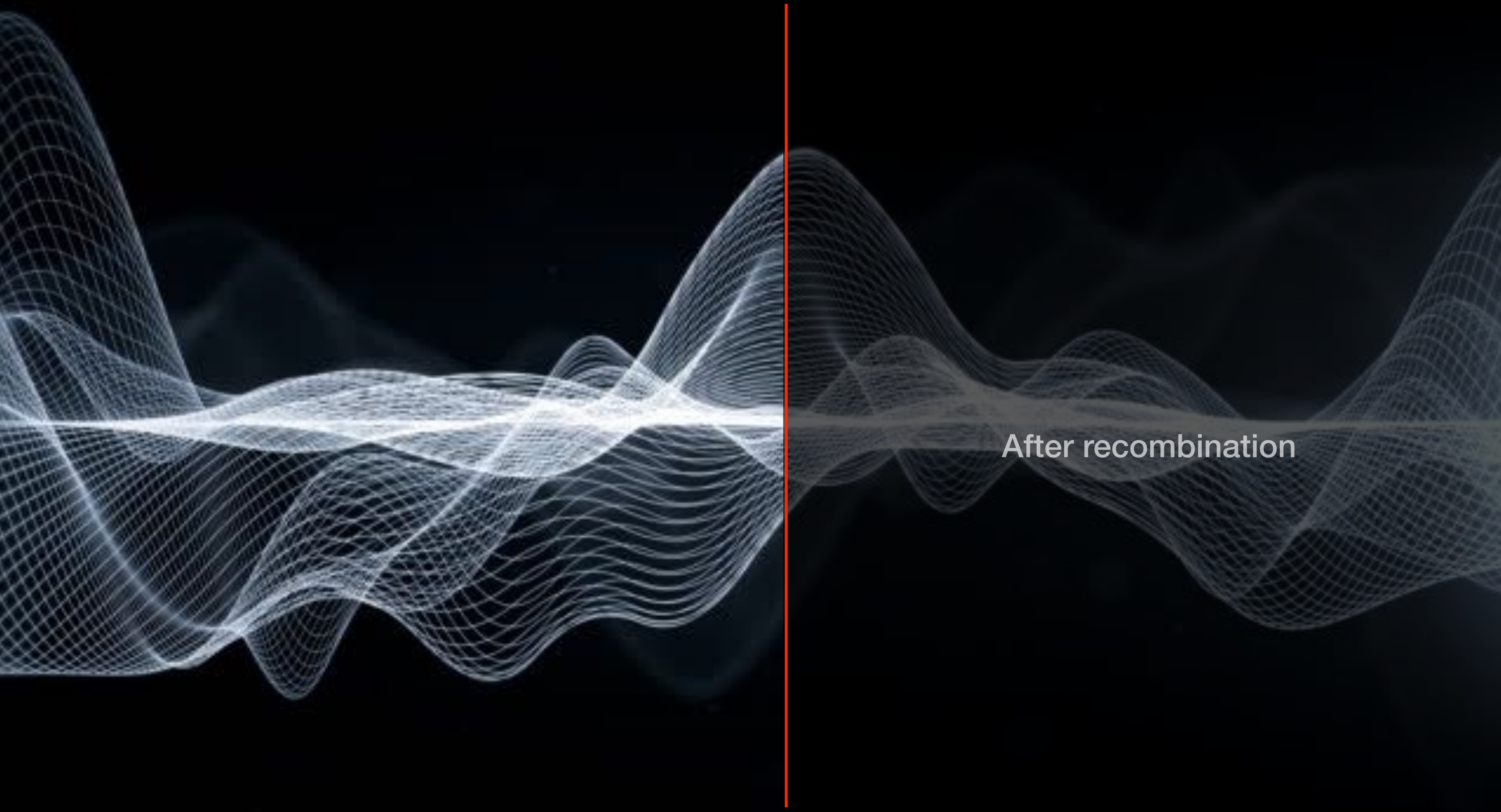
$$c_s^2 = \frac{\partial P}{\partial \rho} = \frac{\gamma k T}{\mu m_H}$$

$$\gamma = C_P/C_V$$

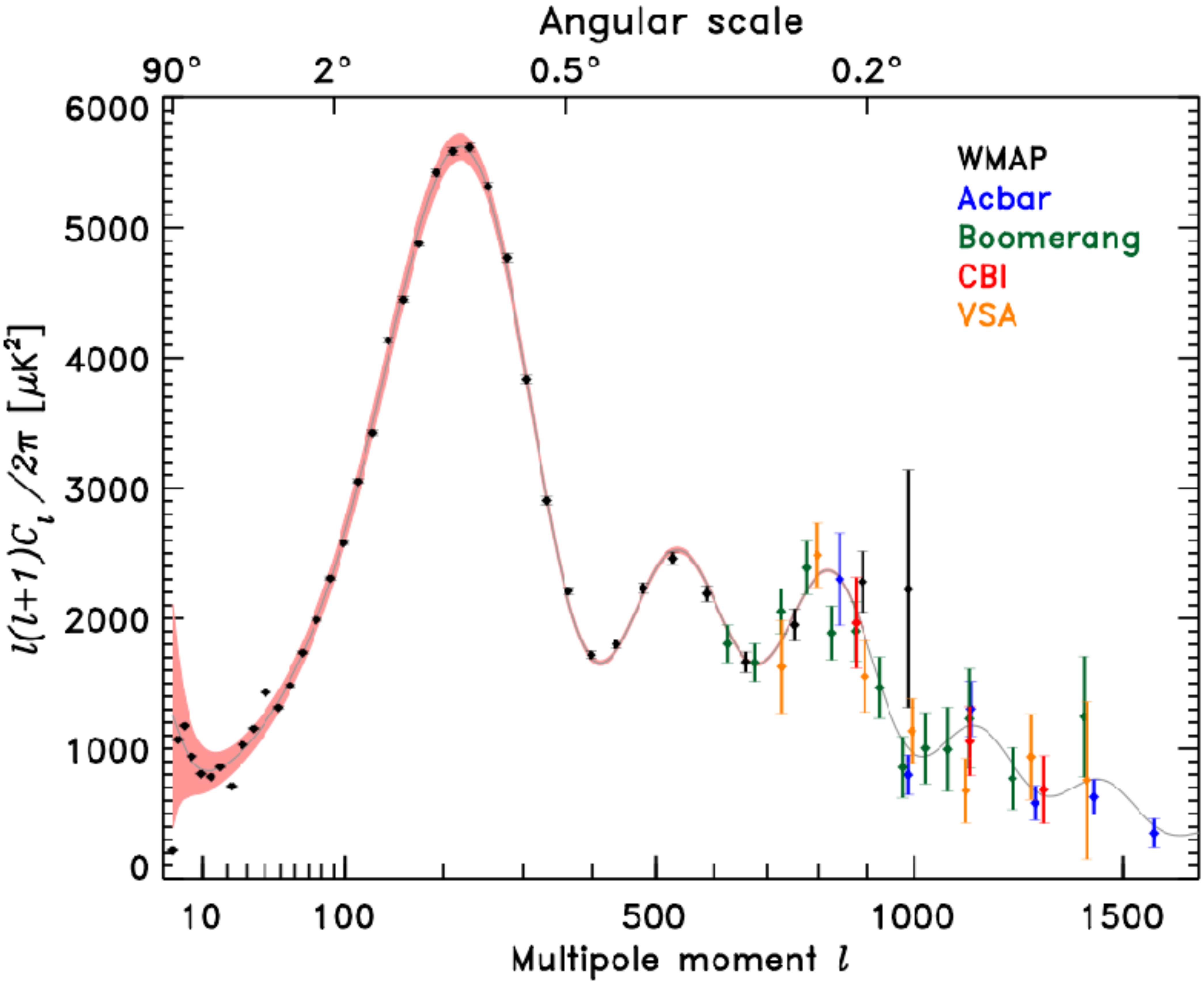
Simple gas cylinder + piston model derivation



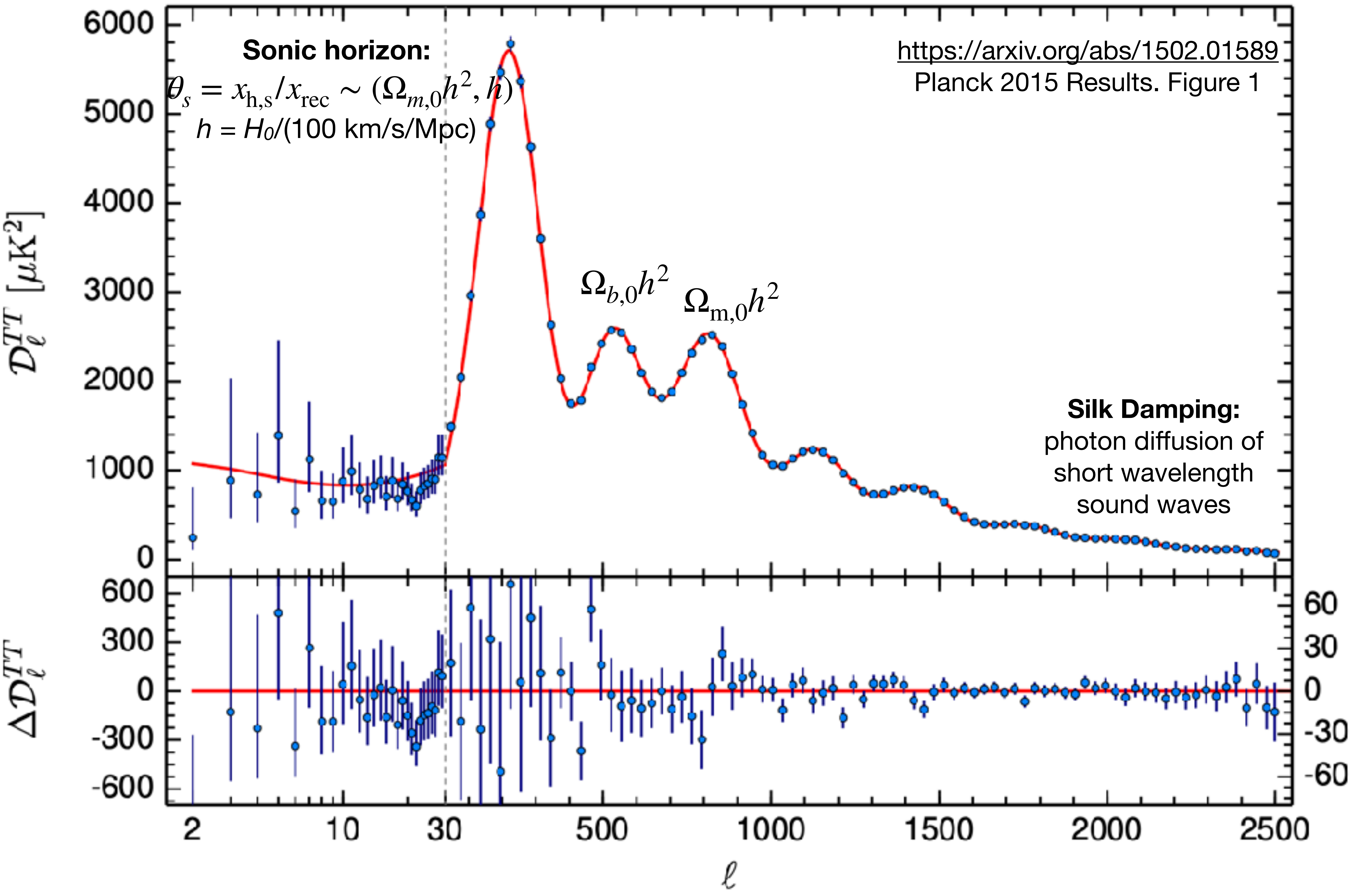
random acoustic oscillations frozen at recombination



The Power Spectrum of the CMB from WMAP (2003)



The Power Spectrum of the CMB from Planck (2015)



Sonic horizon:

$\theta_s = x_{\mathrm{h,s}}/x_{\mathrm{rec}} \sim (\Omega_{m,0}h^2, h)$

$h = H_0/(100$ km/s/Mpc$)$

$\Omega_{b,0}h^2$

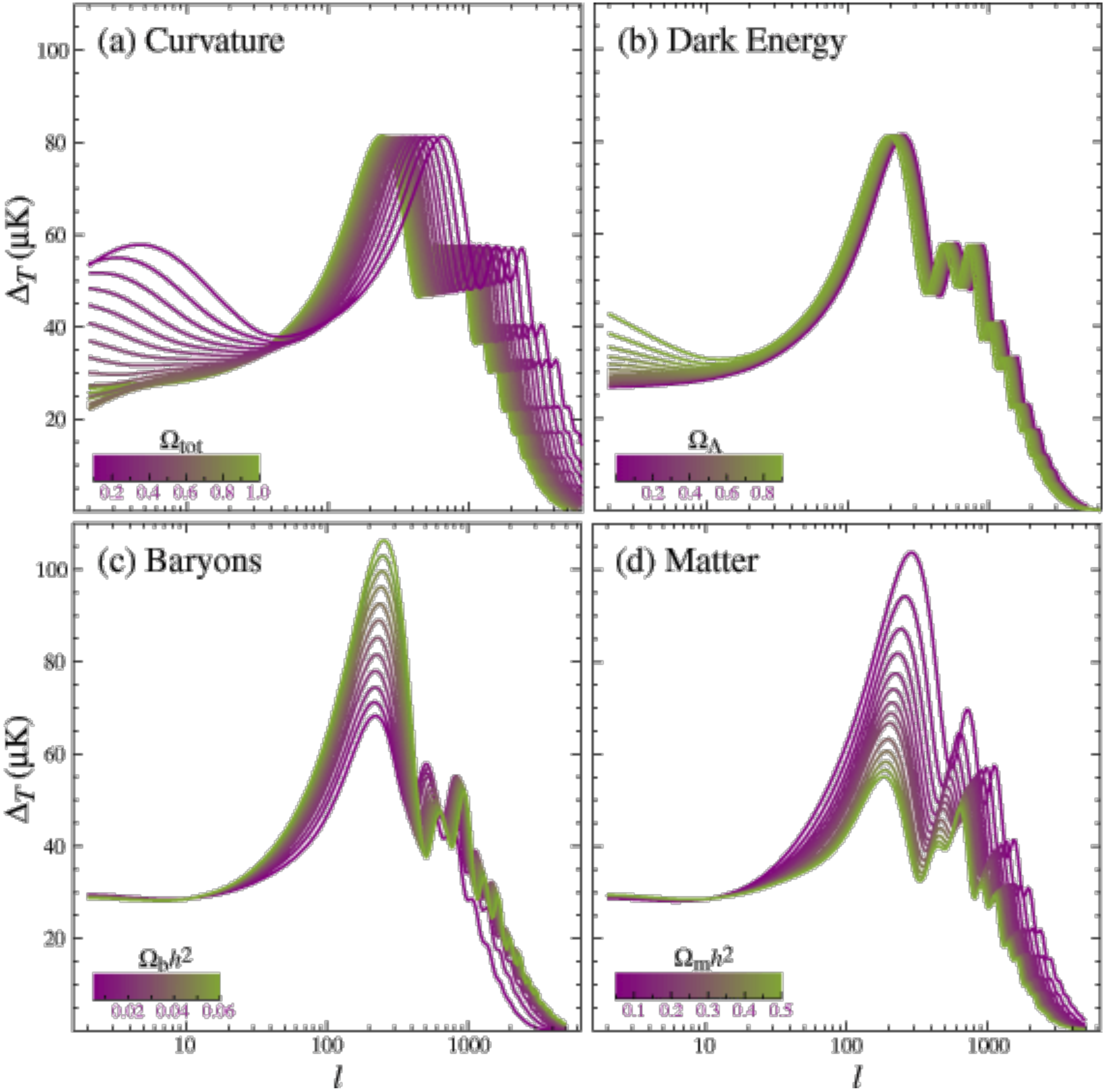
$\Omega_{\mathrm{m},0}h^2$

https://arxiv.org/abs/1502.01589

Planck 2015 Results. Figure 1

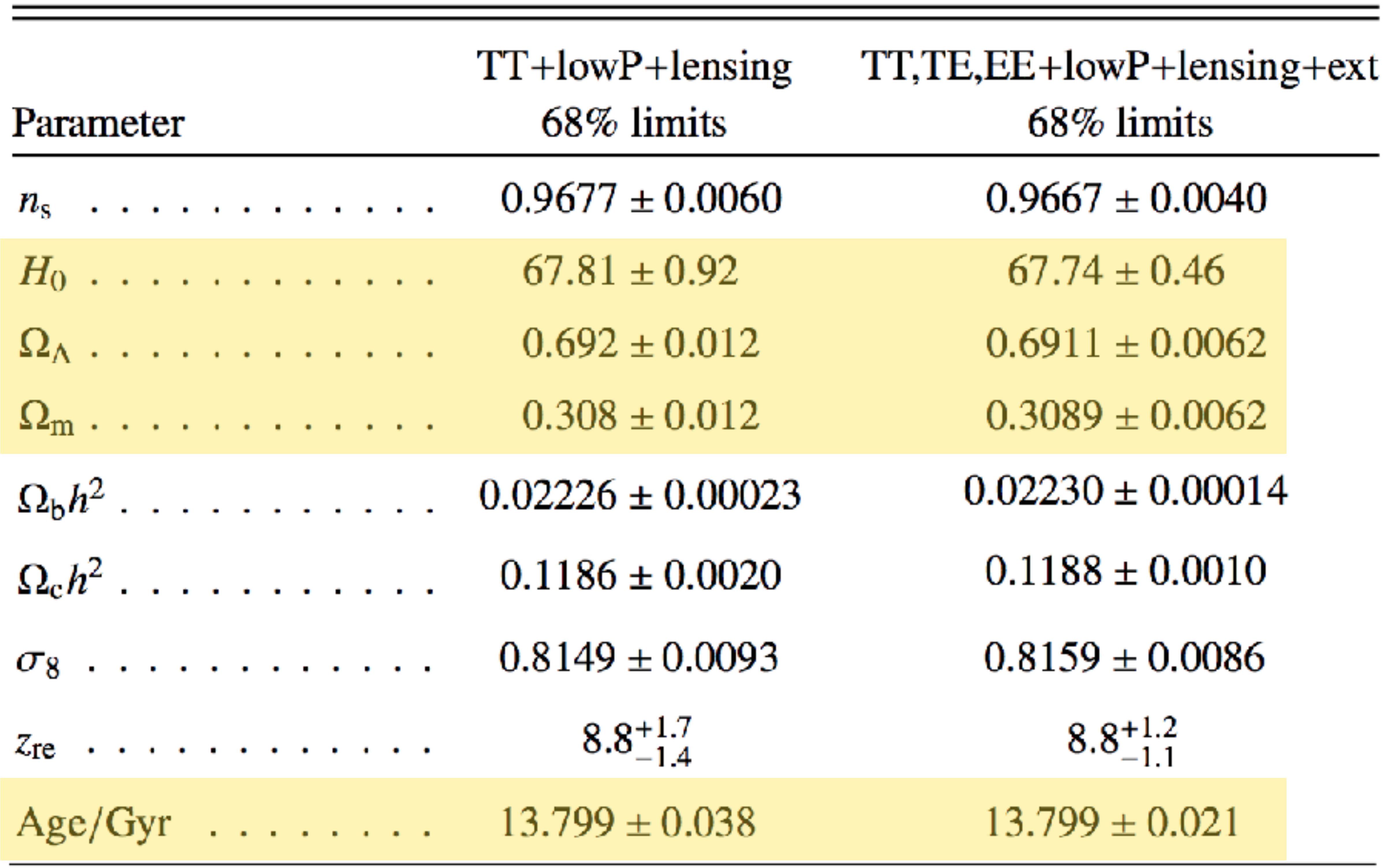
Silk Damping:
photon diffusion of
short wavelength
sound waves

The Power Spectrum: Sensitivities to Cosmological Parameters



Hu & Dodelson (2002)

The Era of Precision Cosmology (1-2% errors)

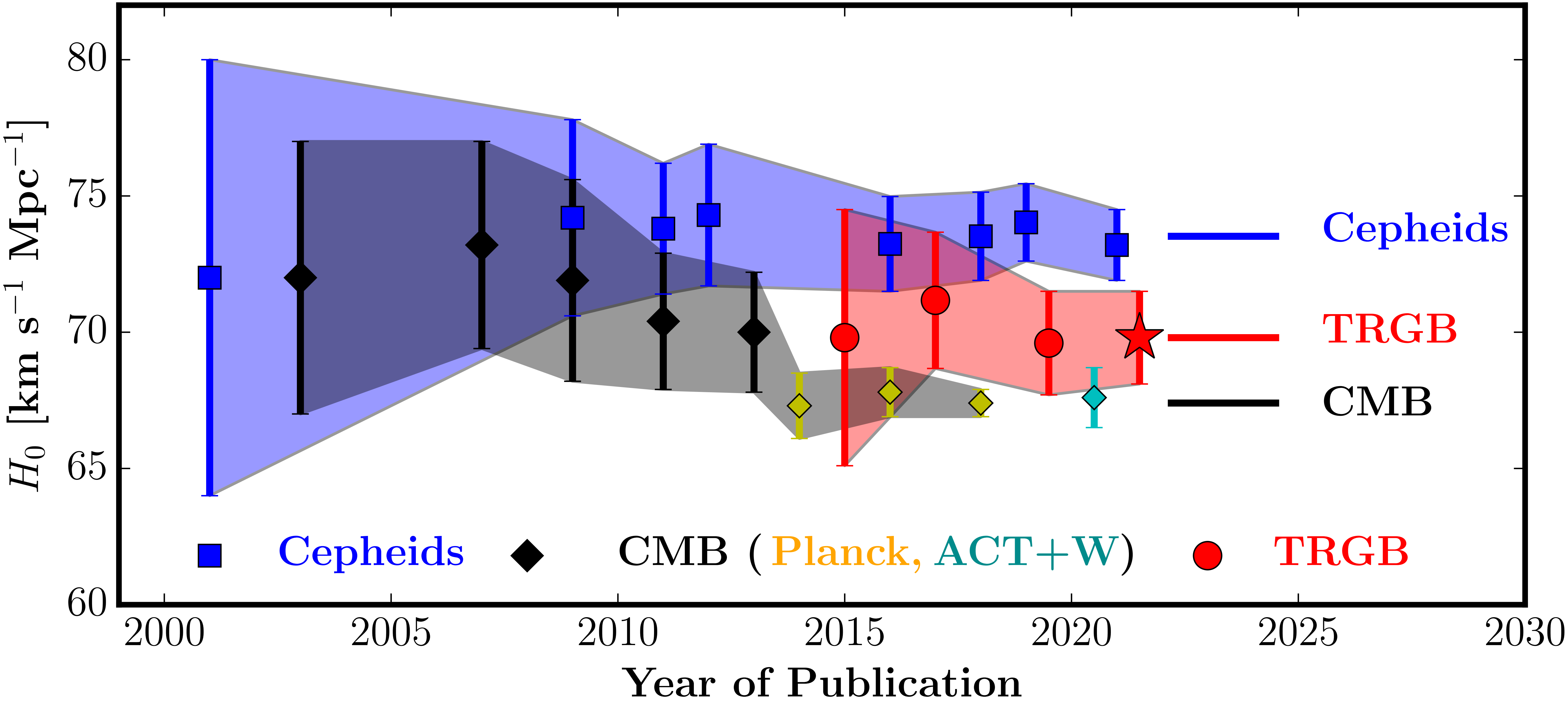


Parameter	TT+lowP+lensing 68% limits	TT,TE,EE+lowP+lensing+ext 68% limits
n_s	0.9677 ± 0.0060	0.9667 ± 0.0040
H_0	67.81 ± 0.92	67.74 ± 0.46
Ω_Λ	0.692 ± 0.012	0.6911 ± 0.0062
Ω_m	0.308 ± 0.012	0.3089 ± 0.0062
$\Omega_b h^2$	0.02226 ± 0.00023	0.02230 ± 0.00014
$\Omega_c h^2$	0.1186 ± 0.0020	0.1188 ± 0.0010
σ_8	0.8149 ± 0.0093	0.8159 ± 0.0086
z_{re}	$8.8^{+1.7}_{-1.4}$	$8.8^{+1.2}_{-1.1}$
Age/Gyr	13.799 ± 0.038	13.799 ± 0.021

https://arxiv.org/abs/1502.01589
Planck 2015 Results. Table 4

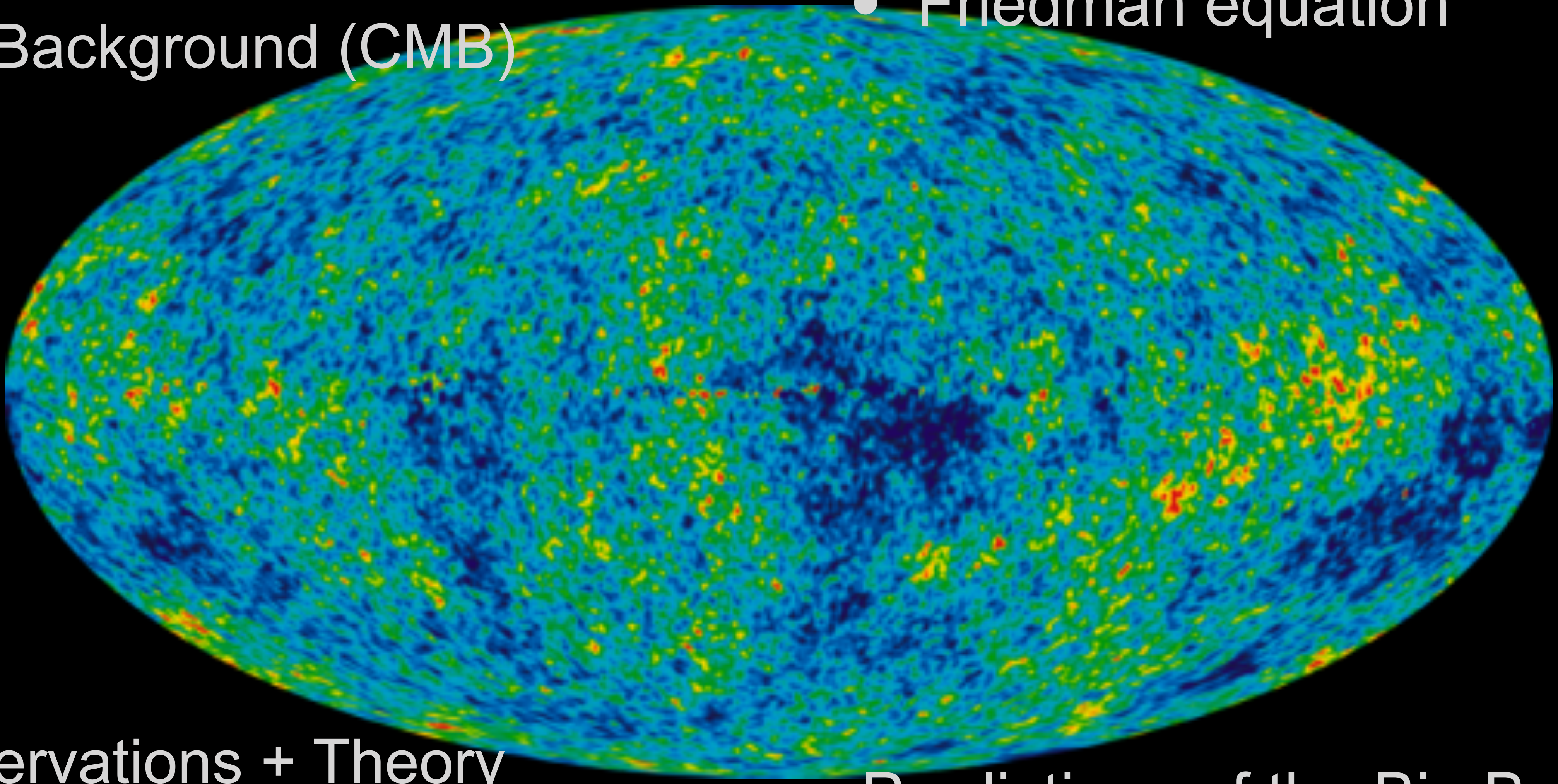
The Tension between local and CMB measurements of H_0

The tension between **68** and **73 km/s/Mpc** in ***H_0***
could be reconciled by small systematic errors of
0.154 in magnitude or **0.0001** in redshift



Freedman et al. (2019)

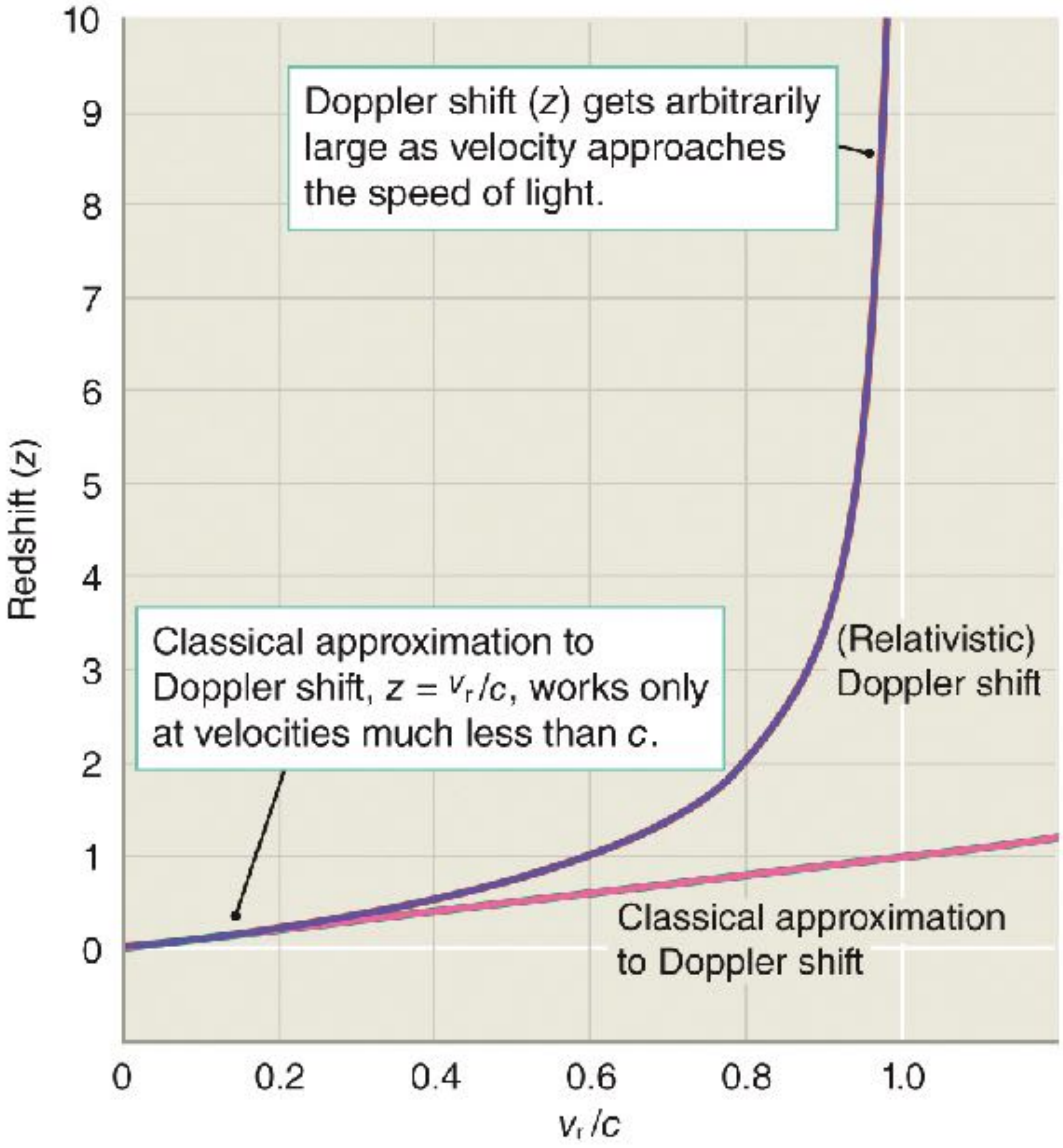
The Expanding Universe



- Observations (facts)
  - Hubble-Lemaître Law & the Hubble "constant"
  - The Cosmic Microwave Background (CMB)
- Interpretations (theories)
  - The cosmological principle
  - Robertson-Walker metric
  - Friedman equation
- Observations + Theory
  - Accelerating Expansion: Evidence of dark energy
  - The cosmic composition
- Predictions of the Big Bang theory: how everything began?

What is redshift? Classical vs. Relativistic Doppler Shift, and Scale Factors

- The **classical Doppler shift formula,**
  $$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_0} = 1 + \frac{v_r}{c}$$
  gives recession velocity:
  $$v_r = cz$$
- The **relativistic Doppler shift formula,**
  $$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_0} = \sqrt{\frac{1 + v_r/c}{1 - v_r/c}}$$
  gives recession velocity:
  $$v_r = c\frac{(1+z)^2 - 1}{(1+z)^2 + 1}$$
- But, **cosmological redshift** should be understood as a ratio of **scale factors**:
  $$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_0} = \frac{1}{R_U(z)}$$



How to Solve the Friedmann Equation? H(z) solution

- The full GR version of the Friedmann (1922) Equation is:

$$H^2\left[1-\left(\frac{\rho_m}{\rho_c}+\frac{\rho_\gamma}{\rho_c}+\frac{\Lambda c^2}{8\pi G\rho_c}\right)\right]R_U^2=-kc^2$$

where $\rho_c=\frac{3H^2}{8\pi G}$ is the **critical density** at redshift z and $H\equiv\dot{R}_U/R_U$ the **Hubble parameter** at time t or redshift z.

- Define Ω's as critical density ratios:
  - $\Omega_m\equiv\rho_m/\rho_c$, **ordinary matter** (baryons and dark matter)
  - $\Omega_\gamma\equiv\rho_\gamma/\rho_c$, **relativistic matter** (light and neutrinos)
  - $\Omega_\Lambda\equiv\Lambda c^2/(8\pi G\rho_c)$, **cosmological constant** (dark energy)
- Replacing those, we have the final Friedmann Equation:

$$H^2\ [1-(\Omega_m+\Omega_\gamma+\Omega_\Lambda)]\ R_U^2=-kc^2$$

- Apply the boundary condition today, the **Hubble parameter** is:

$$H^2=\frac{H_0^2}{R_U^2}[(1-\Omega_0)+\Omega_{m,0}/R_U+\Omega_{\gamma,0}/R_U^2+\Omega_{\Lambda,0}R_U^2]$$

How to Solve the Friedmann Equation? t(z) or R_u(t) solution

- Write down the Friedmann Equation with the boundary condition and replace Hubble parameter with scale factor, $H(t) \equiv \dot{R}_U/R_U$, we have

$$\left(\frac{1}{R_U}\frac{dR_U}{dt}\right)^2 = \frac{H_0^2}{R_U^2}[(1-\Omega_0) + \Omega_{m,0}/R_U + \Omega_{\gamma,0}/R_U^2 + \Omega_{\Lambda,0}R_U^2]$$

- For simplicity, assume a **flat universe:** *k = 0 and* $\Omega_0 = 1$.
- Separate time and scale factor into two sides of the equation:

$$dt = \frac{1}{H_0}\frac{R_U dR_U}{\sqrt{\Omega_{m,0}R_U + \Omega_{\gamma,0} + \Omega_{\Lambda,0}R_U^4}}$$

- Integrating it from R_U=0 (i.e., t=0) to R_U = 1/(1+z) [i.e., t(z)], we can solve for the t(z) relation for *any given values of the density parameters*.
- For example, for **a matter-only flat universe (Einstein-de Sitter universe),** we have solved for both H(z) and t(z):

$$\Omega_0 = \Omega_{m,0} = 1, \Omega_{\gamma,0} = \Omega_{\Lambda,0} = 0$$

$$\Rightarrow H = H_0/R_U^{3/2} = H_0(1+z)^{3/2}$$

$$\Rightarrow t(z) = \frac{2}{3}t_H(1+z)^{-3/2}$$

Temperature-Redshift Relation of the Cosmic Background Radiation

- Wien's displacement law: $\lambda_{\text{peak}} = (2.9\ \text{mm K})/T$
- Cosmological redshift: $\lambda_{\text{obs}} = \lambda_{\text{emit}}(1+z)$
- Combining the above two, we have the relation the blackbody temperature when emitted and the observed temperature:

  $$T_{\text{obs}} = T_{\text{emit}}/(1+z)$$

  in other words, the observed temperature today is much lower than the original temperature due to cosmic expansion.

- The above equation cannot constrain the temperature when the cosmic radiation background first emerged. In fact, infinite number of (T_{emit}, z) combinations could give us the same 3 K observed temperature, for examples:
  - T_{emit} = 10 K, z = 2.3
  - T_{emit} = 100 K, z = 32
  - T_{emit} = 1000 K, z = 330
- We call that **the two parameters (T_{emit}, z)** are **degenerate.**