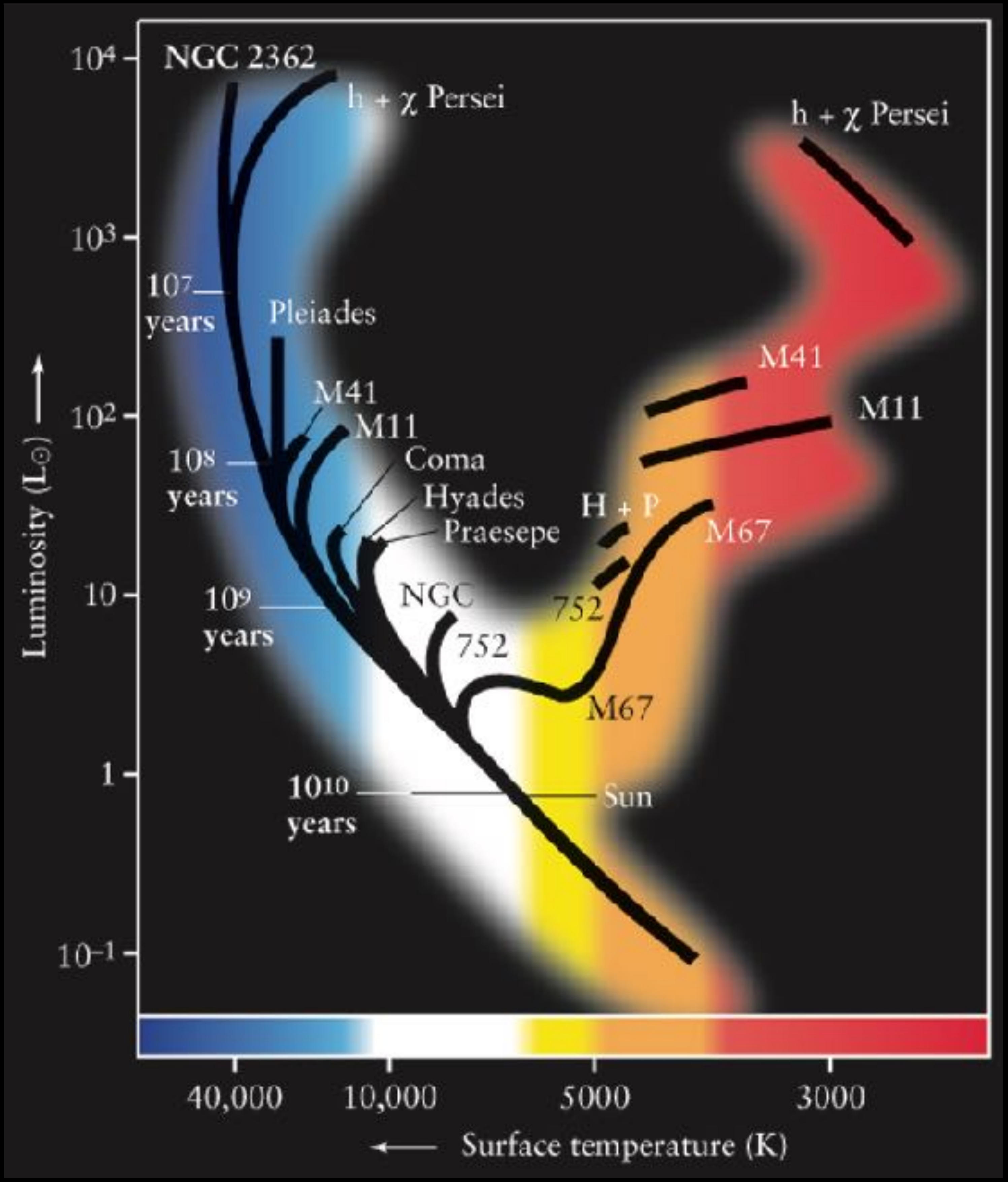
Chap 3: The Evolution of Low-Mass Stars

Chap 3: Key Concepts

- Observations
  - Nothing last forever, even stars
  - H-R diagram of star clusters
- Numerical Models
  - Equations of stellar structure and evolution
  - Stellar evolutionary tracks
- Fine-Tune Models
  - Isochrones (equal-age lines)
  - Fitting cluster H-R diagrams
  - Cluster age estimates
- Model Inferences
  - Main stages and rough lifetimes
  - Changes in the interiors of the stars: e- degenerate core + fusion shells
- Mass-Transfer Binaries
  - Roche Lobe, Lagrange Points
  - Novae, Type Ia SNe, Blue Stragglers



The Virial Theorem:

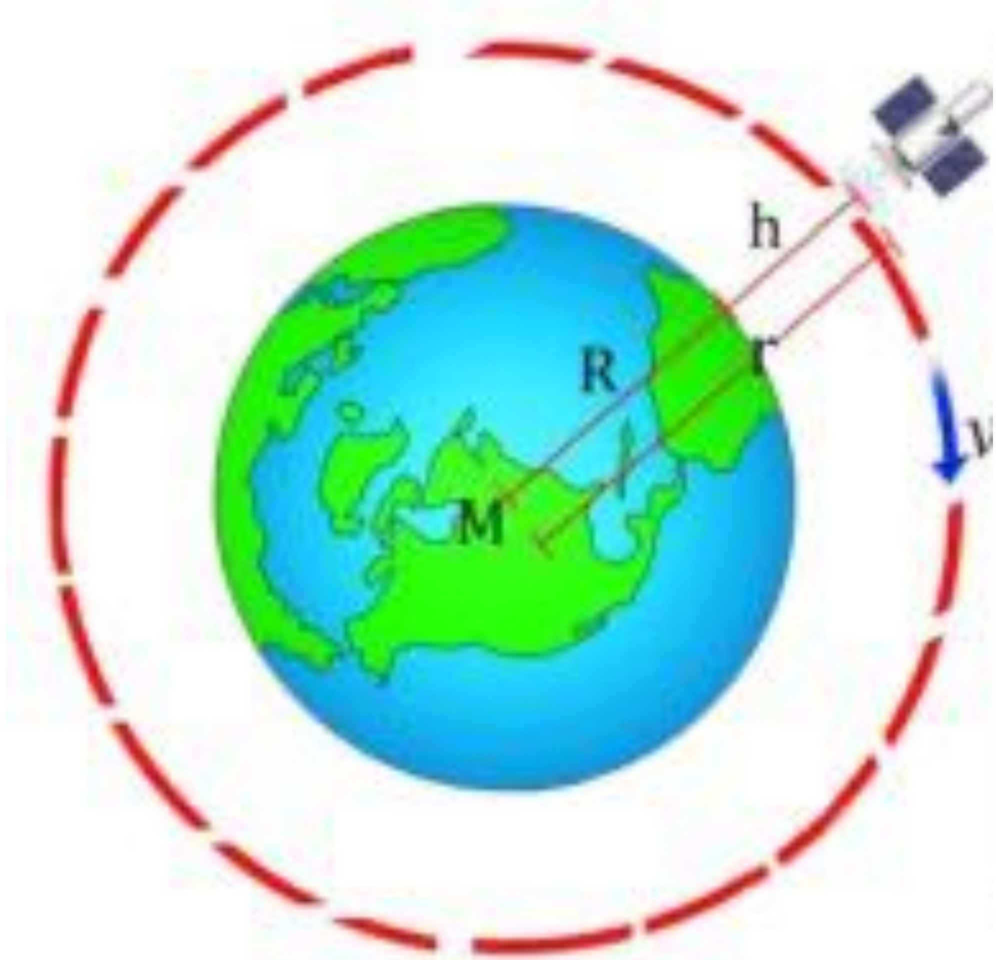
$$2K + U = 0$$

Kinetic and Potential Energies of an Object in Circular Orbit

circular orbital velocity: $v_{\text{circ}} = \sqrt{GM/r}$

Kinetic energy: $K = \dfrac{1}{2}\dfrac{GMm}{r}$

Gravitational potential energy: $U = -\dfrac{GMm}{r}$

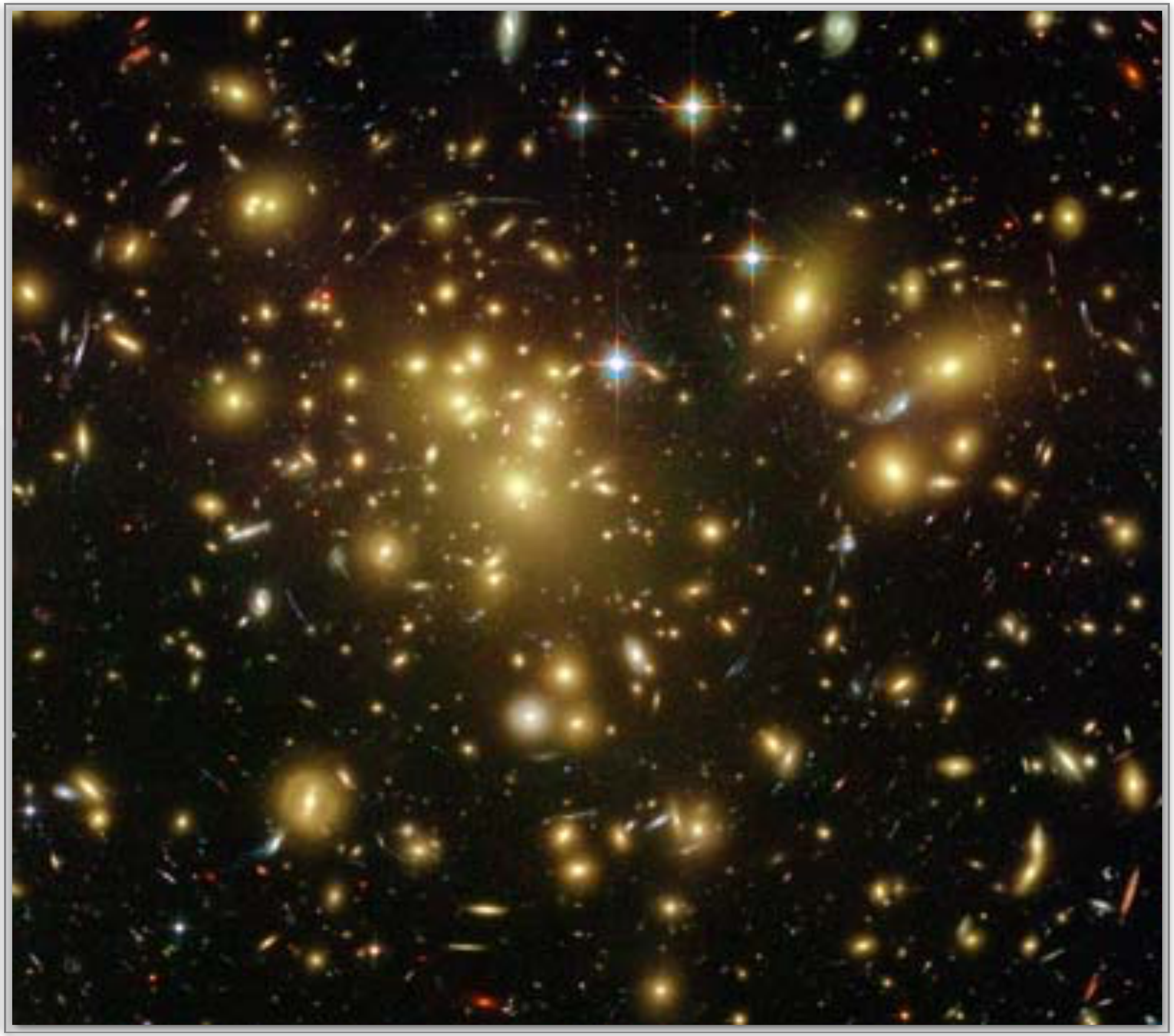


All self-gravitating systems (N-body) obey the Virial Theorem

$$2\bar{K} + \bar{U} = 0 \Rightarrow \bar{v}^2 = \frac{G\tilde{M}}{\bar{R}} \;\&\; G\tilde{M} = \bar{v}^2\bar{R}$$

This applies to all self-gravitating systems:

planetary systems, molecular clouds, stars, star clusters, galaxies, galaxy clusters



As N-body systems, stable gas clouds should follow the Virial Theorem

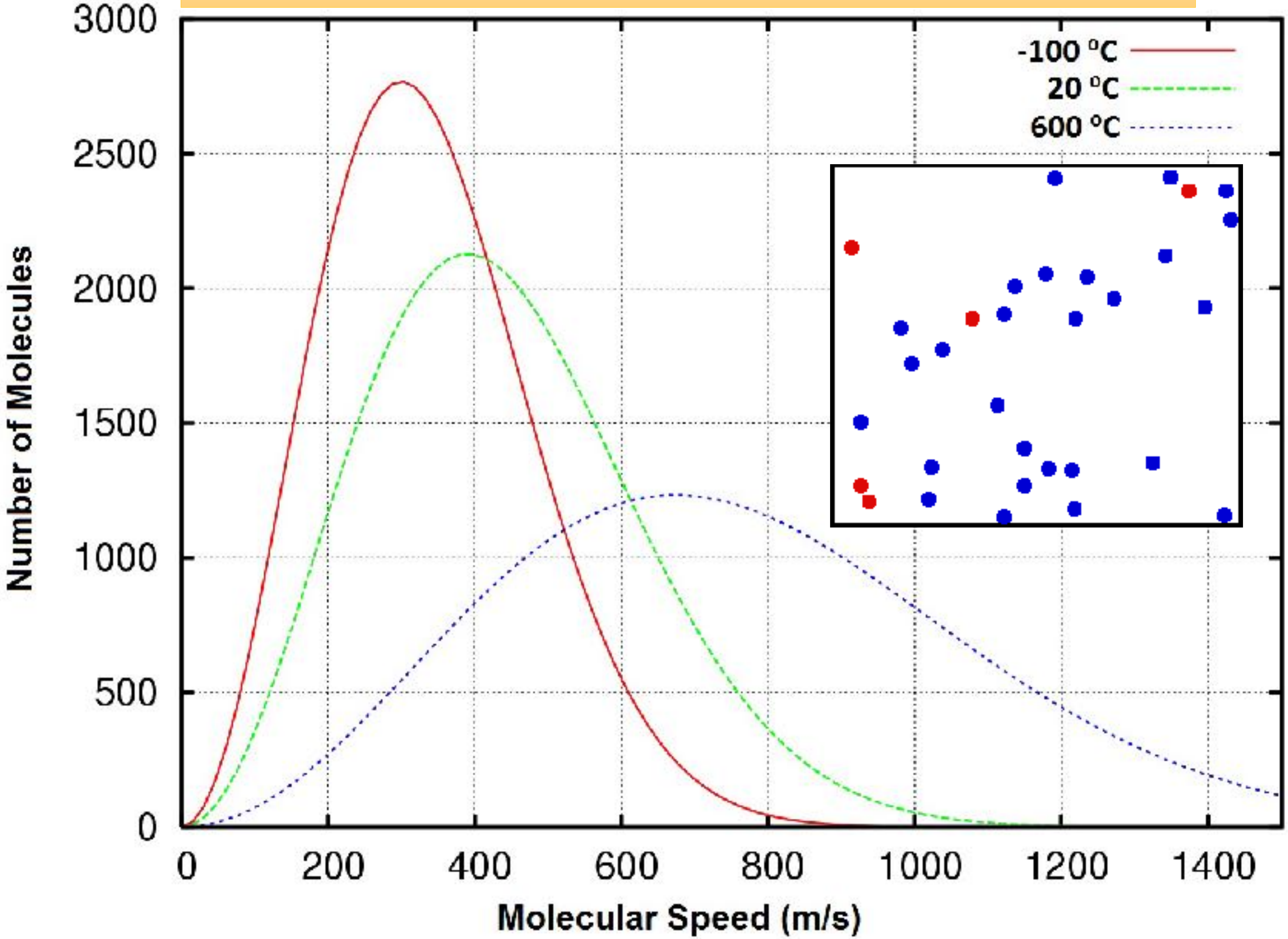
Virial theorem relates **the time-averaged kinetic energy** of a **stable system** of discrete particles, bound by potential forces, with that of **the time-averaged potential energy** of the system. For a **gravitationally bounded stable** system:

2 x Kinetic Energy + Gravitational Potential Energy = 0

- First, we need to know how to express the **total kinetic energy** of the gas particles.
- Then, we need to know how to express the **potential energy** of a spherical gas cloud.

Random Motions of Gas Particles follow the Boltzmann Distribution

Mean Kinetic Energy per Particle = 3kT/2 for monatomic ideal gas



Internal Kinetic Energy of a Gas Cloud

Kinetic Energy Density is: $u = n(\frac{3}{2}kT)$

Total Kinetic Energy is:

$$U = N_{\text{particle}} \cdot \frac{3}{2}kT = \frac{3}{2}\frac{M_{\text{gas}}}{\mu m_H}kT$$

where μm_H gives the **average mass per particle,** and μ is the **mean molecular weight**

Side note: recall the ideal gas law:

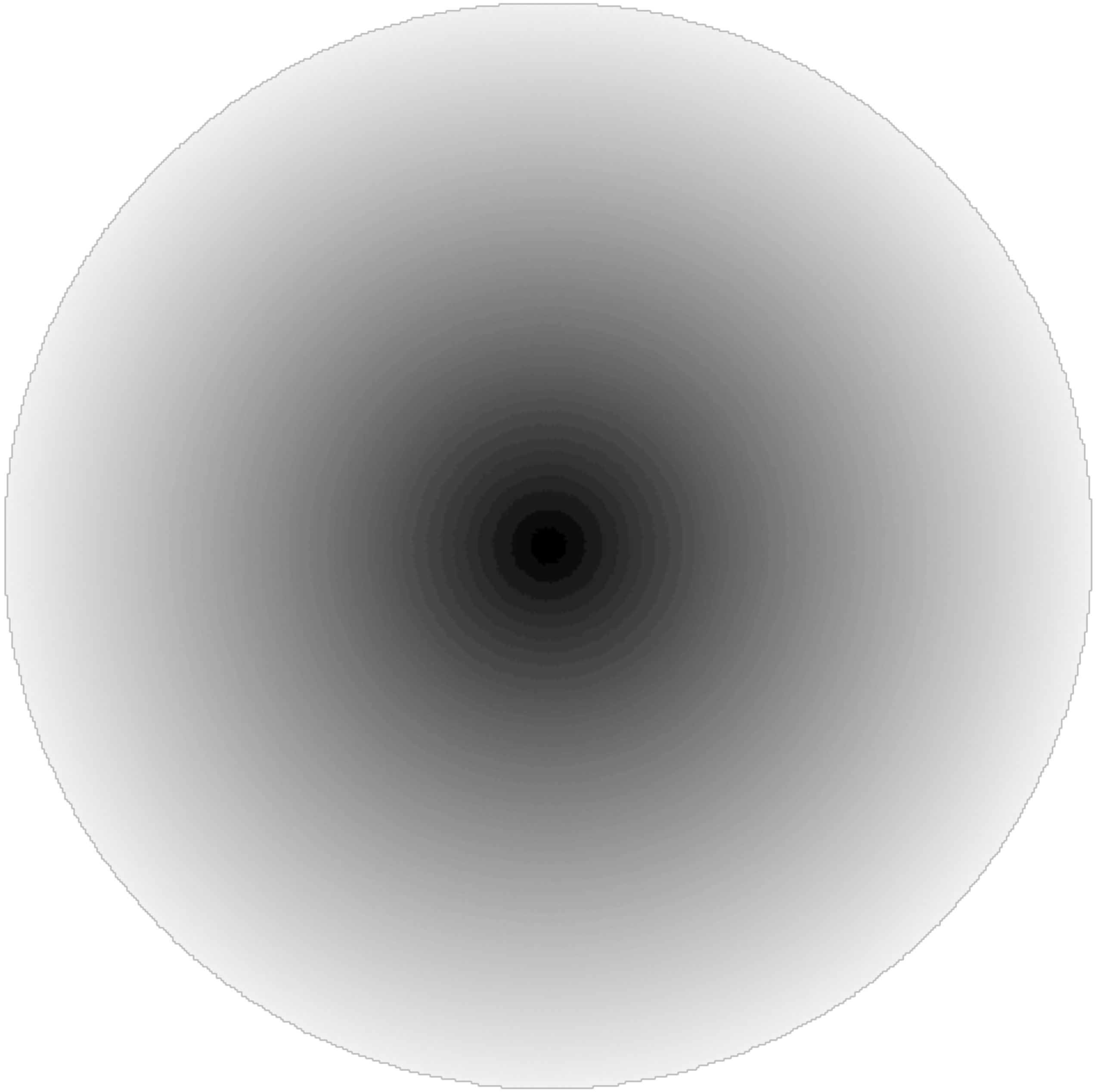
$$P = nkT = \frac{2}{3}u$$

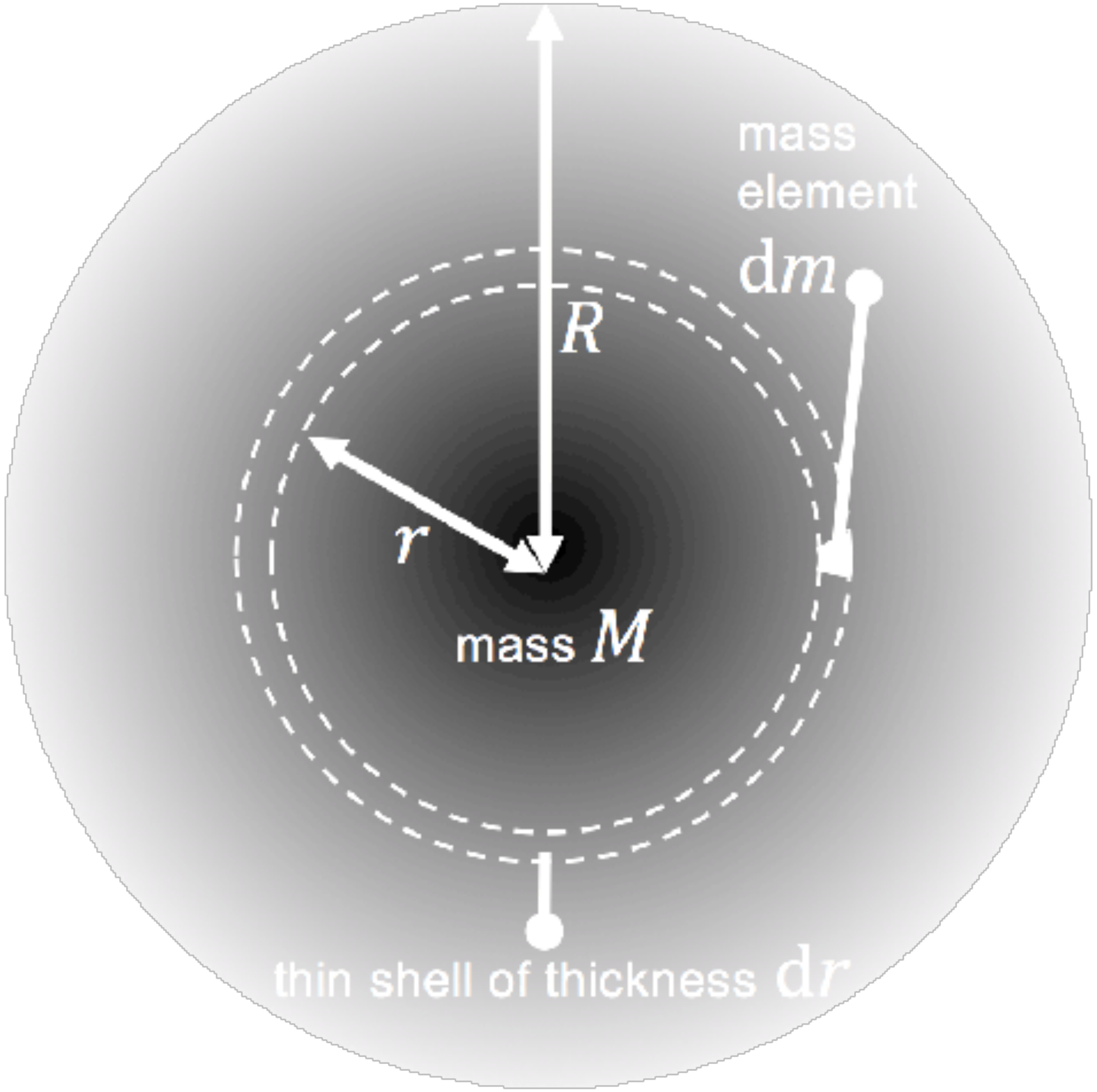
so, **pressure** is simply 2/3 of the **kinetic energy density**

Potential energy of a spherical shell at radius r

Start by writing down the gravitational potential energy of a shell with mass of *dm* at radius of *r*

$$dU_r = -\frac{GM_r dm}{r} = -G(\rho \cdot \frac{4}{3}\pi r^3)(\rho \cdot 4\pi r^2 dr)/r$$

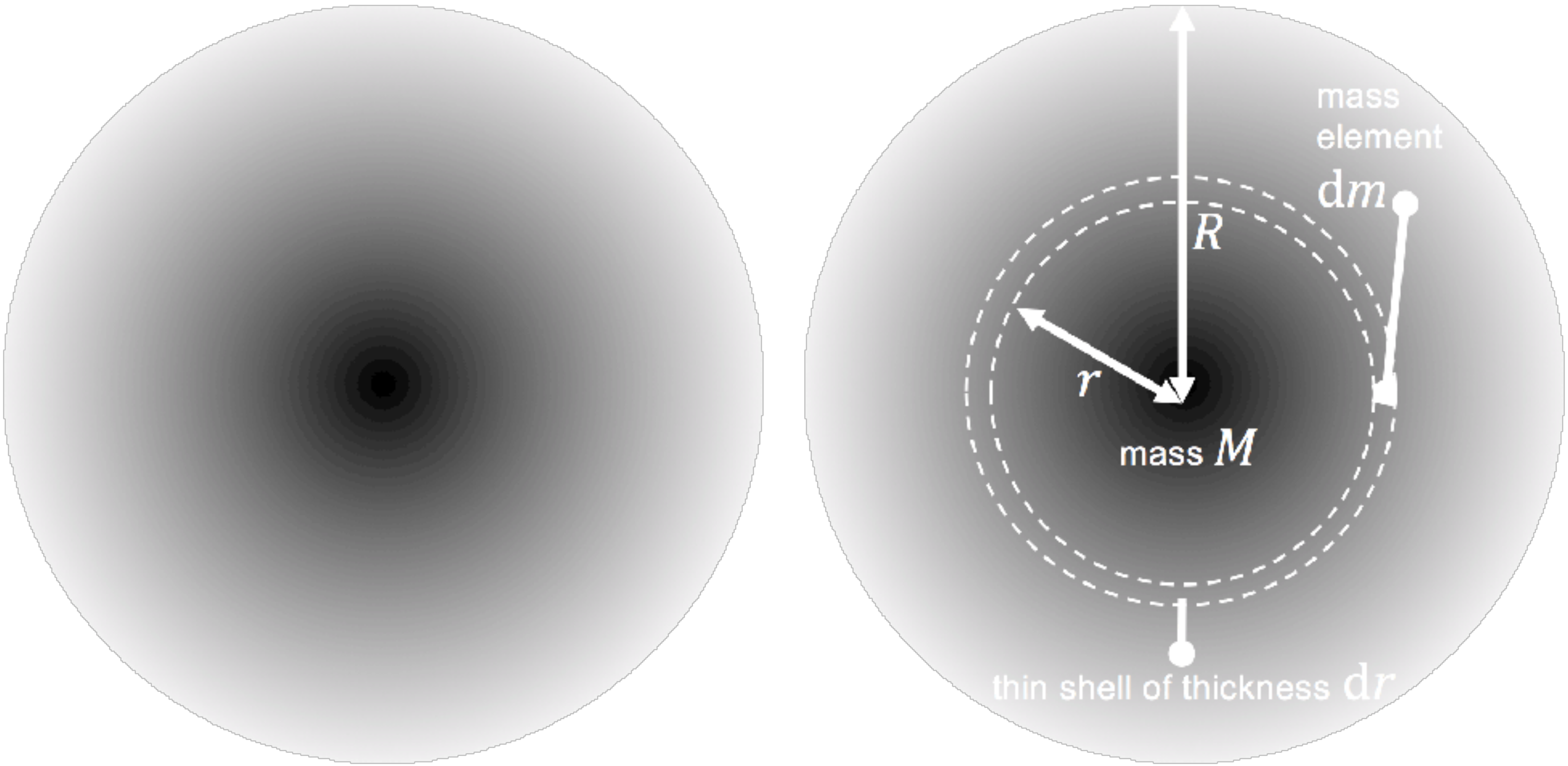




integrate over the sphere for uniform density

Next, we sum up the potential energy of all shells up to radius of R, this operation is an integral:

$$U = \int_0^R dU_r = -\frac{16\pi^2}{3} G\rho^2 \int_0^R r^4 dr = -\frac{3}{5}\frac{GM^2}{R}$$



Virial Theorem applied to a spherical cloud

Now we can put both equations together and then write down the **virial theorem** for a uniform spherical gas cloud:

$$K = \frac{3}{2}\frac{M}{\mu m_H}kT \qquad U = -\frac{3}{5}\frac{GM^2}{R}$$

Virial theorem applies IF the cloud is stable:

$$2K = -U \Rightarrow \frac{3kT}{\mu m_H} = \frac{3GM}{5R}$$

Although 2K and -U both increase as R increases, they don't increase at the same rate ($K \sim R^3$, $U \sim R^5$). So beyond some point, the virial theorem is violated as $2K < -U$.

Application I: Heavier stars have higher core temperature

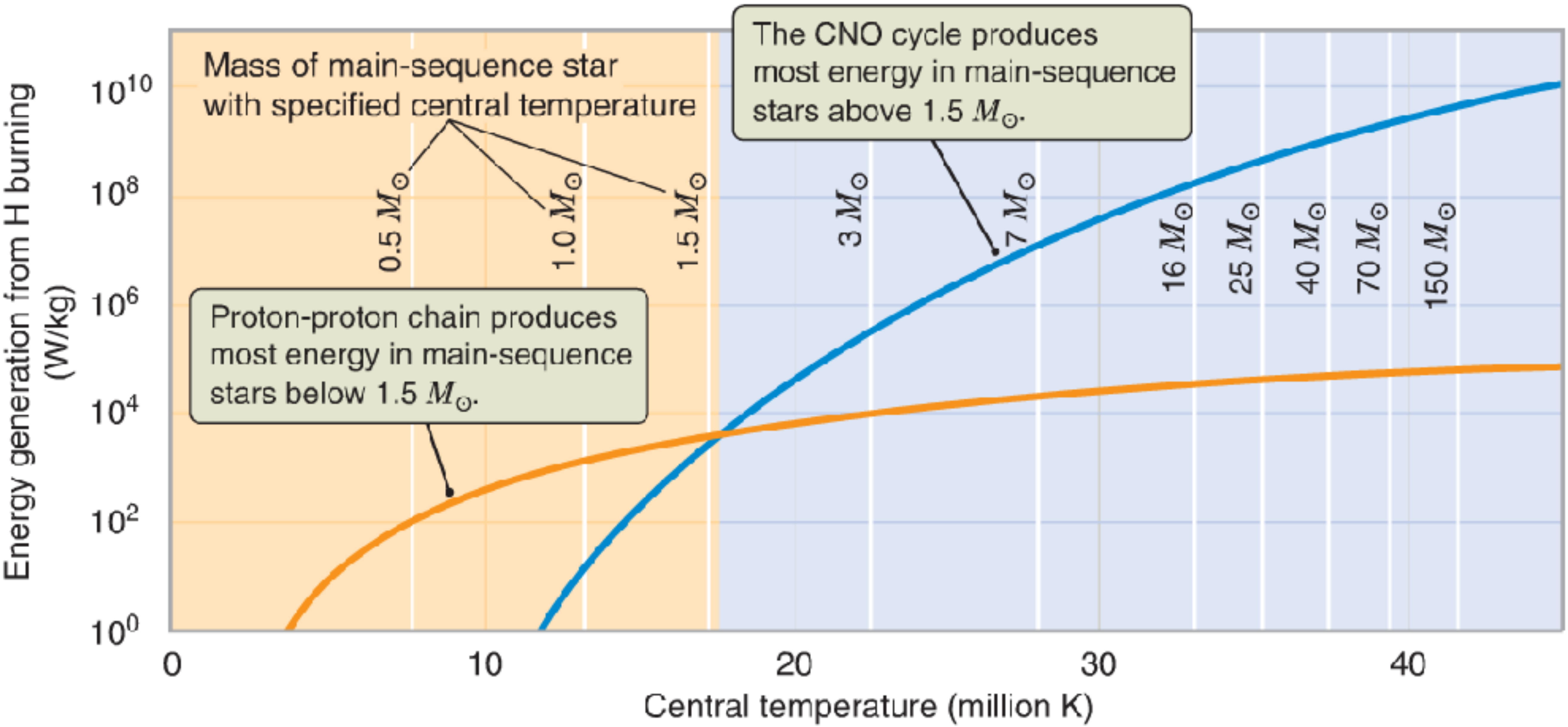
- Core temperature can be estimated using the **virial theorem**:

$$kT_c \approx GM\mu m_H/R$$

- Main sequence stars show a **mass-radius relation** of:

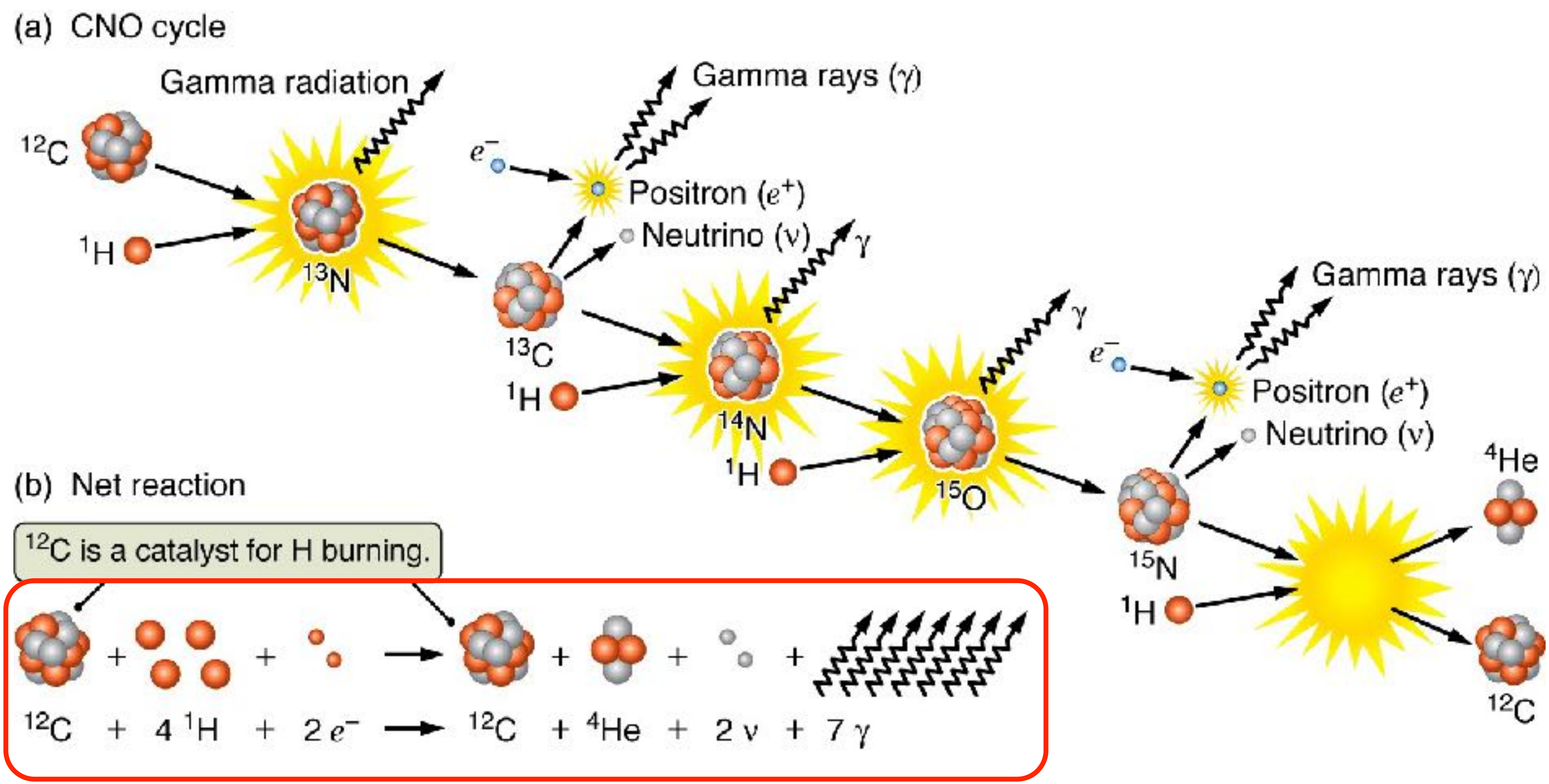
$$R \propto M^{0.7}$$

- Therefore, $T_c \propto M^{0.3}$



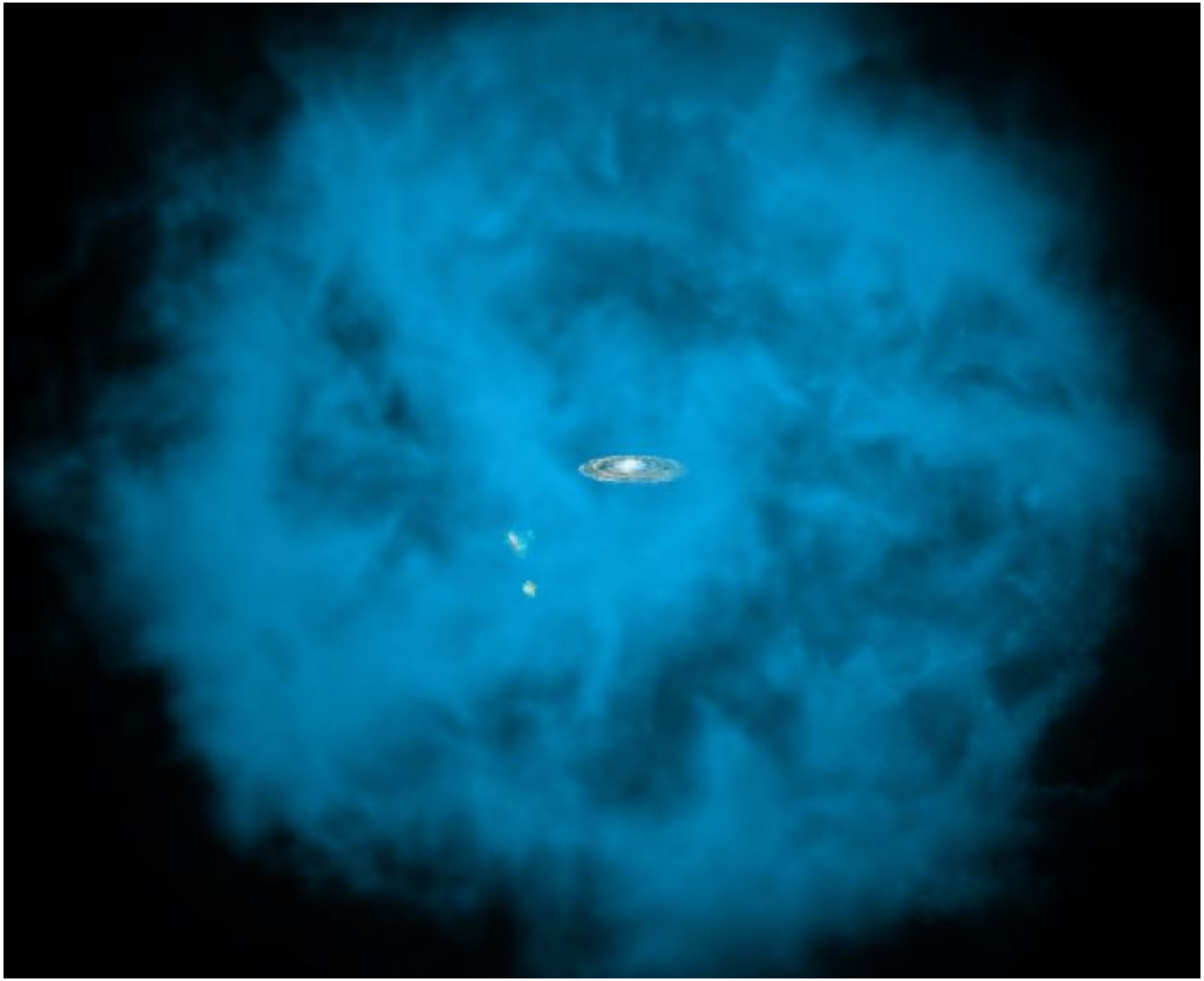
Net reaction of the CNO cycle

- In high-mass stars and the midlife Sun, hydrogen burning proceeds in the CNO cycle instead of the pp chain, due to higher core temperatures.
- The net result is the same as the pp chain: 4 H -> 1 He



Application II: Why the Sun appears to have a sharp edge?

- Wouldn't a spherical gas cloud look like fuzzy on the edges? Like pictured below?

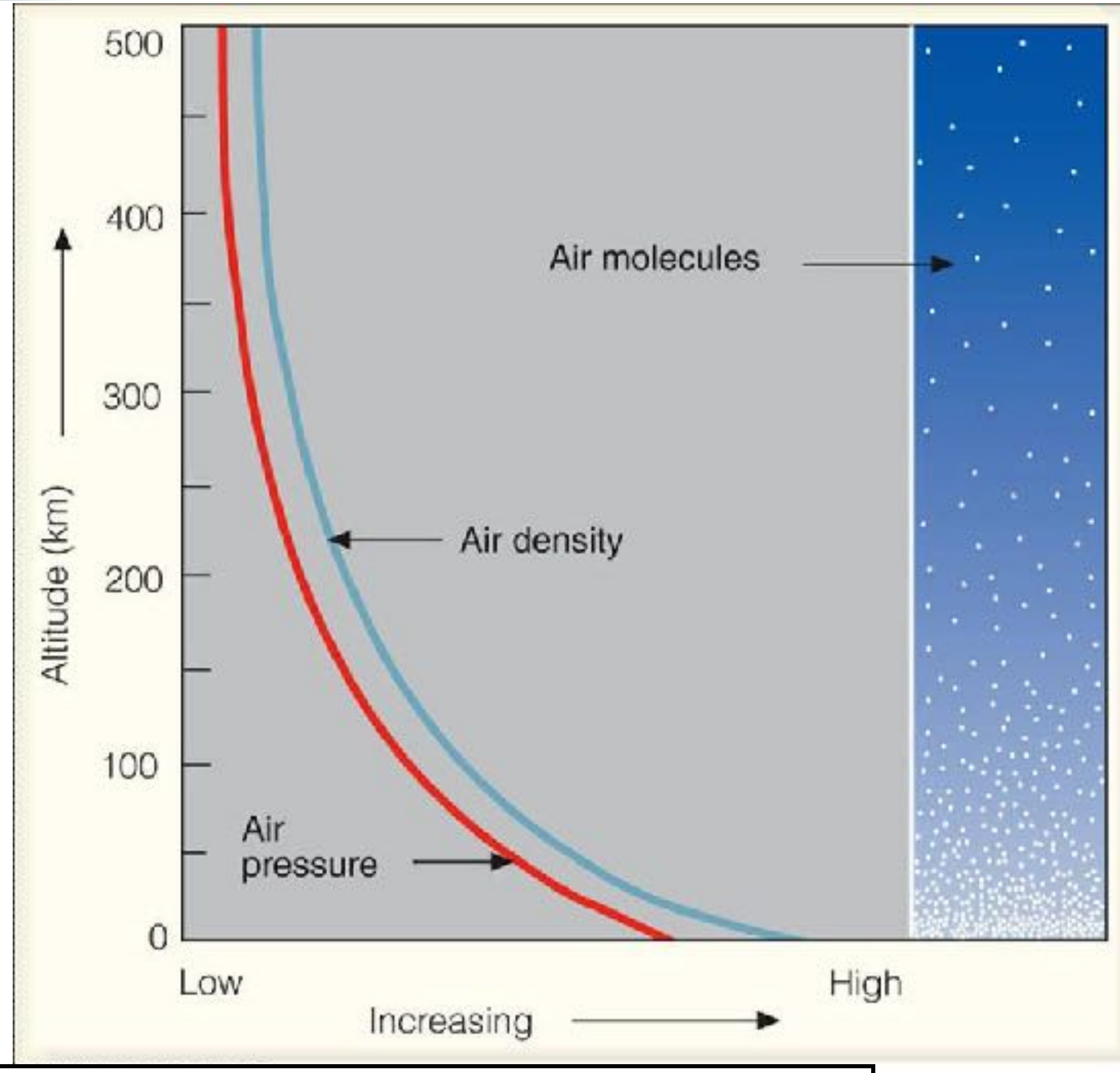


Recall the scale height of an isothermal atmosphere

Solved density profile:

$$n(h) = n_0 \exp\left(-\frac{h}{h_S}\right)$$

where $h_S = \dfrac{kT}{\mu m_H g}$ is the **scale height of the atmosphere**



The value of the expression $\dfrac{k_B \cdot 5800\ \mathrm{K}}{m_H \cdot \dfrac{G \cdot M_\odot}{R_\odot^2}}$ is approximately

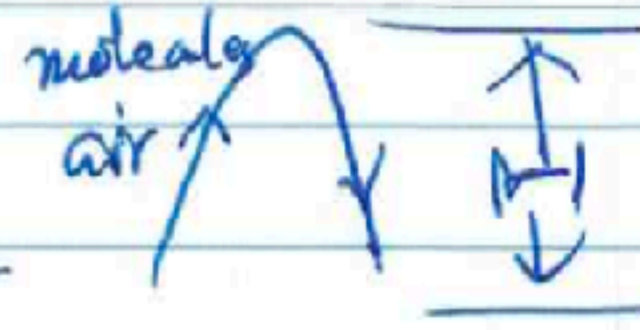
1.745×10^5 m, which represents the **pressure scale height** of the solar

Scale height of the solar photosphere << Solar radius

Scale height for solar atmosphere.

$$\bar{m} g H = k T_s$$

$\bar{m} g H$ ← potential energy; kT_s ← kinetic energy



H is small → the sun looks like a sharp edge

$$\frac{H}{R} = \frac{kT_s}{\bar{m} g R}$$

Virial theorem: $kT_c \sim \frac{GM}{R}\bar{m} \sim \bar{m} g \cdot R$

$$\Rightarrow \frac{H}{R} = \frac{kT_s}{\bar{m} g R} = \frac{kT_s}{kT_c} = \frac{T_s}{T_c} \sim 10^{-4}$$

For Earth. $\frac{H}{R} \sim 10^{-3}$. The Sun appears even sharper.

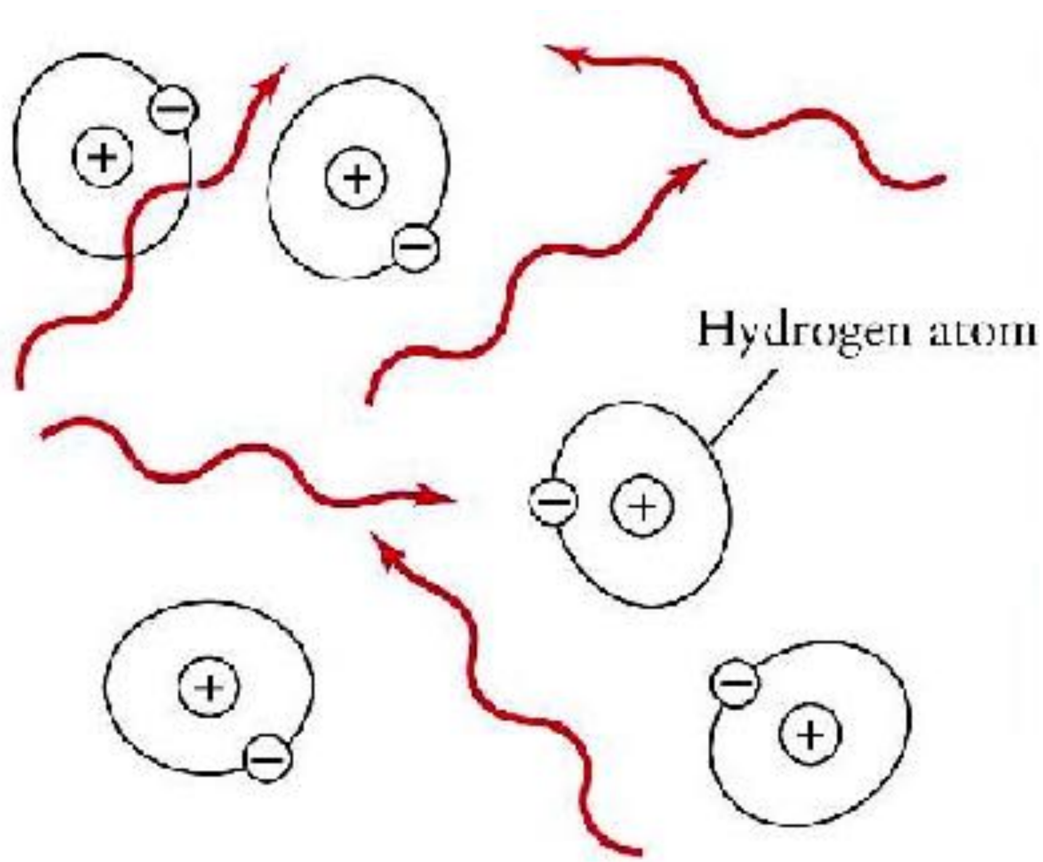
How to calculate μ?

the Mean Molecular Weight

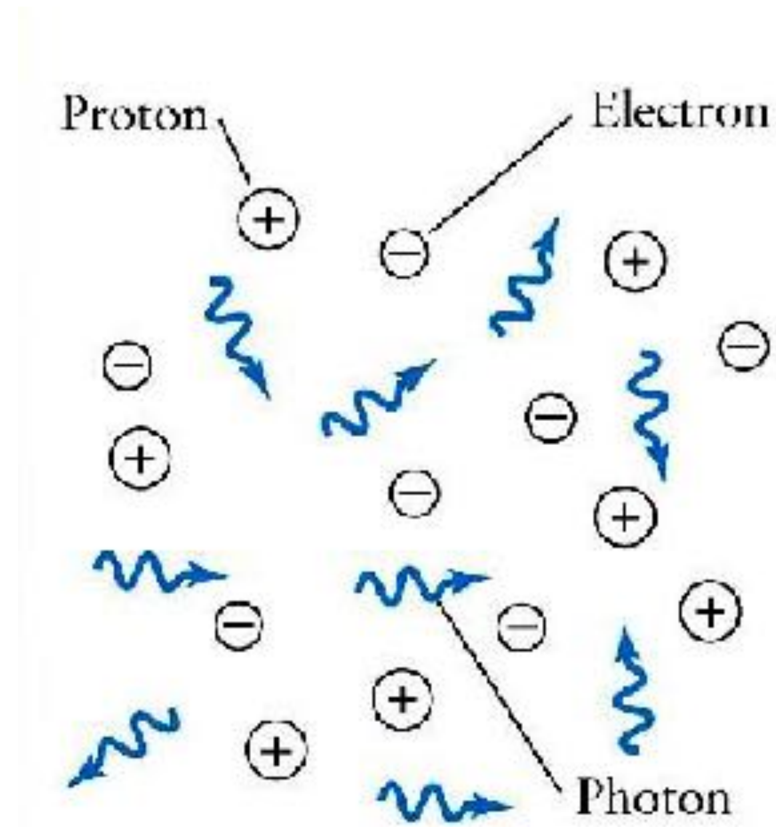
The Mean Molecular Weight: Neutral vs. Ionized Hydrogen

The mean molecular weight is a dimensionless quantity, and it is defined as the ratio between the mean mass per particle and the mass of a single hydrogen atom (1.67e-27 kg):

$$\mu \equiv \frac{\bar{m}}{m_H} = \frac{M_{\rm gas}/N_{\rm particle}}{m_H}$$



neutral atomic hydrogen



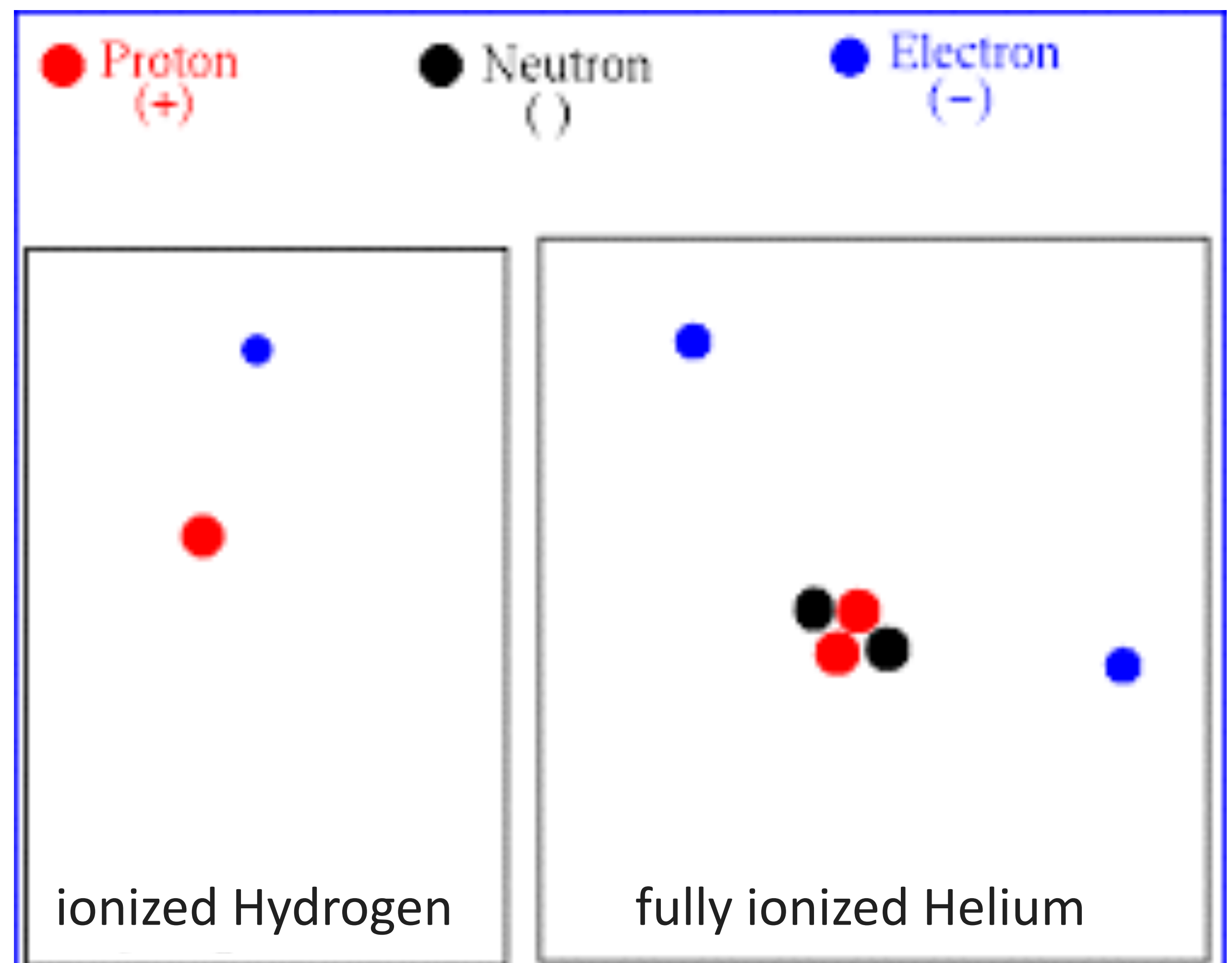
ionized hydrogen

The Mean Molecular Weight: Neutral vs. Ionized Helium

The mean molecular weight is a dimensionless quantity, and it is defined as the ratio between the mean mass per particle and the mass of a single hydrogen atom (1.67e-27 kg):

$$\mu \equiv \frac{\bar{m}}{m_H} = \frac{M_{\text{gas}}/N_{\text{particle}}}{m_H}$$

- *What about **neutral** helium, **partially ionized** helium, and **fully ionized** helium gases?*
- *μ = 4, 4/2, 4/3, respectively*
- *simplest way to calculate this is to imagine 100 Helium atoms and count the mass in proton mass, and count the number of **all particles***

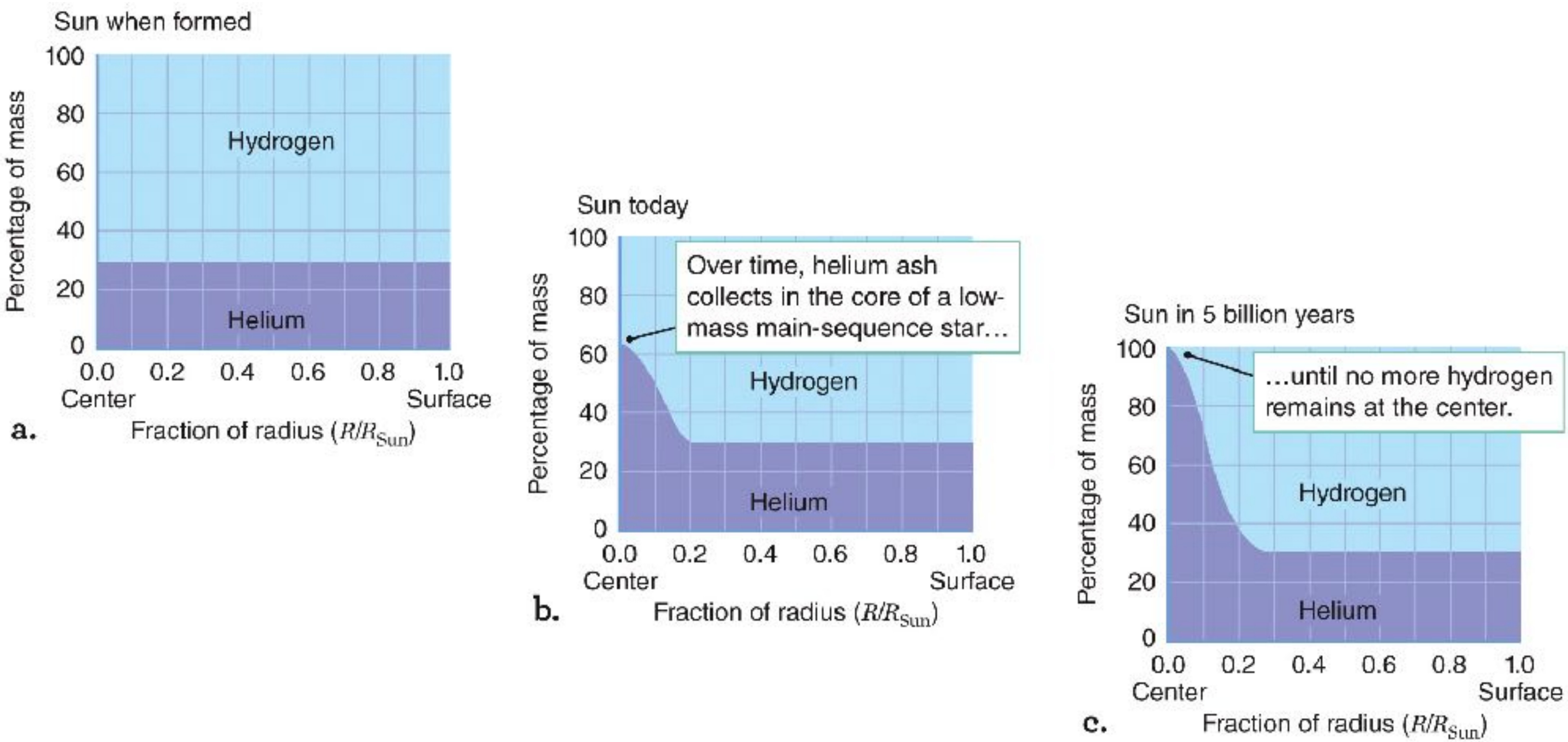


Why do we think stars must evolve?

I. from our understanding of the Sun

Changes on the Main Sequence due to Fuel Exhaustion

- The chemical composition inside a star changes over time as hydrogen is fused into helium.
- The Sun started with 70 percent hydrogen by mass, but now contains only 35 percent hydrogen in the core.
- What will happen when the hydrogen is exhausted in the core?



Main-Sequence Lifetime of a Star Depends on Its Initial Mass

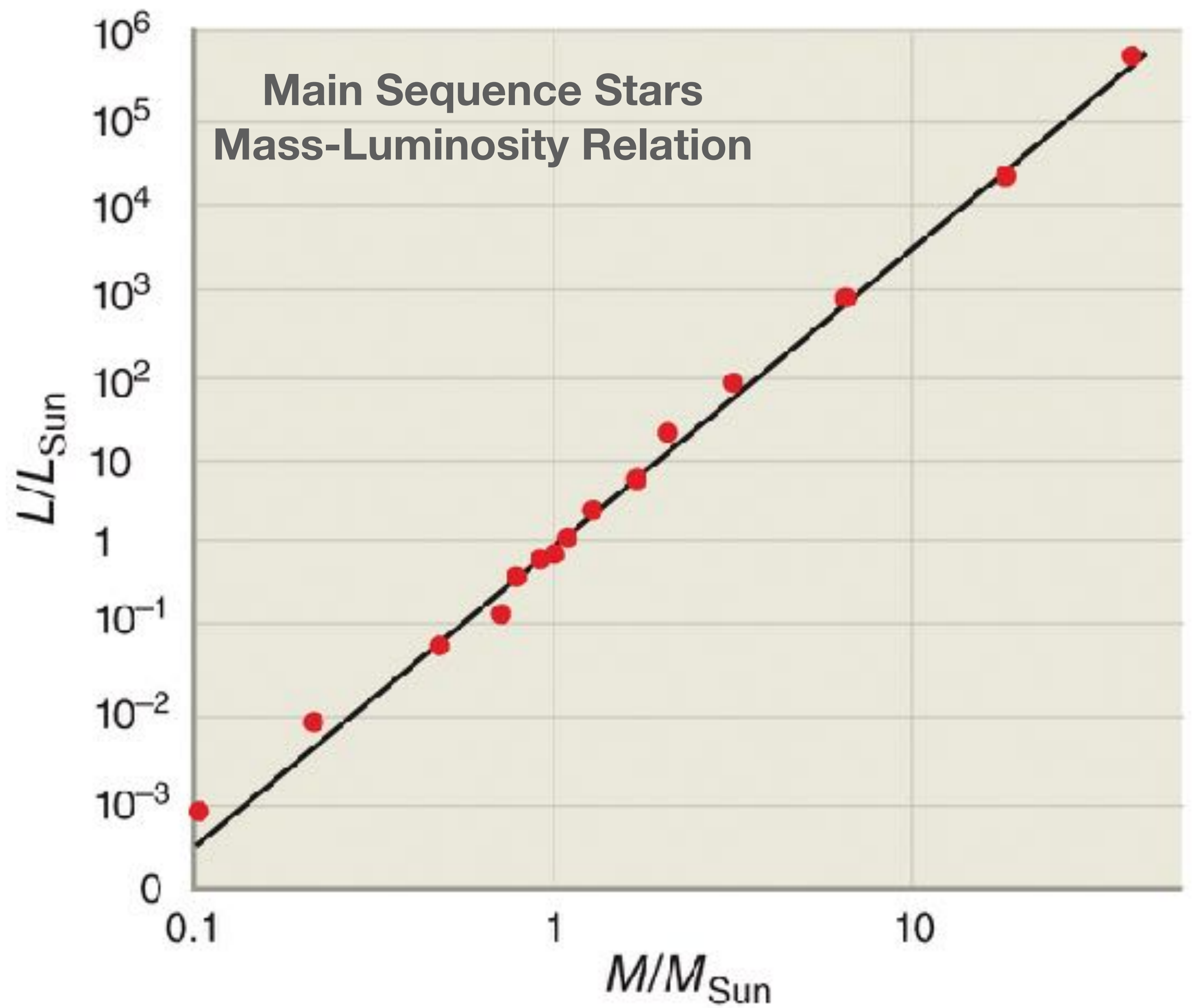
- The main-sequence lifetime of a star is the amount of time that it spends fusing hydrogen as its primary source of energy.

$$\text{Lifetime of star} = \frac{\text{Amount of fuel (} \propto \text{ mass of star)}}{\text{Rate fuel is used (} \propto \text{ luminosity of star)}}$$

- Stars with high masses have shorter lifetimes.
- Higher-mass stars have more fuel, but they use it more quickly:

$$L \propto M^{3.5}$$

=> MS lifetime ~ $M^{-2.5}$



Stars Must Change and The Rate of Change Depends on Initial Mass

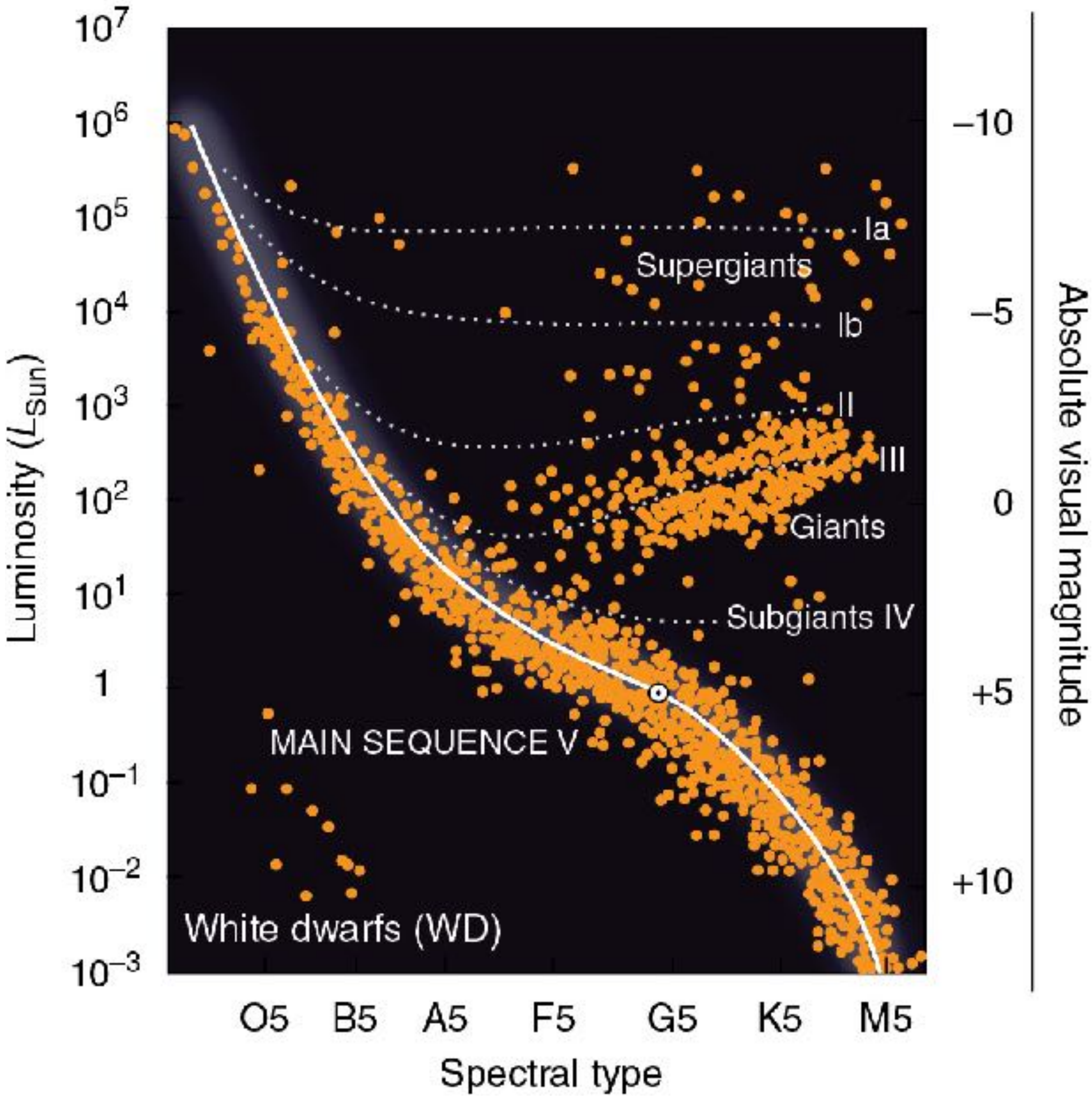
- A star's life depends on mass and composition because the rates and types of fusion depend on the star's mass.
- Stars of different masses evolve differently. There are three categories of stars:
  - **low-mass stars** (Mass < 3 M_{Sun})
  - **intermediate-mass stars** (Mass between 3 M_{Sun} and 8 M_{Sun})
  - **high-mass stars** (Mass > 8 M_{Sun})
- Virial theorem: core temperature increases with stellar mass
  - $2K = -U \Rightarrow \frac{3MkT_c}{\mu m_H} = \frac{3GM^2}{5R} \Rightarrow T_c \propto M/R$
  - For Main-Sequence Stars, $R \propto M^{0.7}$, therefore $T_c \propto M^{0.3}$

Name	High-mass stars	Medium-mass stars	Low-mass stars	Very low-mass stars	Brown dwarfs
Spectral type	O, B	B	A, F, G, K	M	M, L, T, Y
Minimum mass	8 M_{Sun}	3 M_{Sun}	0.5 M_{Sun}	0.08 M_{Sun}	~0.01 M_{Sun} (~13 $M_{Jupiter}$)

How do we know stars evolve?

II. H-R diagram of star clusters

HR Diagram of Solar Neighborhood: Luminosity Classes vs. Spectral Types

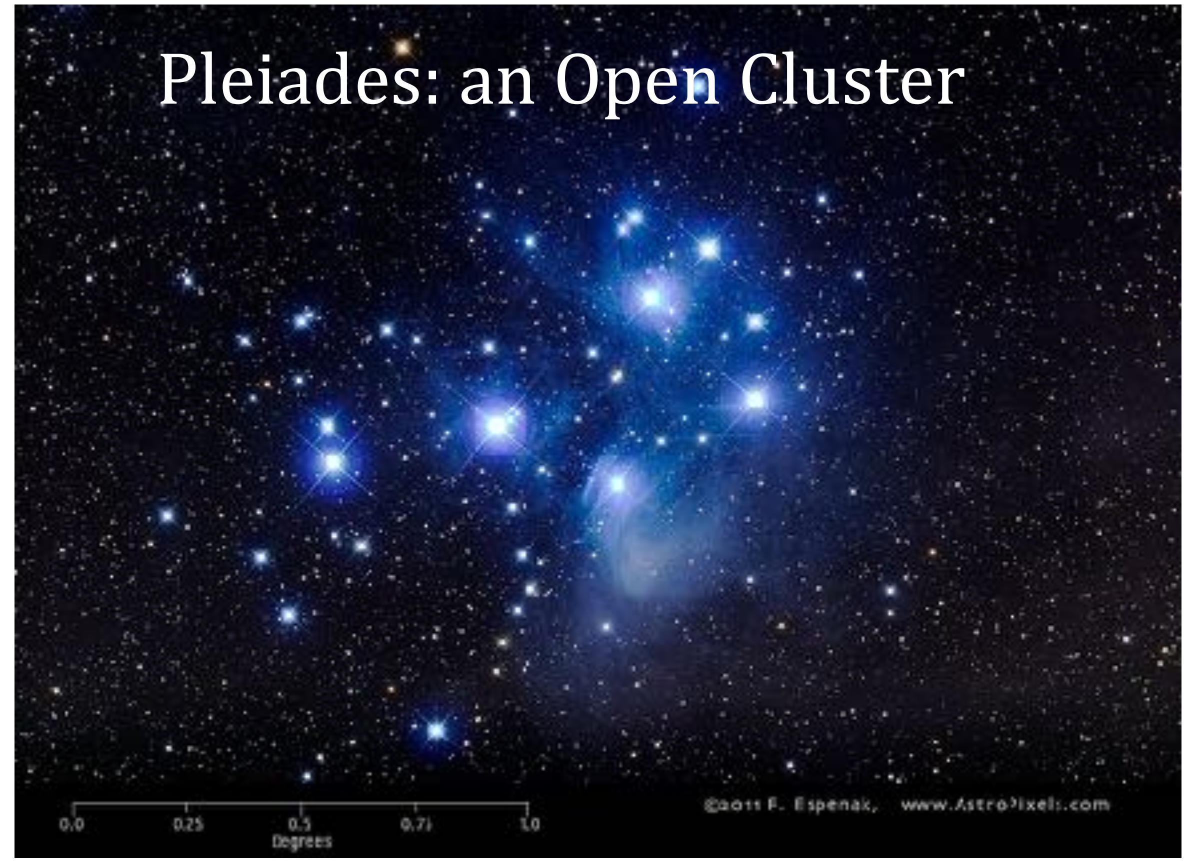


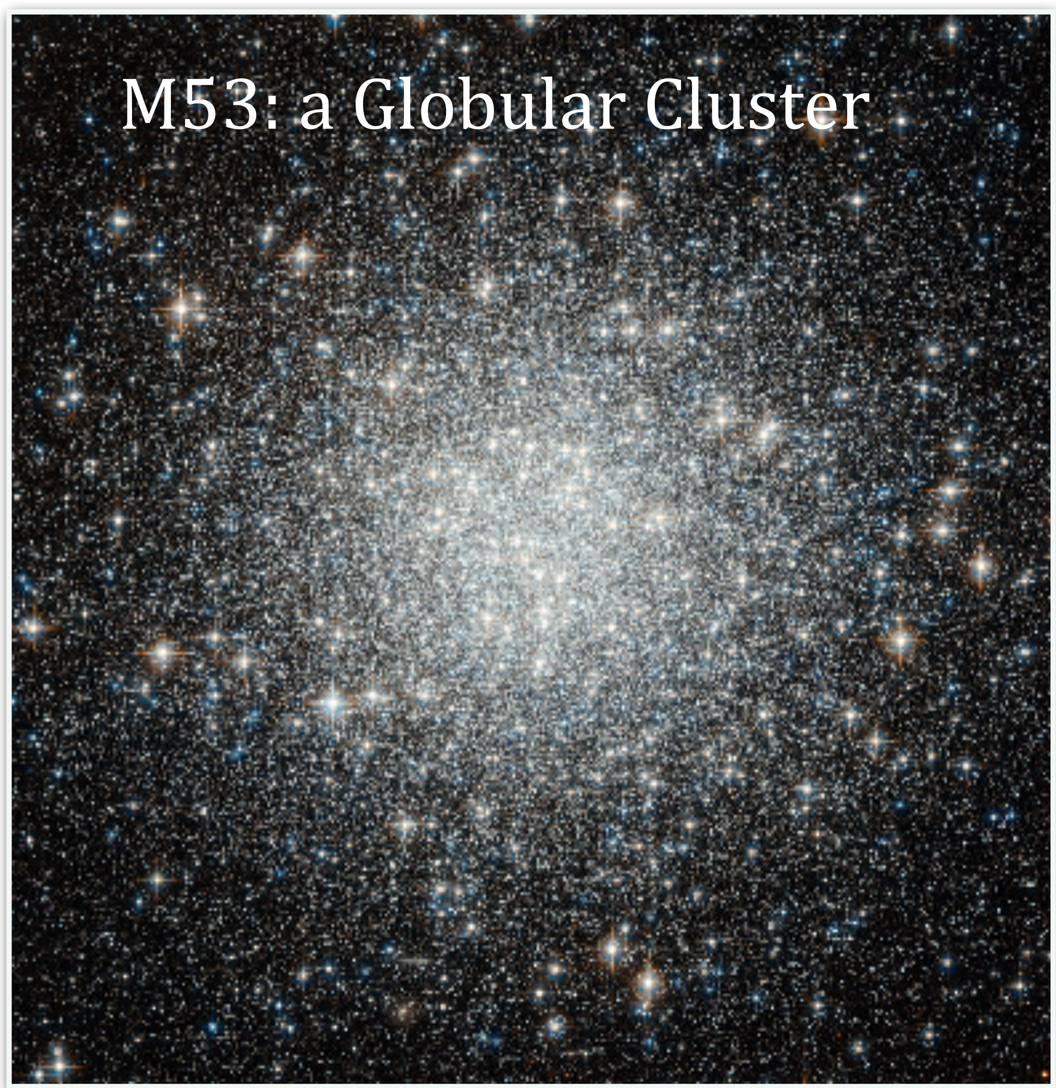
- **Spectral types** correlate with temperature and color: OBAFGKM
- *Broad* **luminosity classes** are defined roughly along the luminosity axis.
- This makes spectral classification of stars in a **two-dimensional** parameter space: T & L

- The Sun is a **G2V** star:
  G2 - spectral type
  V - luminosity class
- Betelgeuse is a **M1Ia**:
  M1 - spectral type
  Ia - luminosity class

Star clusters are ideal laboratories to study stellar evolution

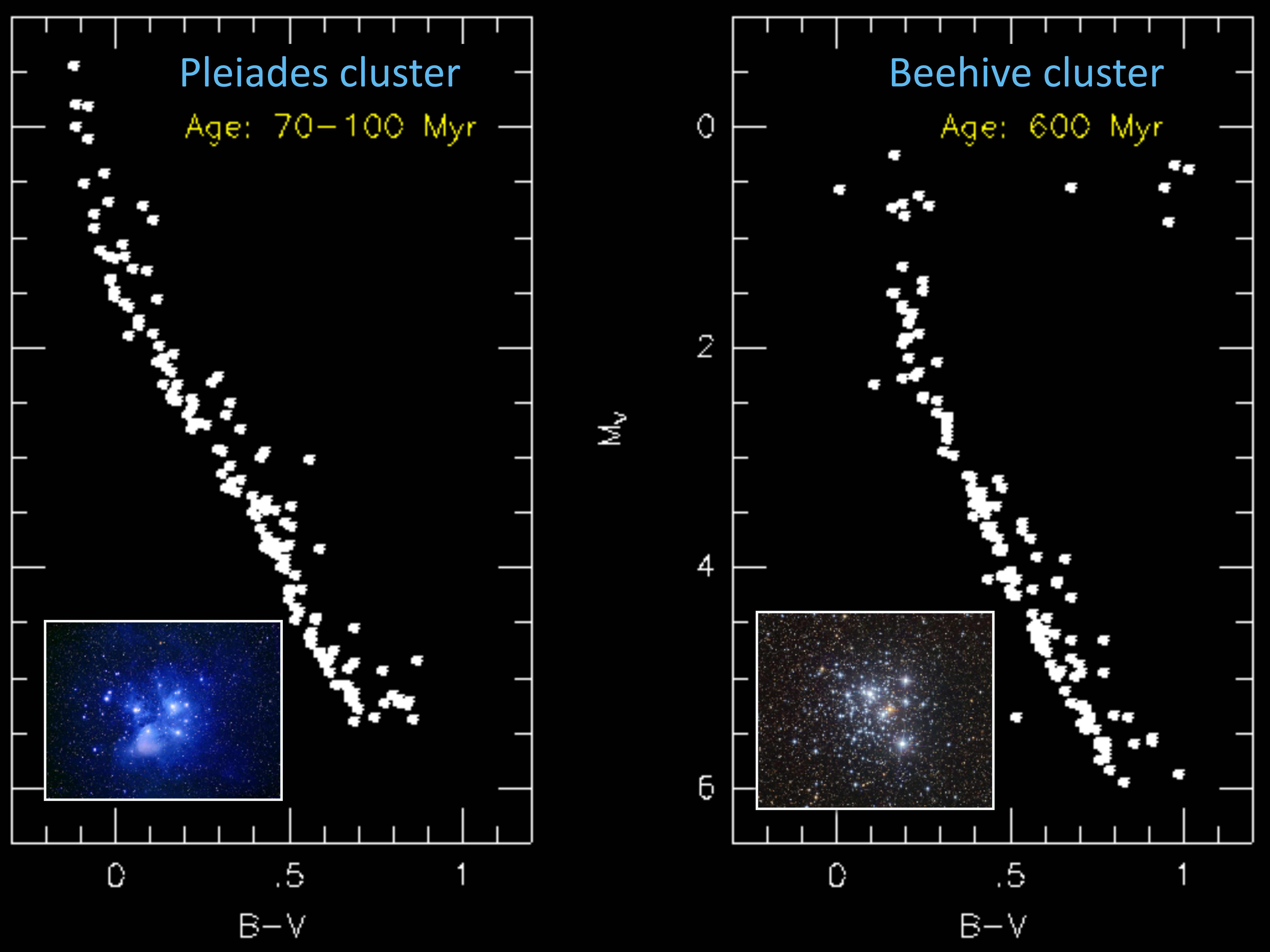
- Star clusters are bound groups of stars, all made at **the same time**.





- Each star **evolves at a rate set by its mass:**
  High-mass stars evolve more quickly along their evolutionary tracks than low mass stars.

Open Clusters in the Milky Way Galaxy



Pleiades cluster
Age: 70–100 Myr
Beehive cluster
Age: 600 Myr
0
2
4
6
M_V
0
.5
1
B–V
0
.5
1
B–V

Dependency on Age: log Age/yr

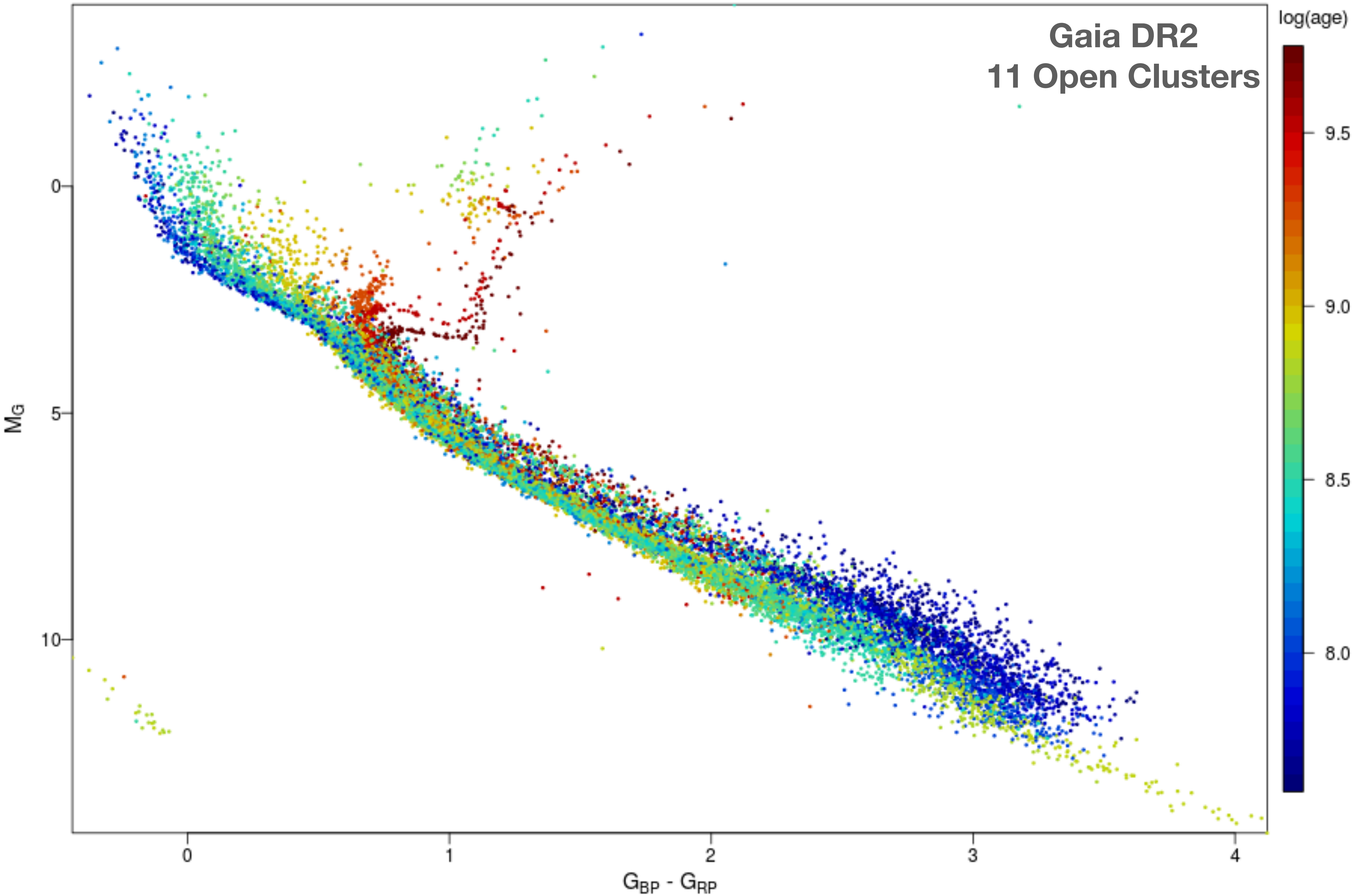
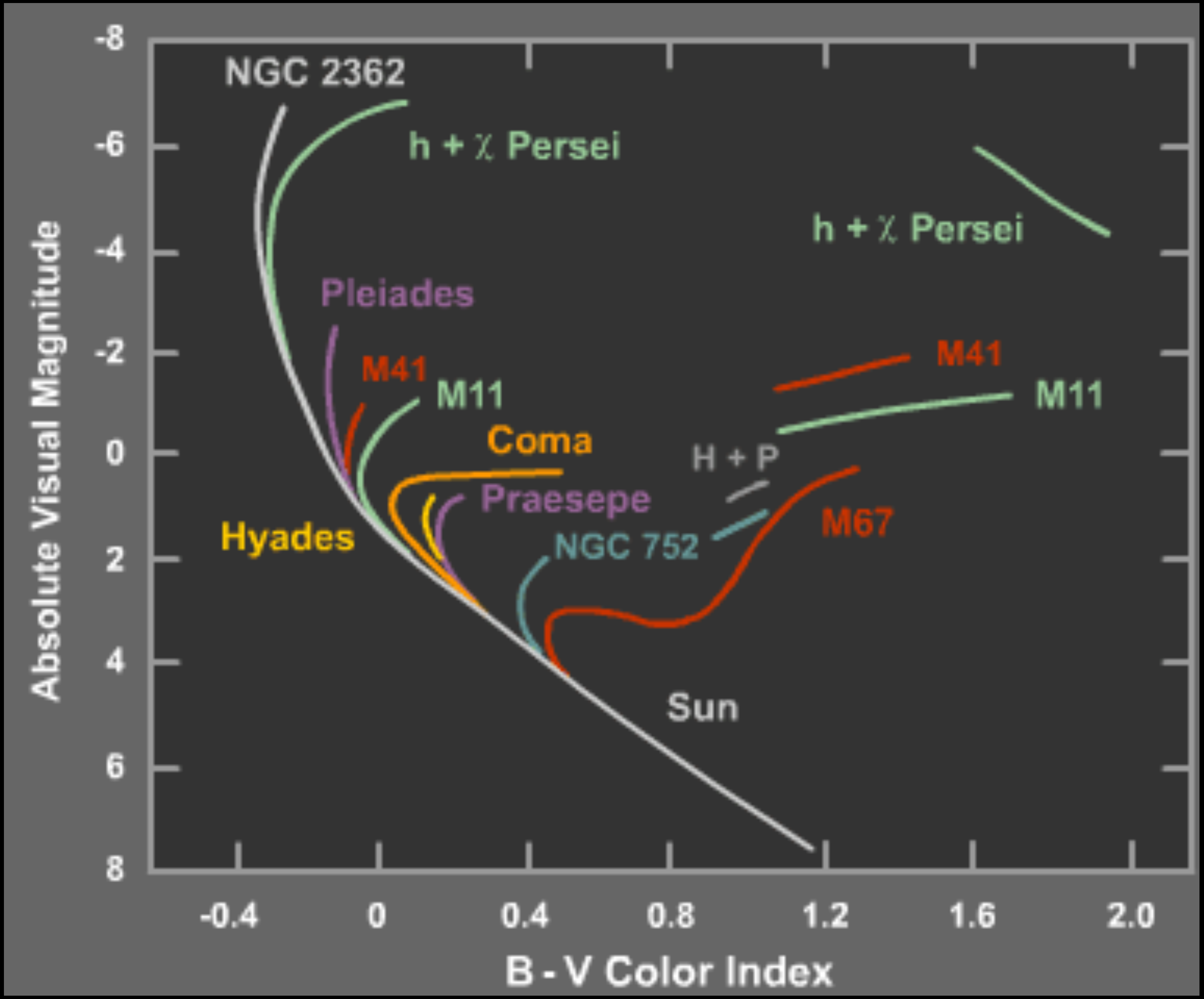


Table 2. Overview of reference values used in constructing the composite HRD for open clusters (Fig. 2).

Cluster	DM	log(age)	[Fe/H]	$E(B-V)$	Memb
Hyades	3.389	8.90	0.13	0.001	480
Coma Ber	4.669	8.81	0.00	0.000	127
Pleiades	5.667	8.04	−0.01	0.045	1059
IC 2391	5.908	7.70	−0.01	0.030	254
IC 2602	5.914	7.60	−0.02	0.031	391
α Per	6.214	7.85	0.14	0.090	598
Praesepe	6.350	8.85	0.16	0.027	771
NGC 2451A	6.433	7.78	−0.08	0.000	311
Blanco 1	6.876	8.06	0.03	0.010	361
NGC 6475	7.234	8.54	0.02	0.049	874
NGC 7092	7.390	8.54	0.00	0.010	248

Simplified H-R diagram of open clusters in the MW galaxy

Can we build a single model to explain all of the star clusters?
If so, what parameter makes different clusters look different on the HRD?



Globular clusters in the Milky Way Galaxy

Dependency on metallicity: $[Fe/H] = \log N_{Fe}/N_H - \log (N_{Fe}/N_H)_{sun}$

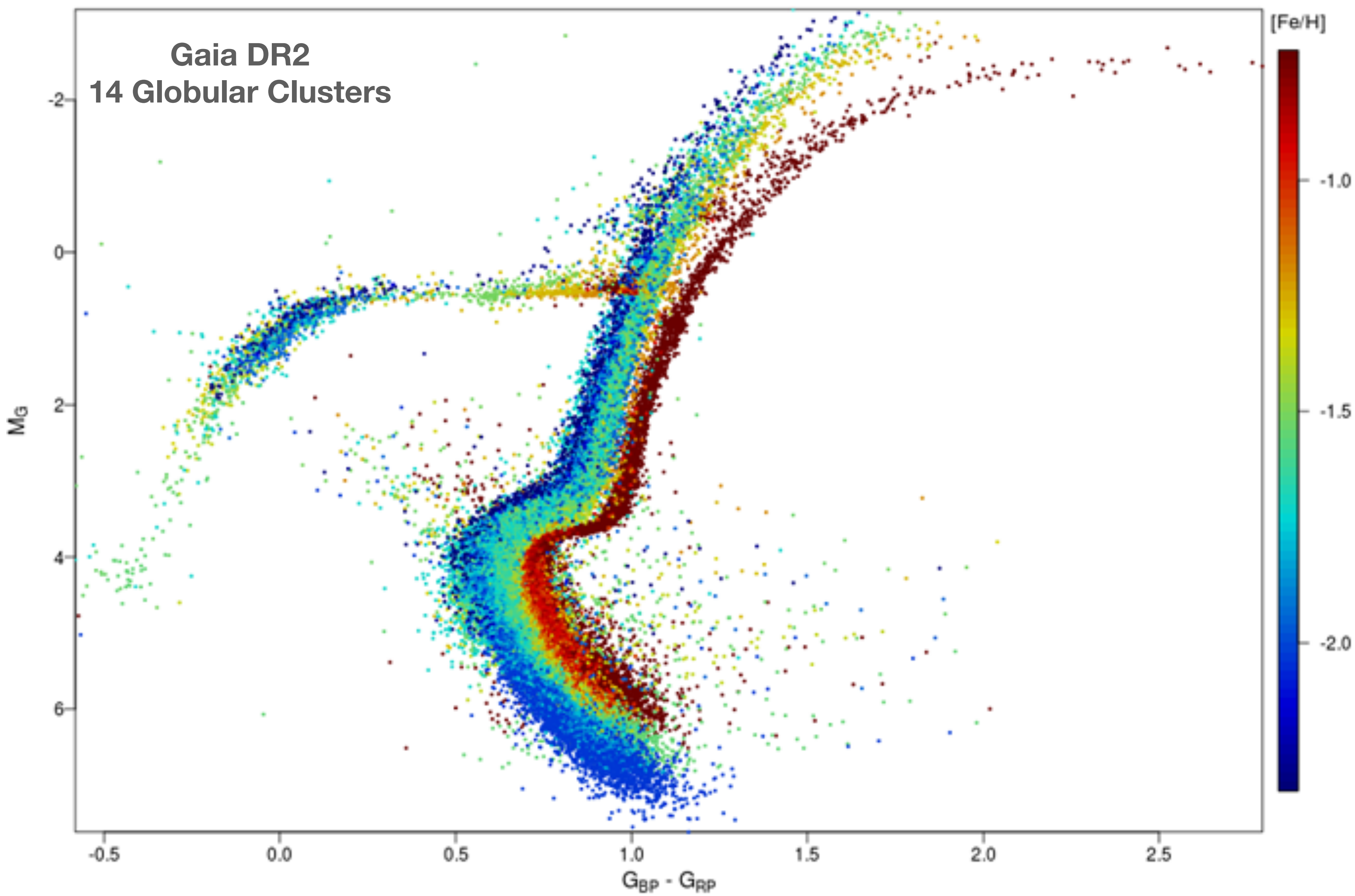


Table 3. Reference data for 14 globular clusters used in the construction of the combined HRD (Fig. 3).

NGC	DM	Age (Gyr)	[Fe/H]	$E(B-V)$	Memb
104	13.266	12.75[a]	−0.72	0.04	21580
288	14.747	12.50[a]	−1.31	0.03	1953
362	14.672	11.50[a]	−1.26	0.05	1737
1851	15.414	13.30[c]	−1.18	0.02	744
5272	15.043	12.60[b]	−1.50	0.01	9086
5904	14.375	12.25[a]	−1.29	0.03	3476
6205	14.256	13.00[a]	−1.53	0.02	10311
6218	13.406	13.25[a]	−1.37	0.19	3127
6341	14.595	13.25[a]	−2.31	0.02	1432
6397	11.920	13.50[a]	−2.02	0.18	10055
6656	12.526	12.86[c]	−1.70	0.35	9542
6752	13.010	12.50[a]	−1.54	0.04	10779
6809	13.662	13.50[a]	−1.94	0.08	8073
7099	14.542	13.25[a]	−2.27	0.03	1016

How do we model stellar evolution?

Computational code of stellar evolution

Stellar Structure Models - Basic Equations

$$\frac{dP}{dr} = -G\frac{M_r \rho}{r^2}$$

HYDROSTATIC EQUILIBRIUM

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

MASS CONSERVATION

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon$$

ENERGY EQUATION

$$\left.\frac{dT}{dr}\right|_{rad} = -\frac{3}{4ac}\frac{\overline{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}$$

RADIATIVE TRANSPORT

$$\left.\frac{dT}{dr}\right|_{\mathrm{ad}} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2}$$

ADIABATIC CONVECTION

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Stellar Structure Models - Constitutive Relations

CONSTITUTIVE RELATIONS (CR)

$$P = \frac{\rho k T}{\mu m_{\mathrm{H}}} + \frac{1}{3} a T^4$$

$$\overline{\kappa} = \begin{cases} \overline{\kappa}_{bf} = \text{bound-free} \\ \overline{\kappa}_{ff} = \text{free-free} \\ \overline{\kappa}_{es} = \text{electron scattering} \end{cases}$$

$$\epsilon = \begin{cases} \epsilon_{\text{pp-chain}} \\ \epsilon_{\text{CNO cycle}} \\ \epsilon_{3\alpha} \end{cases}$$

FROM TABLES OR FITTED TO A FUNCTION

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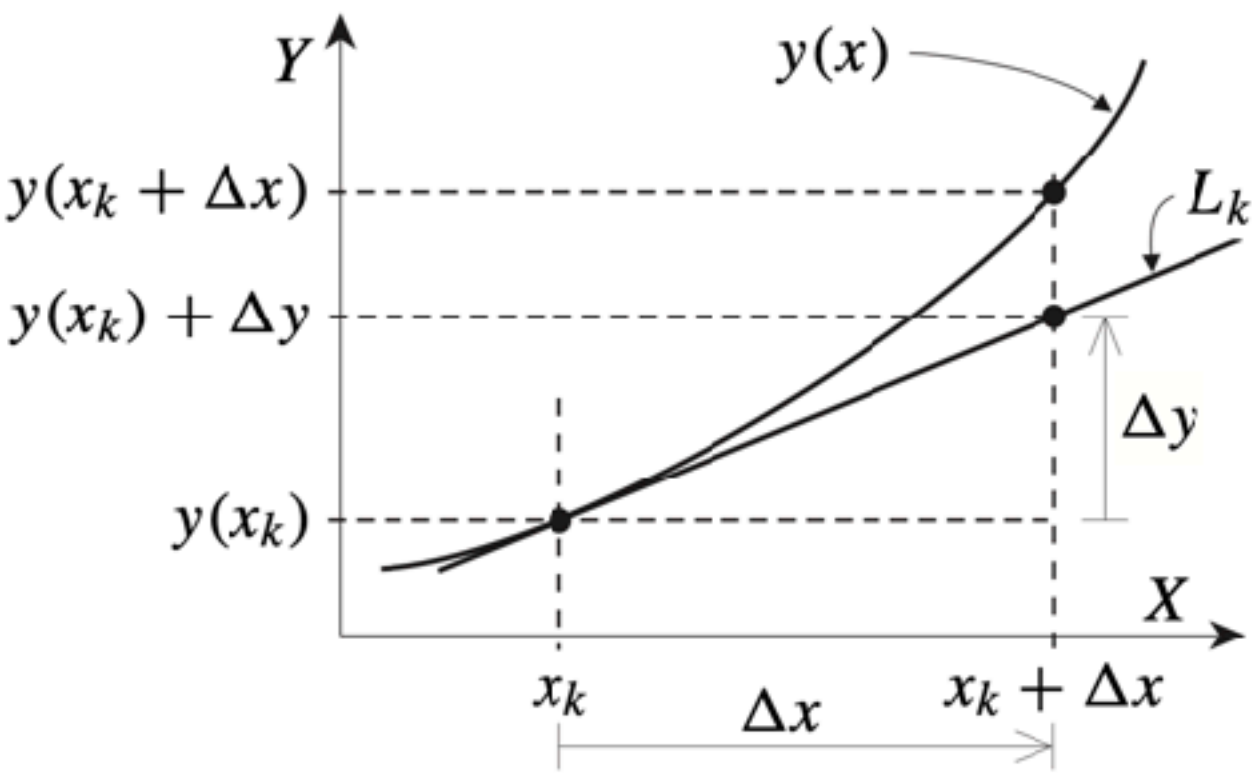
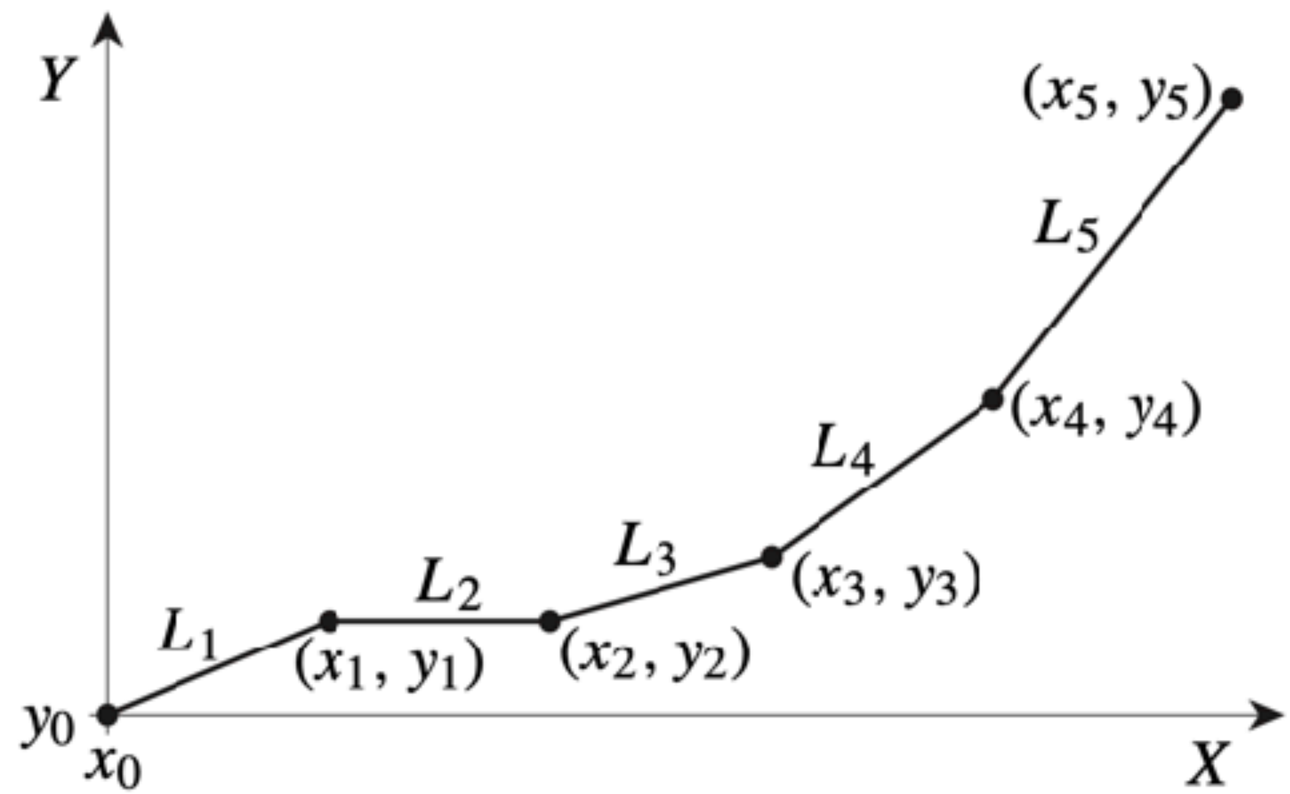
Euler Method: a numerical procedure to solve differential equations

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_k + \Delta x) \approx y_k + \Delta x \cdot f(x_k, y_k)$$





$$\text{Example : Solve } \frac{dn}{dh} = -\frac{\mu m_p g}{kT} n \equiv -\frac{n}{h_s}$$

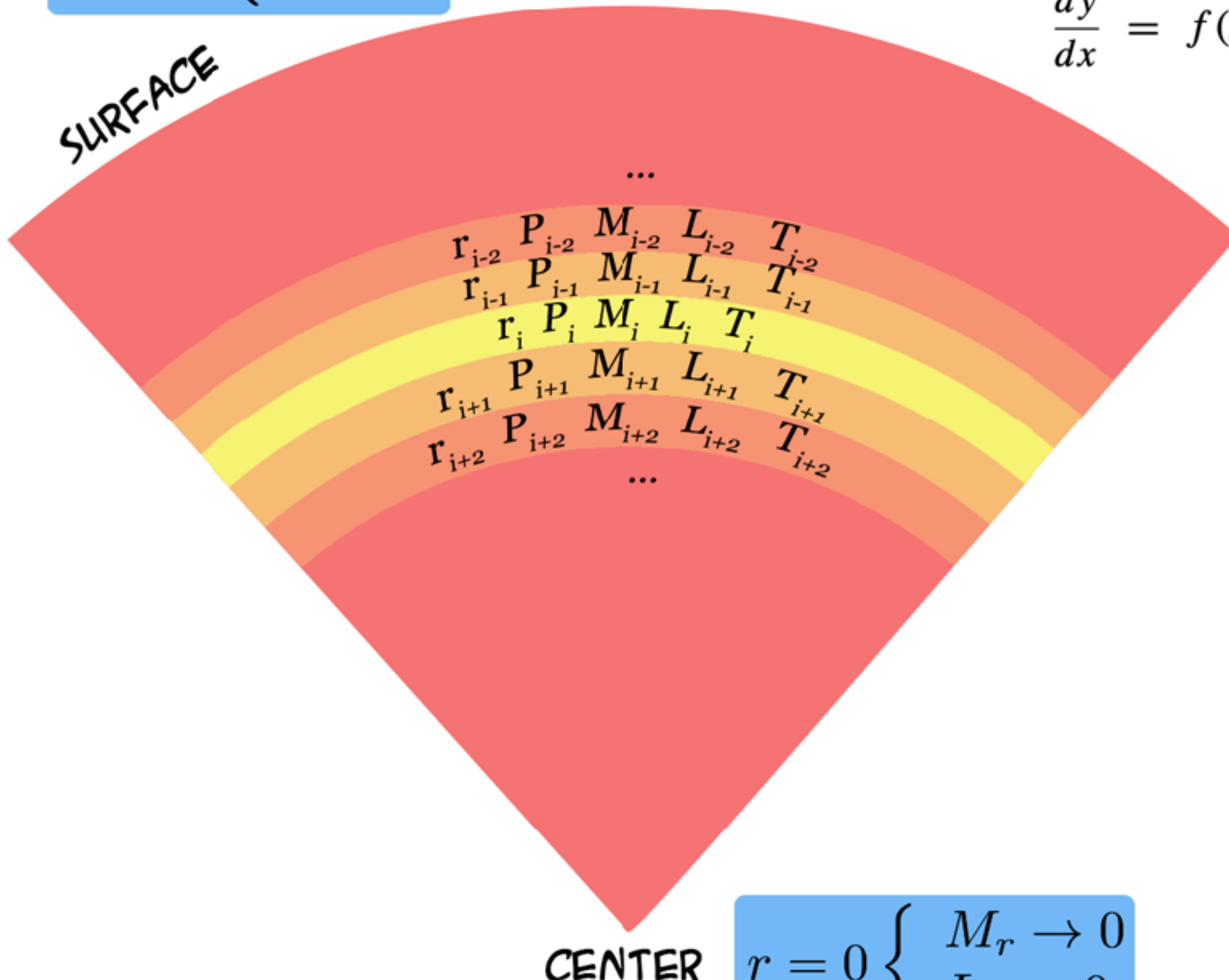
https://www.uah.edu/images/people/faculty/howellkb/DEText-Ch9.pdf

$$r = R^* \begin{cases} T \to 0 \\ P \to 0 \\ \rho \to 0 \end{cases}$$

Euler Method: a numerical procedure to solve differential equations

$$y(x_k + \Delta x) \approx y_k + \Delta x \cdot f(x_k, y_k)$$

$$\frac{dy}{dx} = f(x, y)$$



$$r = 0 \begin{cases} M_r \to 0 \\ L_r \to 0 \end{cases}$$

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Modules for Experiments in Stellar Astrophysics (MESA)

https://docs.mesastar.org/en/release-r22.11.1/index.html

http://user.astro.wisc.edu/~townsend/static.php?ref=mesa-web-submit

Motivation

Stellar evolution calculations (i.e., stellar evolution tracks and detailed information about the evolution of internal and global properties) are a basic tool that enable a broad range of research in astrophysics. Areas that critically depend on high-fidelity and modern stellar evolution include asteroseismology, nuclear astrophysics, stellar populations, chemical evolution and population synthesis, astrobiology, binary stars, variable stars, supernovae, novae, compact objects, tidal disruption events, stellar hydrodynamics, and stellar activity.

New observational capabilities are emerging in these fields that place a high demand on exploration of stellar dependencies on mass, metallicity and age. So, even though one dimensional stellar evolution is a mature discipline, we continue to ask new questions of stars. Some important aspects of stars are truly three-dimensional, such as convection, rotation, and magnetism. These aspects remain in the realm of research frontiers with evolving understanding and insights, quite often profound. However, much remains to be gained scientifically (and pedagogically) by accurate one-dimensional calculations, and this is the focus of MESA.

MESA-Web Calculation Submission

To submit a *MESA-Web* calculation, simply enter your email address in the *Email Address* field at the bottom of the form below, and then click the *Submit* button.

The default parameters have been chosen to evolve a 1 $M_\odot$ model from pre main-sequence to white dwarf in less than 2 hours of wall time. To obtain more-Detailed information about each parameter, click on the name of the parameter to visit the corresponding entry on the the *MESA-Web Input* page.

After a calculation completes, you will receive an email with link to a Zip archive that contains the output from *MESA-Web* (note that the link expire after one day). For information on the contents of this archive, see the *MESA-Web Output* page.

Initial Properties

Mass: 1.0 $M_\odot$

Metallicity: 0.02

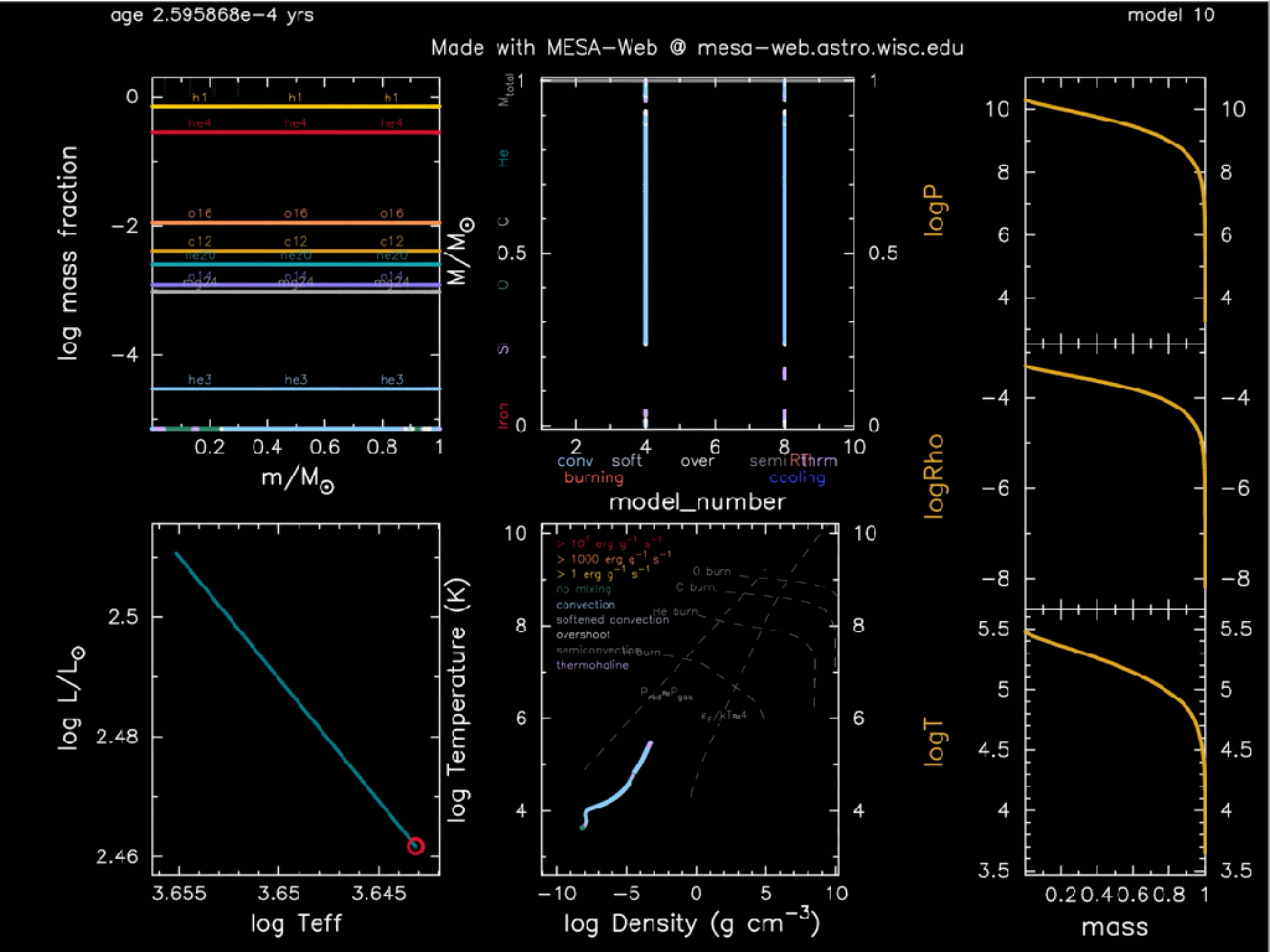
Rotation Rate ($\Omega_{ZAMS}/\Omega_{crit}$): 0.0

Nuclear Reactions

Network: basic

Custom Nuclear Reaction Rate: none

Upload Rate: Choose File no file selected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model 10
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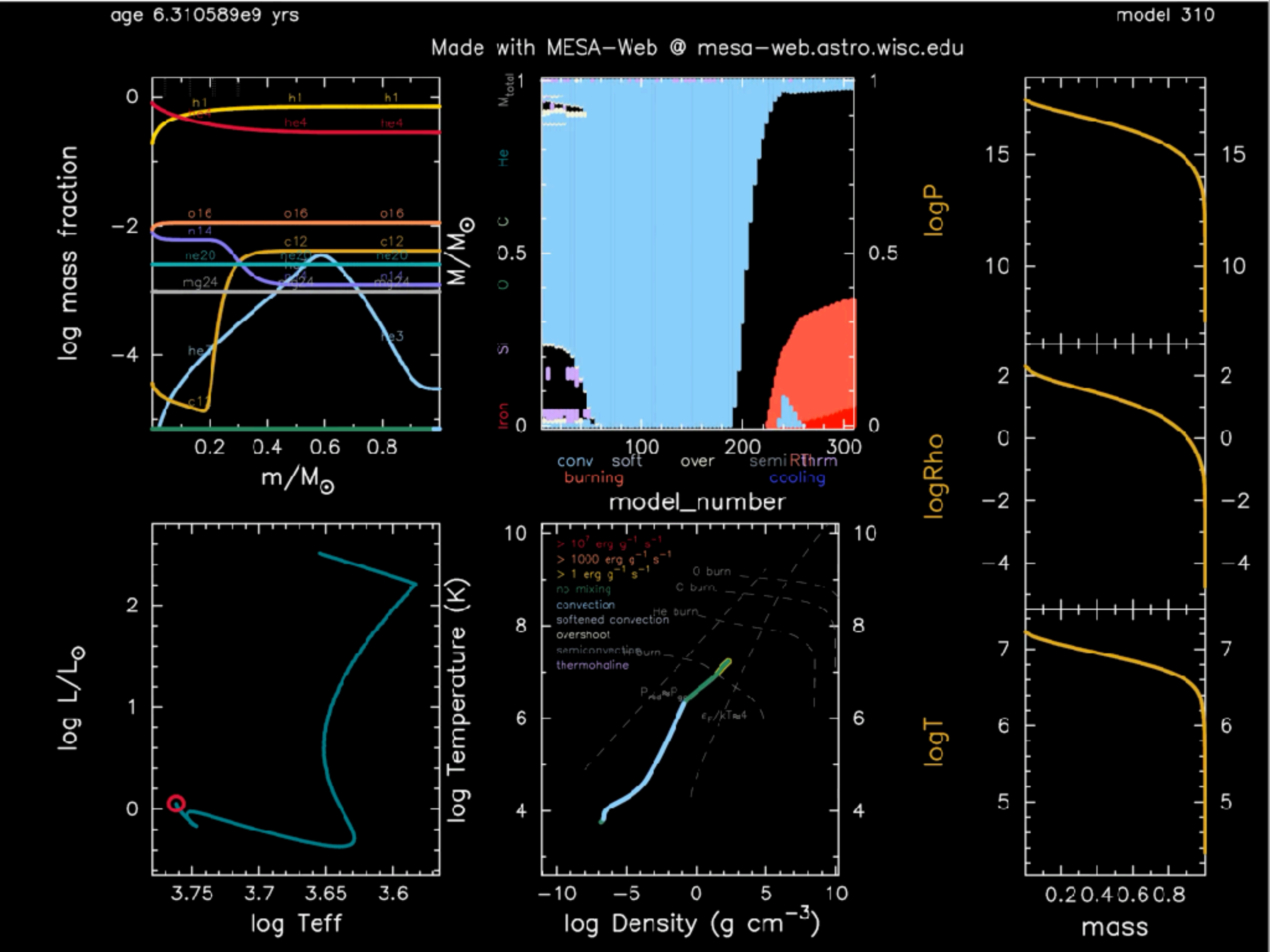
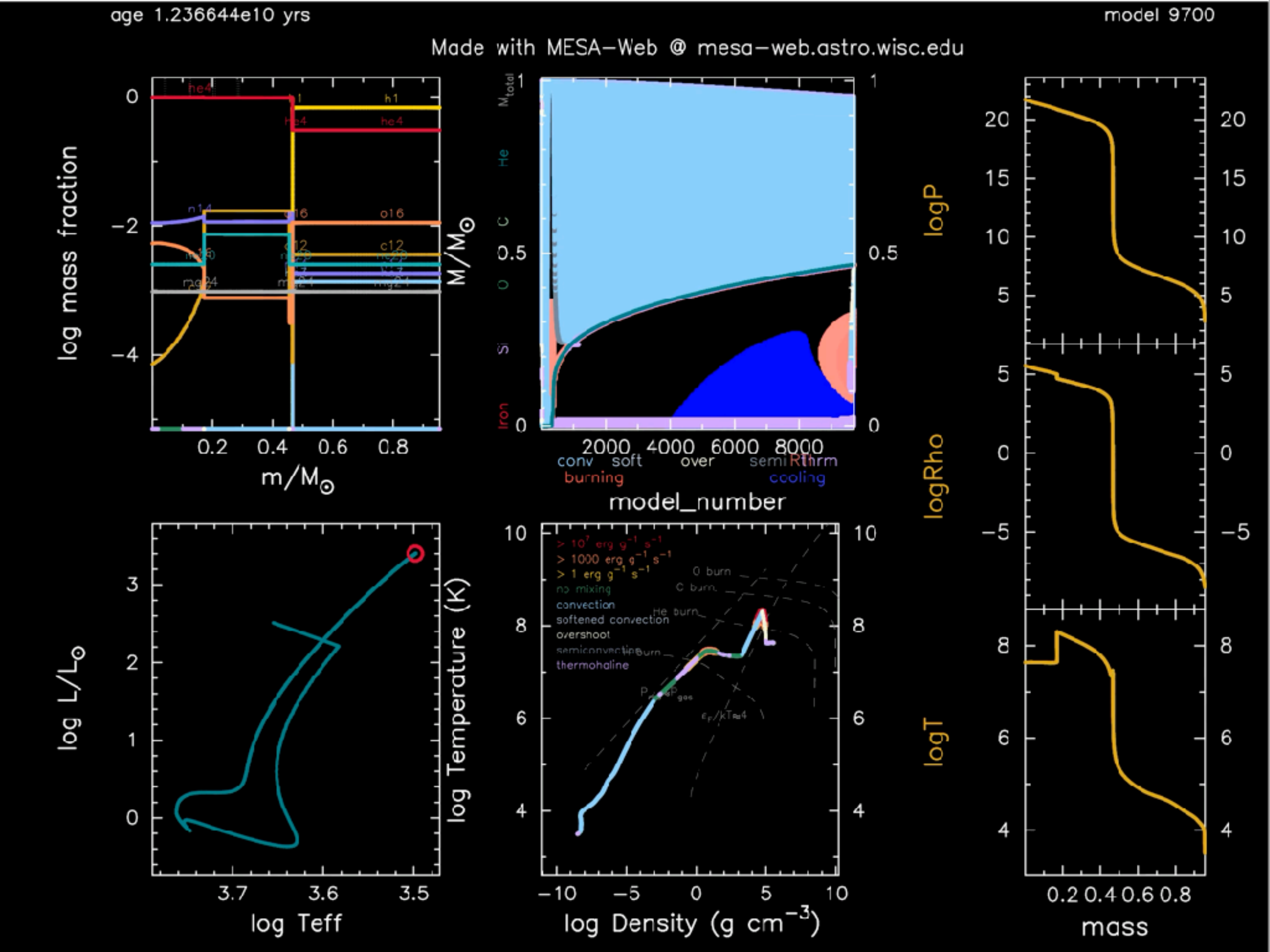
log mass fraction
h1
he4
o16
c12
ne20
n14
mg24
he3
m/M⊙
M/M⊙
Mtotal
He
C
O
Si
Iron
conv
soft
over
semi
RTi
thrm
burning
cooling
model_number
logP
logRho
logT
mass
log L/L⊙
log Teff
log Temperature (K)
log Density (g cm⁻³)
> 10⁷ erg g⁻¹ s⁻¹
> 1000 erg g⁻¹ s⁻¹
> 1 erg g⁻¹ s⁻¹
no mixing
convection
softened convection
overshoot
semiconvection
thermohaline
O burn
C burn
He burn
H burn
Prad≈Pgas
εF/kT≈4


age 6.310589e9 yrs
model 310
Made with MESA-Web @ mesa-web.astro.wisc.edu
log mass fraction
h1
he4
o16
n14
c12
ne20
mg24
he3
m/M$_\odot$
M/M$_\odot$
M$_{total}$
He
C
O
Si
Iron
conv
soft
over
semi
RT
thrm
burning
cooling
model_number
logP
logRho
logT
mass
log L/L$_\odot$
log Teff
log Temperature (K)
log Density (g cm^{-3})
> 10^7 erg g^{-1} s^{-1}
> 1000 erg g^{-1} s^{-1}
> 1 erg g^{-1} s^{-1}
no mixing
convection
softened convection
overshoot
semiconvection
thermohaline
O burn
C burn
He burn
H burn
$P_{rad} \approx P_{gas}$
$\epsilon_F/kT \approx 4$



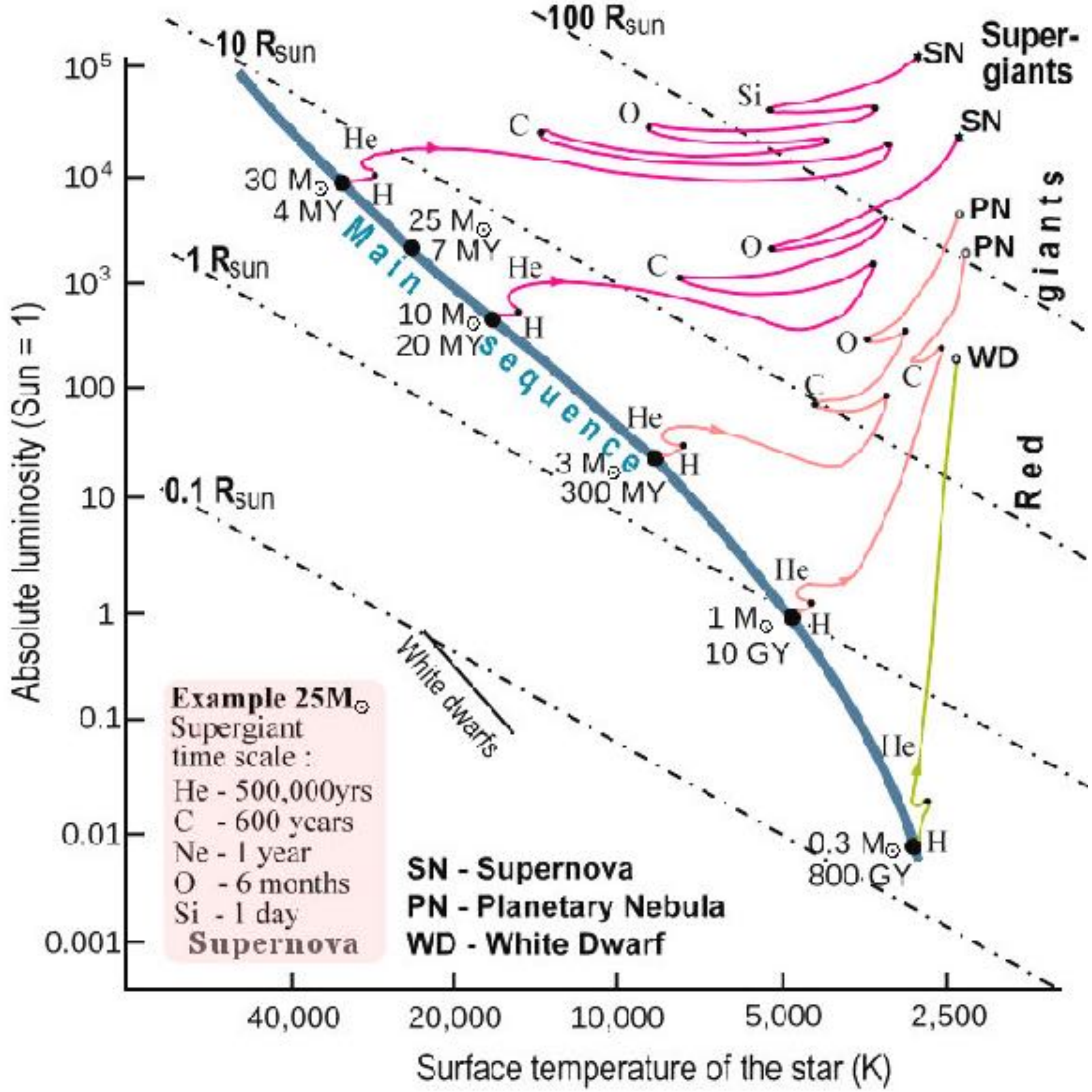
age 1.236644e10 yrs
model 9700
Made with MESA-Web @ mesa-web.astro.wisc.edu
log mass fraction
m/M⊙
M/M⊙
model_number
conv soft over semi RTthrm
burning cooling
logP
logRho
logT
mass
log L/L⊙
log Teff
log Temperature (K)
log Density (g cm⁻³)
no mixing
convection
softened convection
overshoot
semiconvection
thermohaline
O burn
C burn
He burn

How do we check stellar evolution models?

Model Isochrones vs. Cluster H-R Diagrams

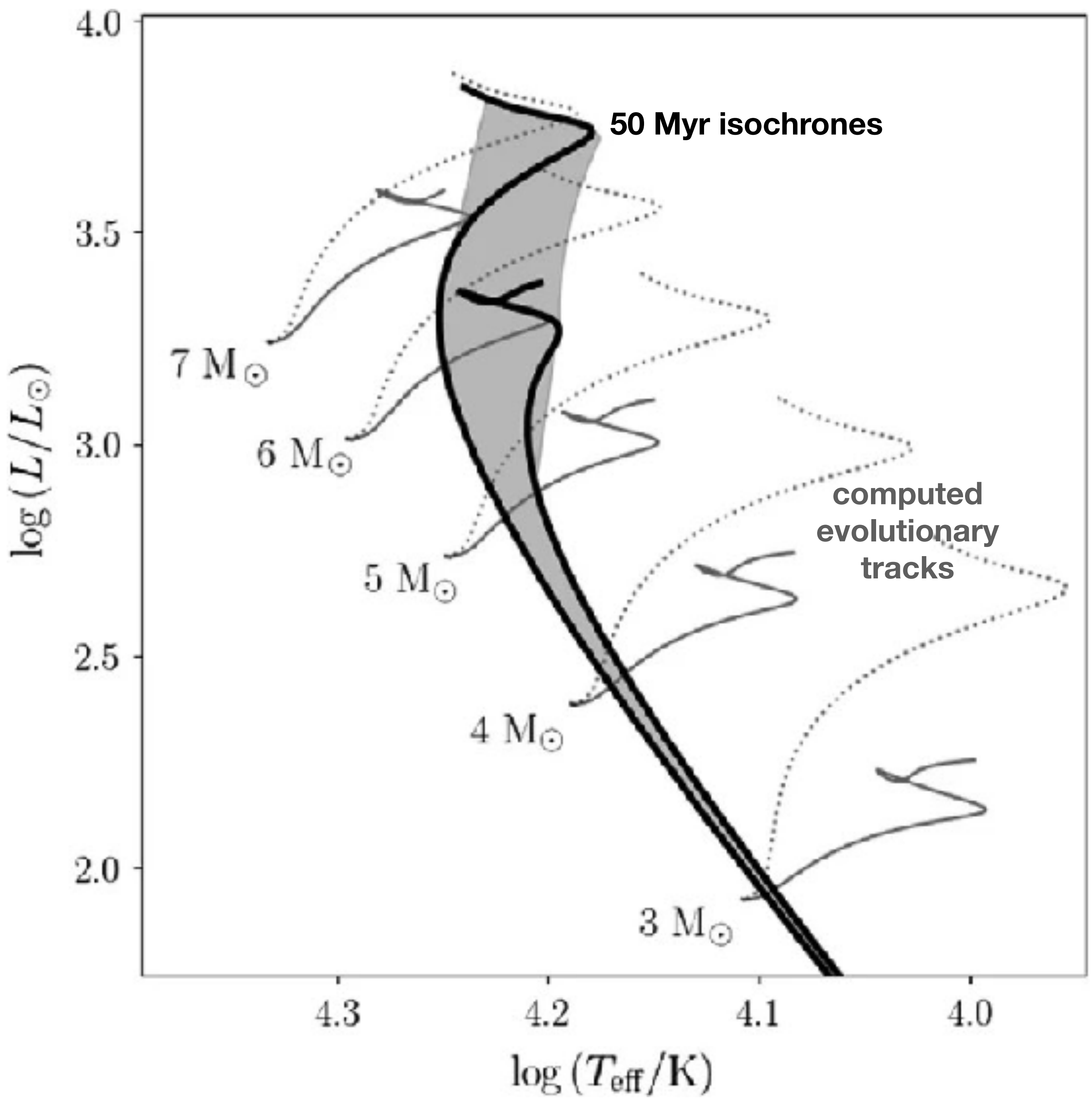
Computed evolutionary tracks of stars with different *initial* masses

- An **evolutionary track** is a computed trajectory of **a single star** on the H-R diagram as it ages over time.

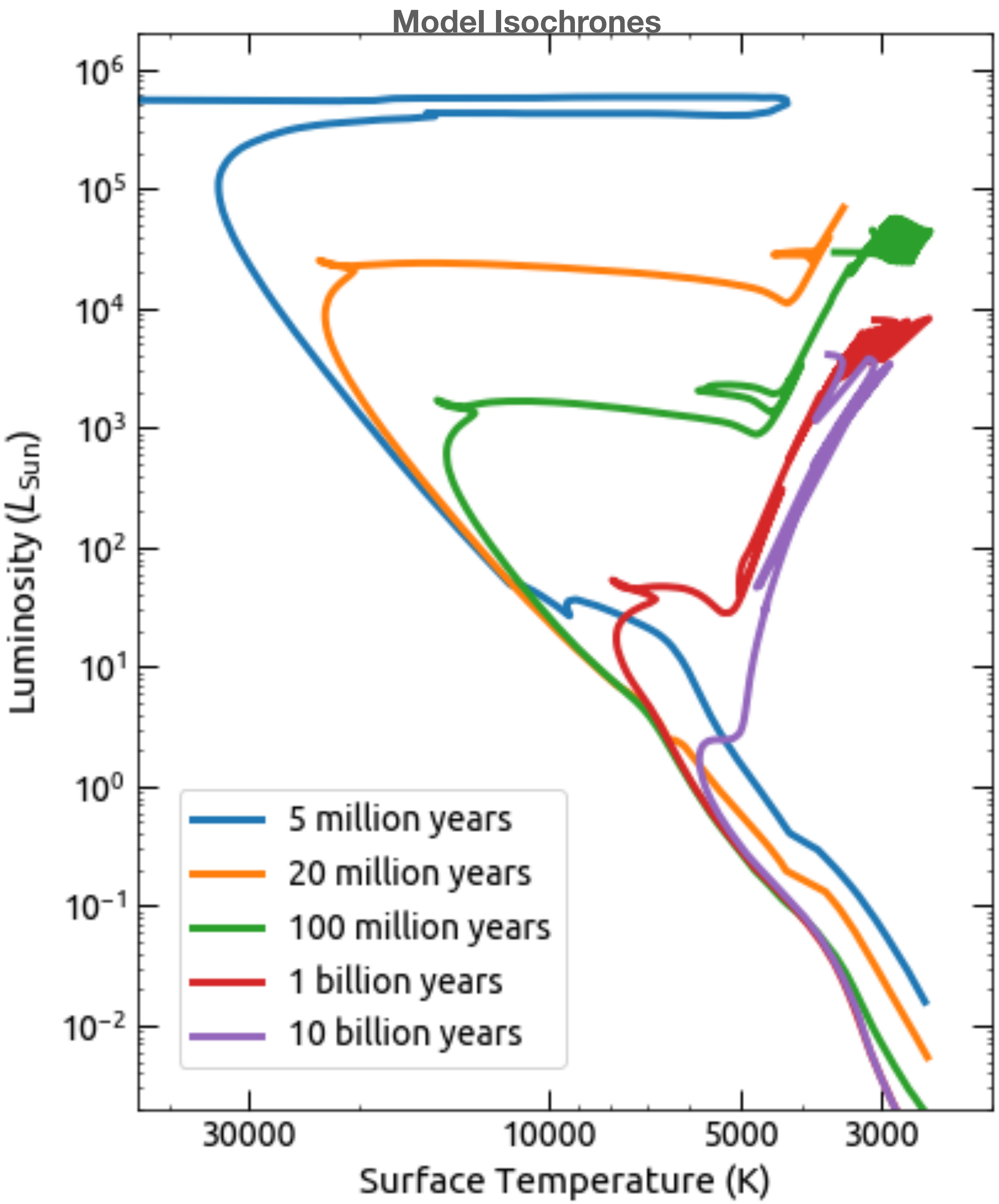


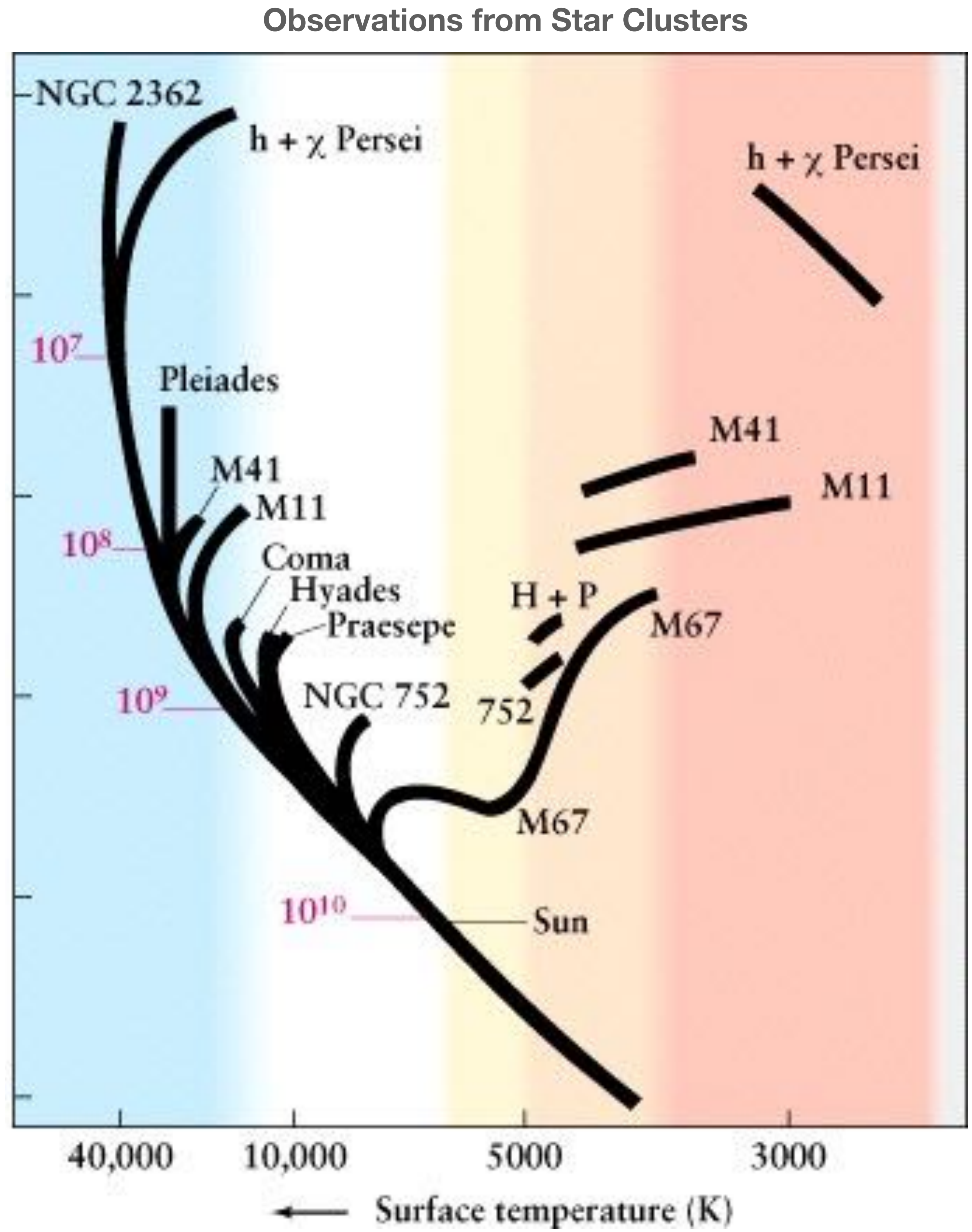
Building isochrones from evolutionary tracks

- An **isochrone** (iso = equal, chrone = time) is a line drawn on the H-R diagram connecting **stars of the same age**



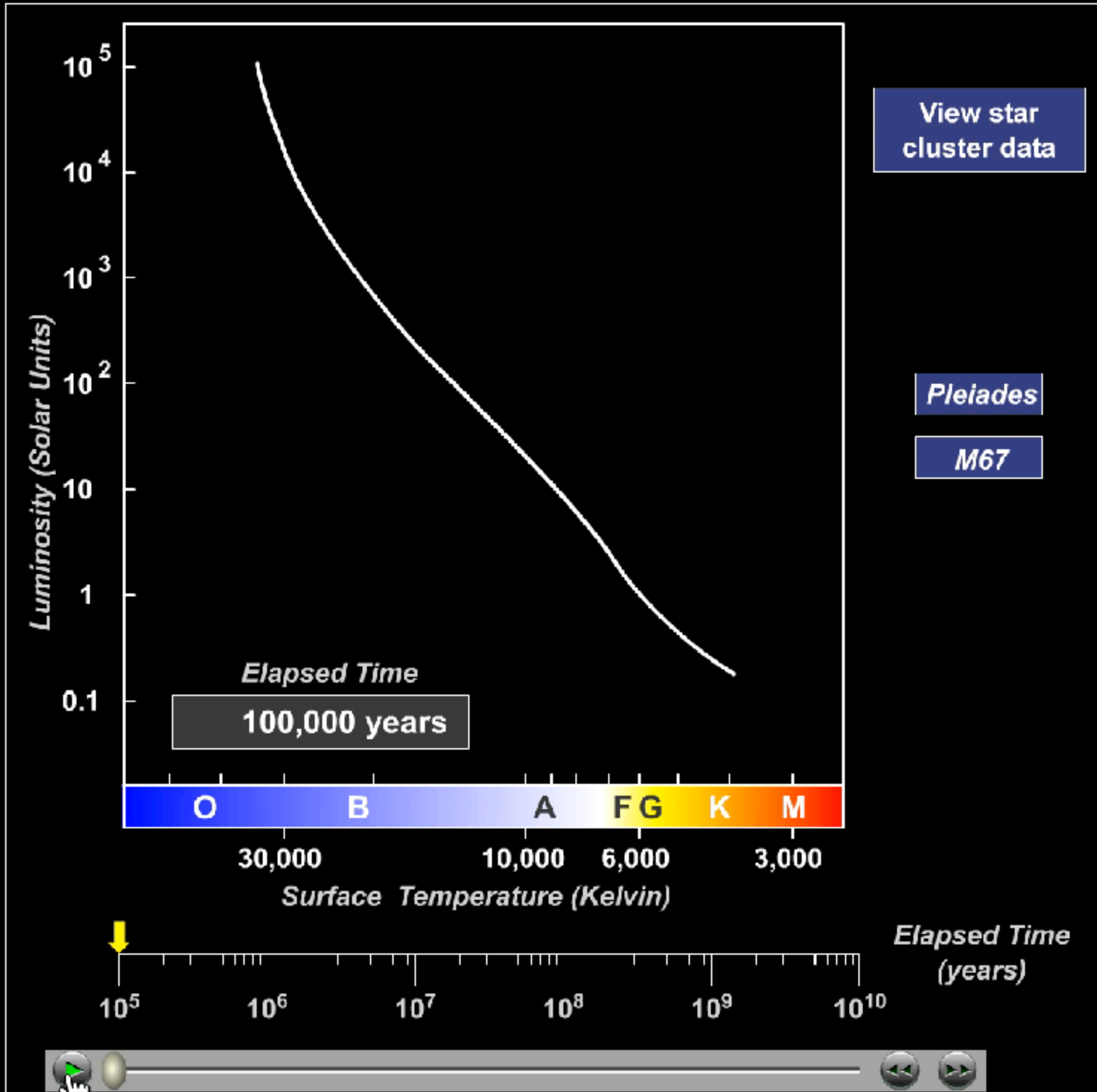
Isochrones change with age: this is how we explain the various different observed H-R diagrams of star clusters



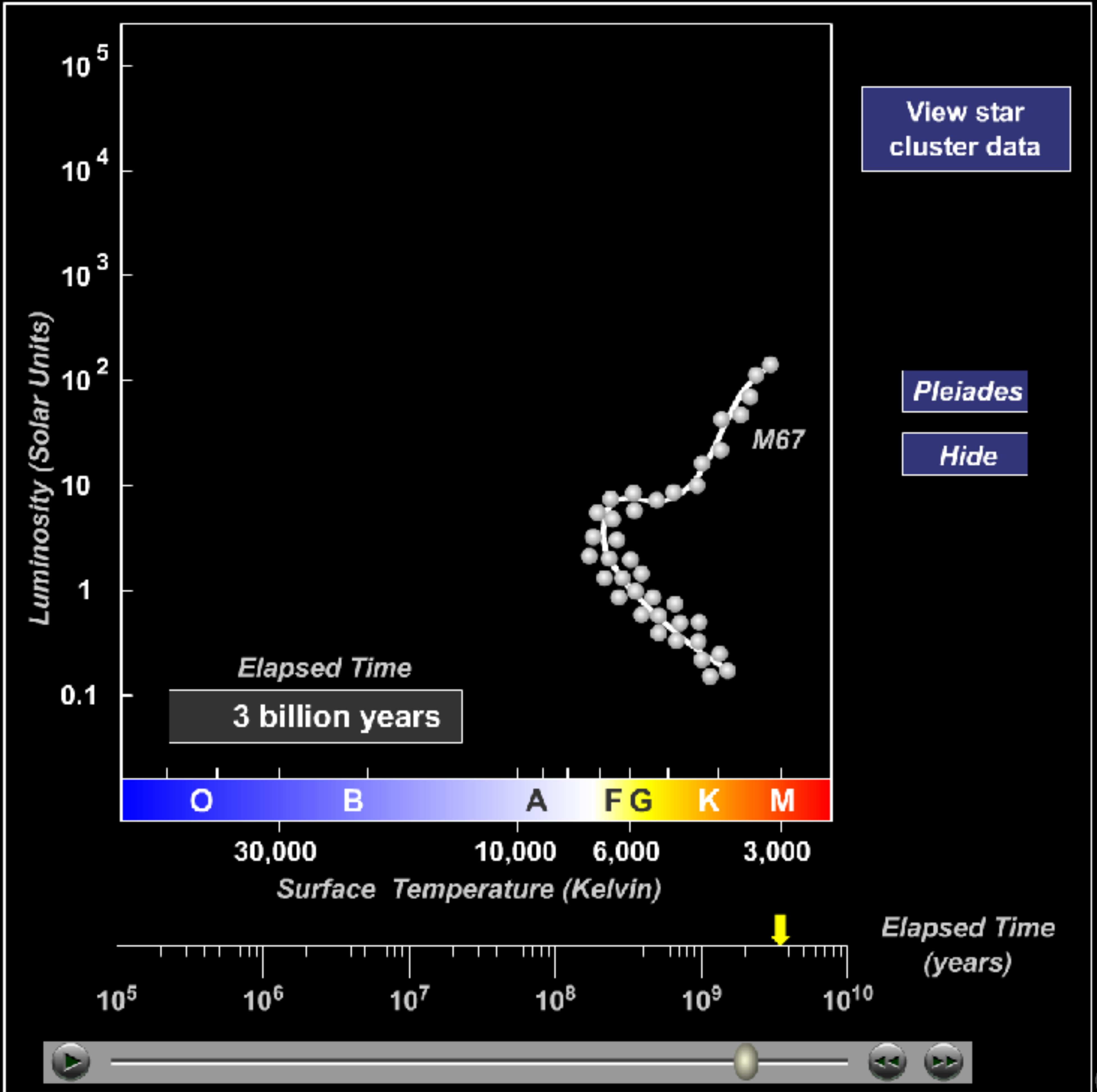


How can you tell if a curve on a HR diagram is an isochrone instead of an evolutionary track?

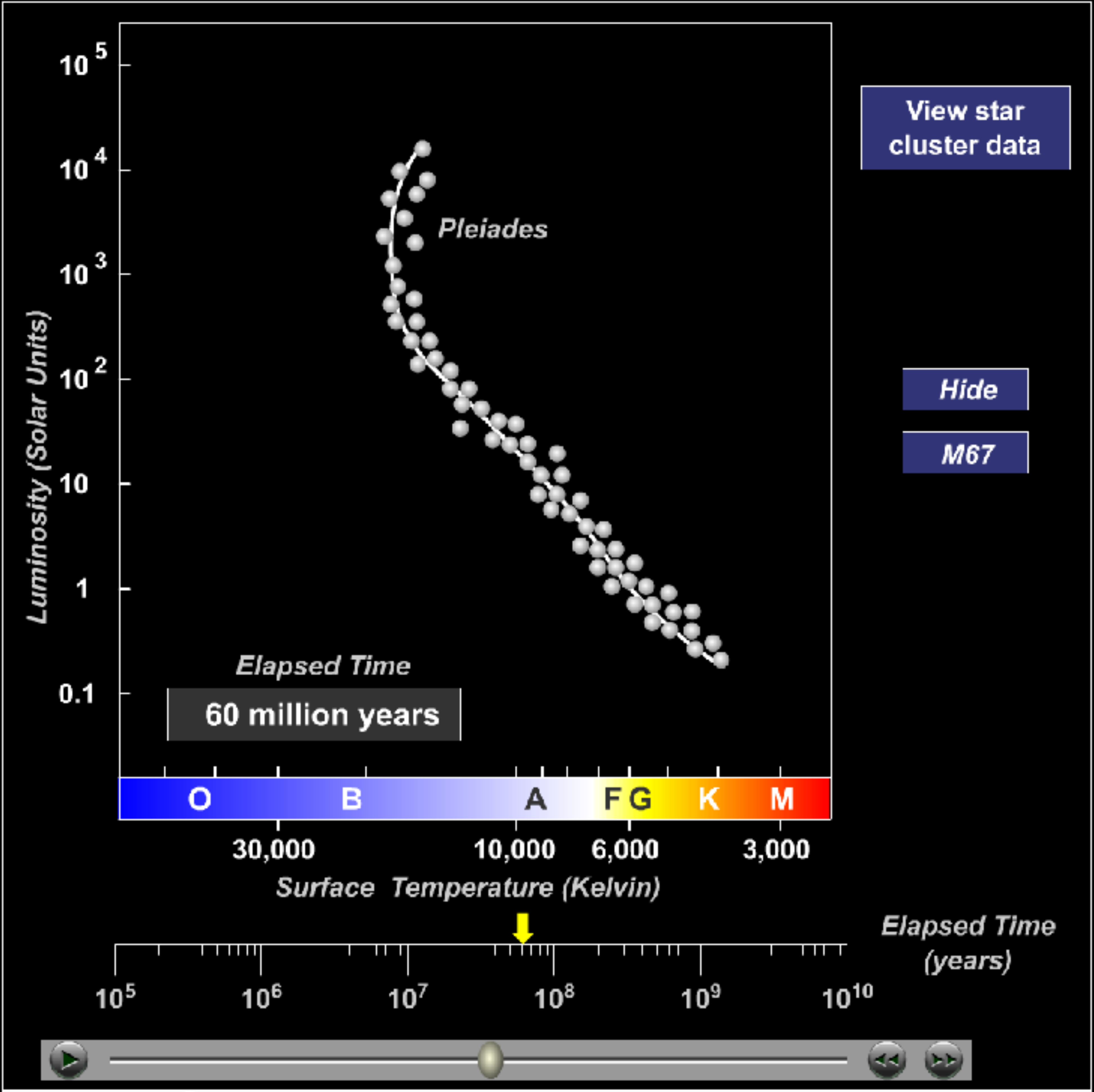
Using the H-R Diagram to Determine the Age of a Star Cluster



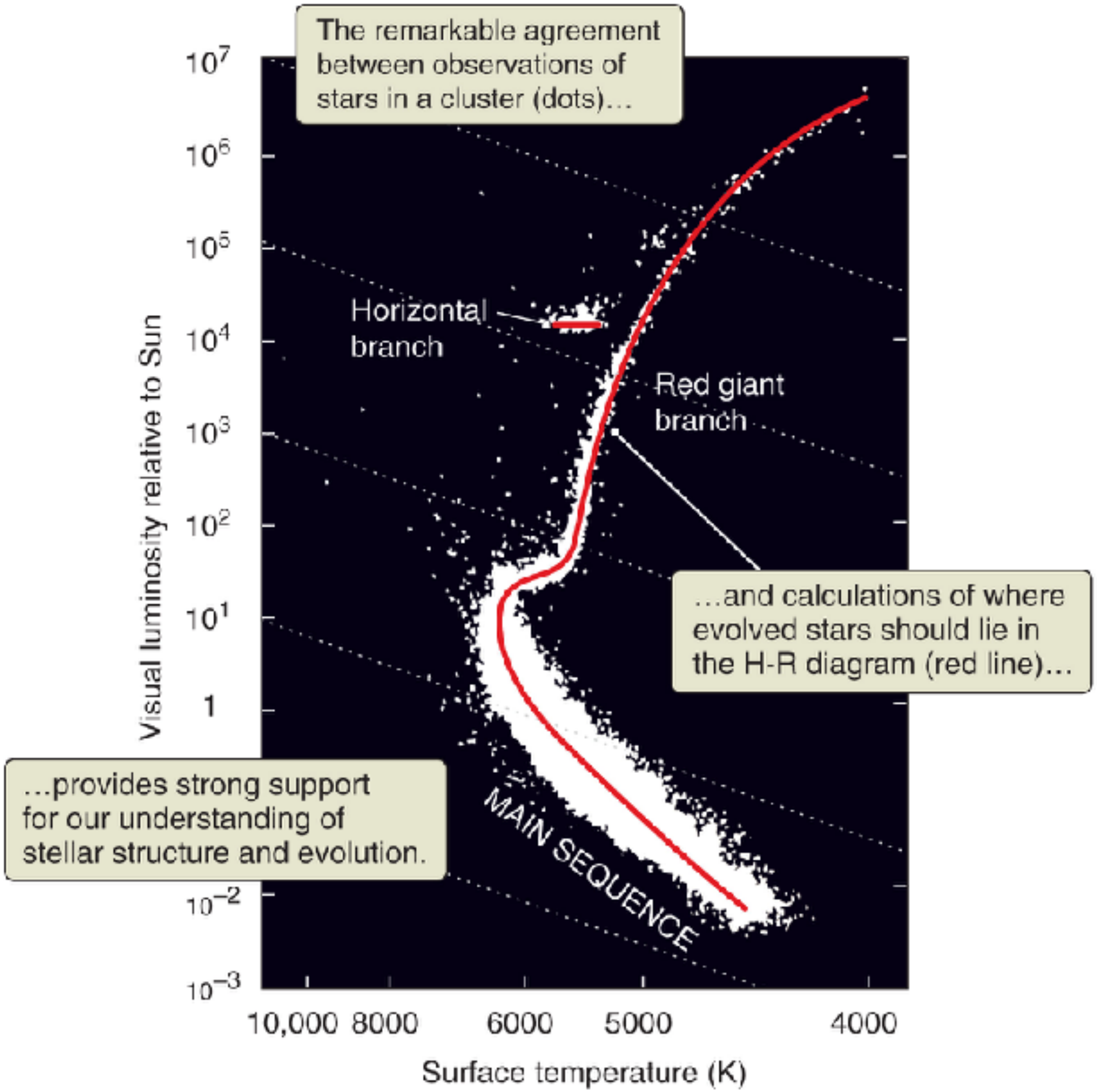
Using the H-R Diagram to Determine the Age of a Star Cluster



Using the H-R Diagram to Determine the Age of a Star Cluster

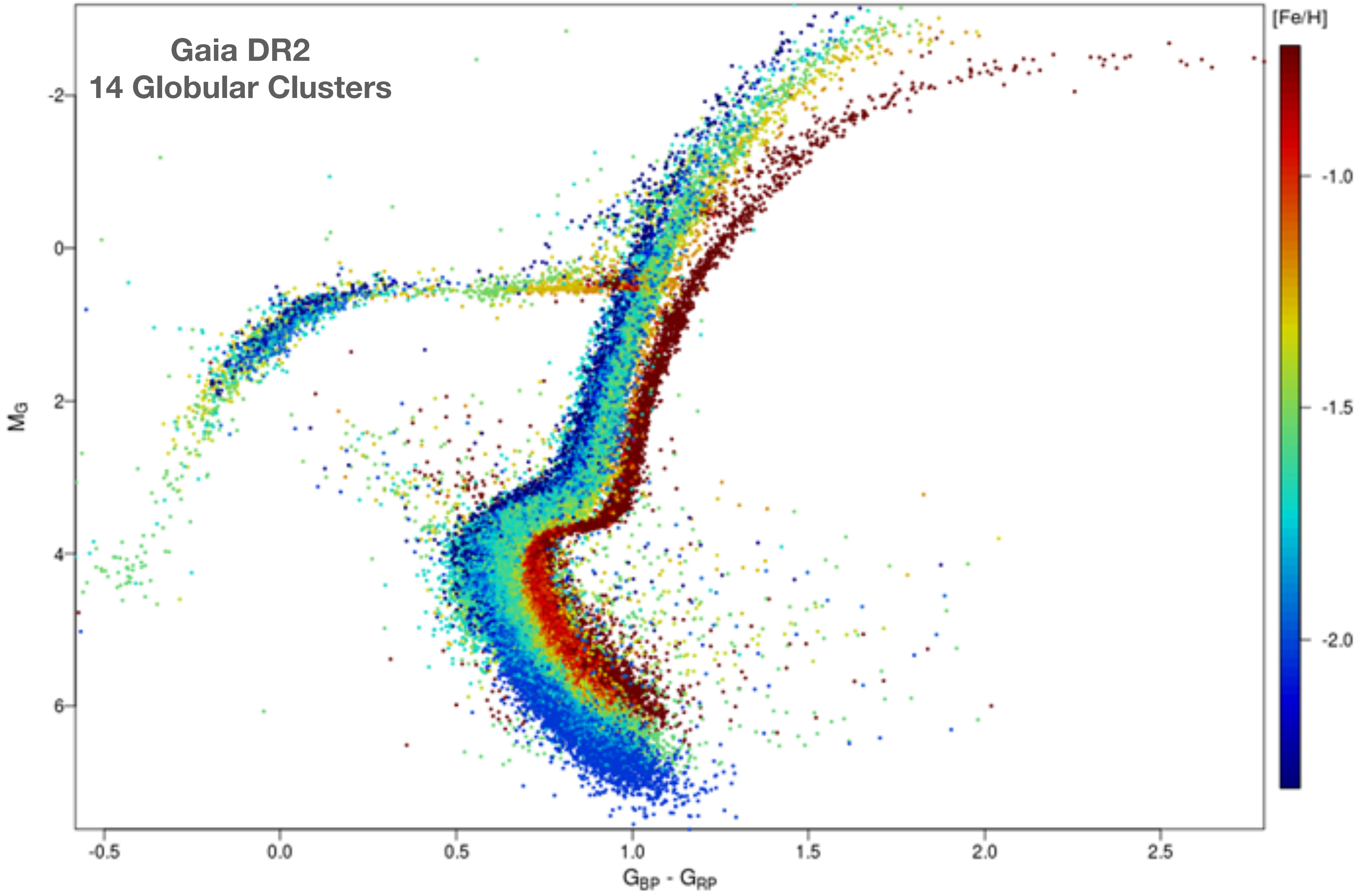


Summary: Model Isochrones vs. Cluster Data

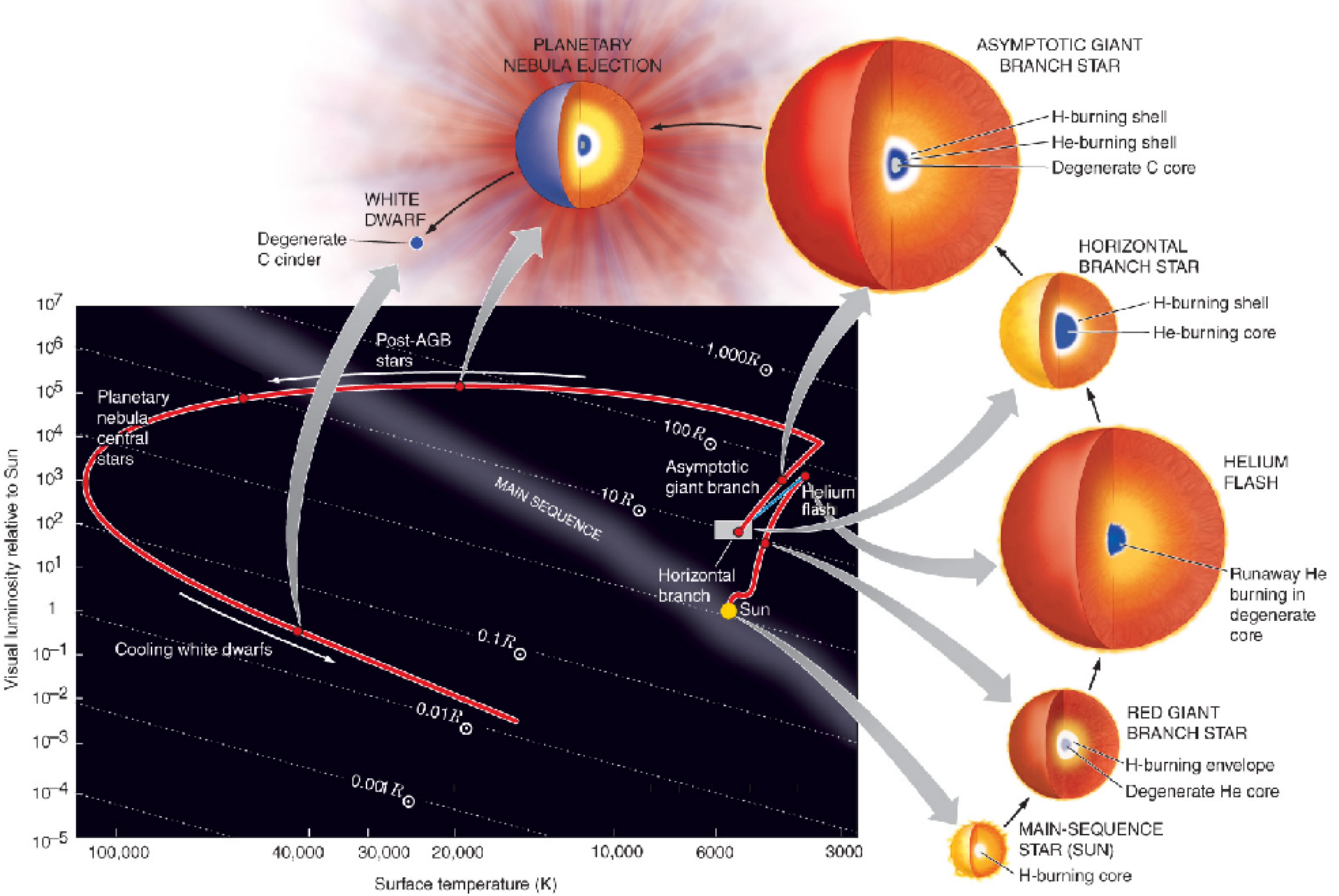


By comparing the distribution of cluster stars on the HR diagram and model isochrones, we can (1) fine-tune stellar evolution models, and (2) estimate age and chemical composition of clusters

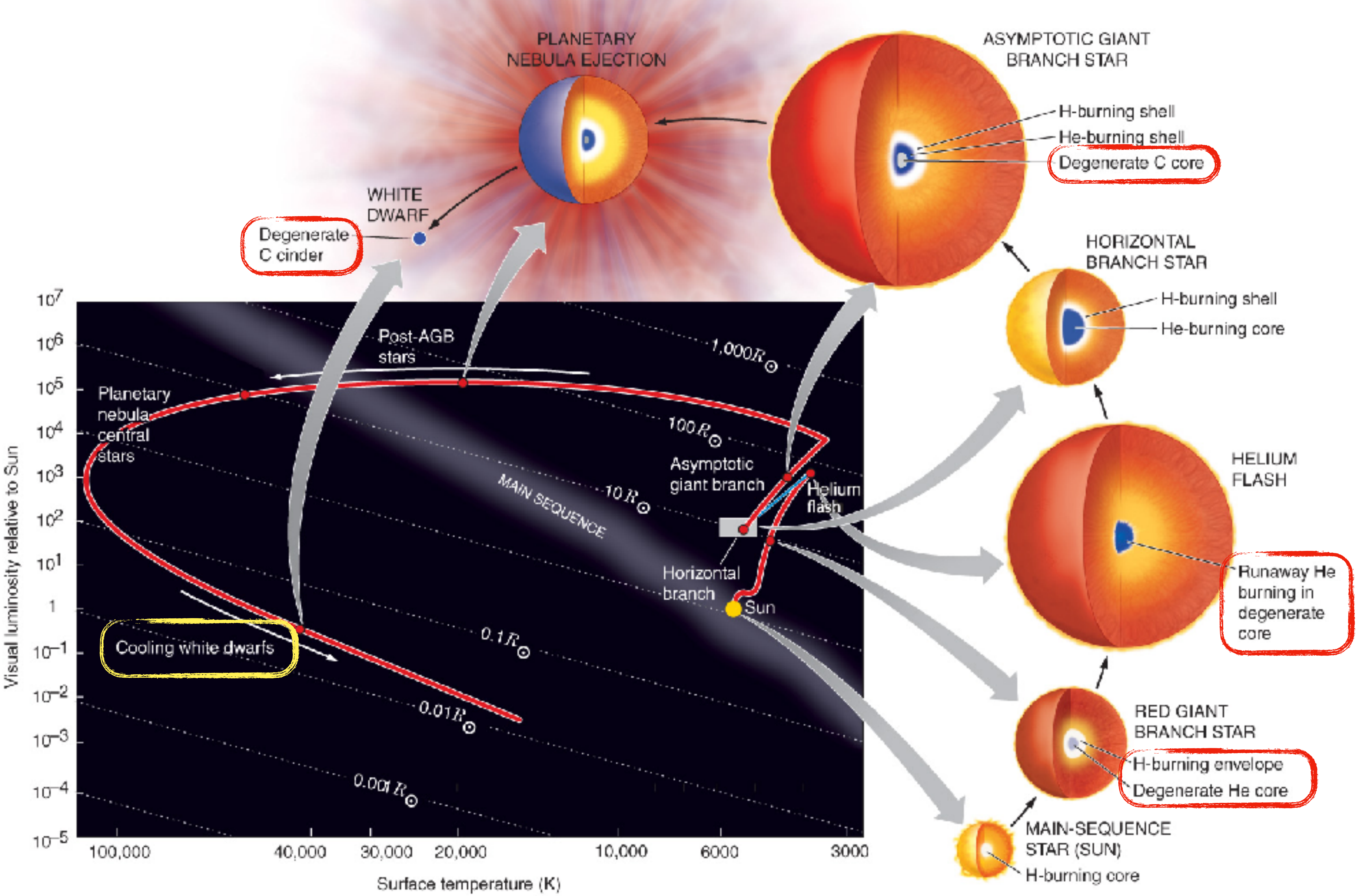
Dependency on metallicity: [Fe/H] = $\log N_{Fe}/N_H - \log (N_{Fe}/N_H)_{sun}$



Evolution track of the Sun predicted by numerical model

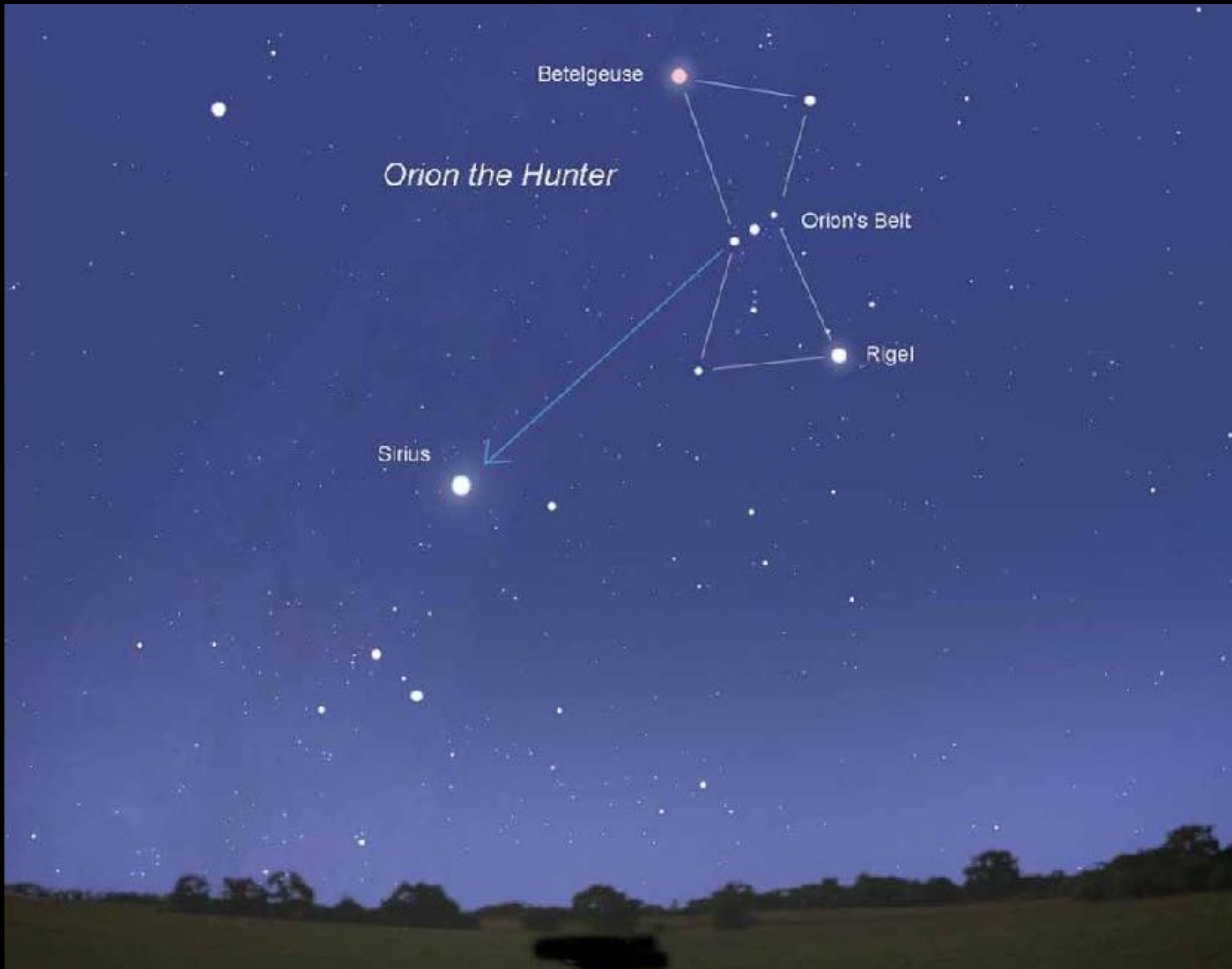


The post-MS core is degenerate in various stages



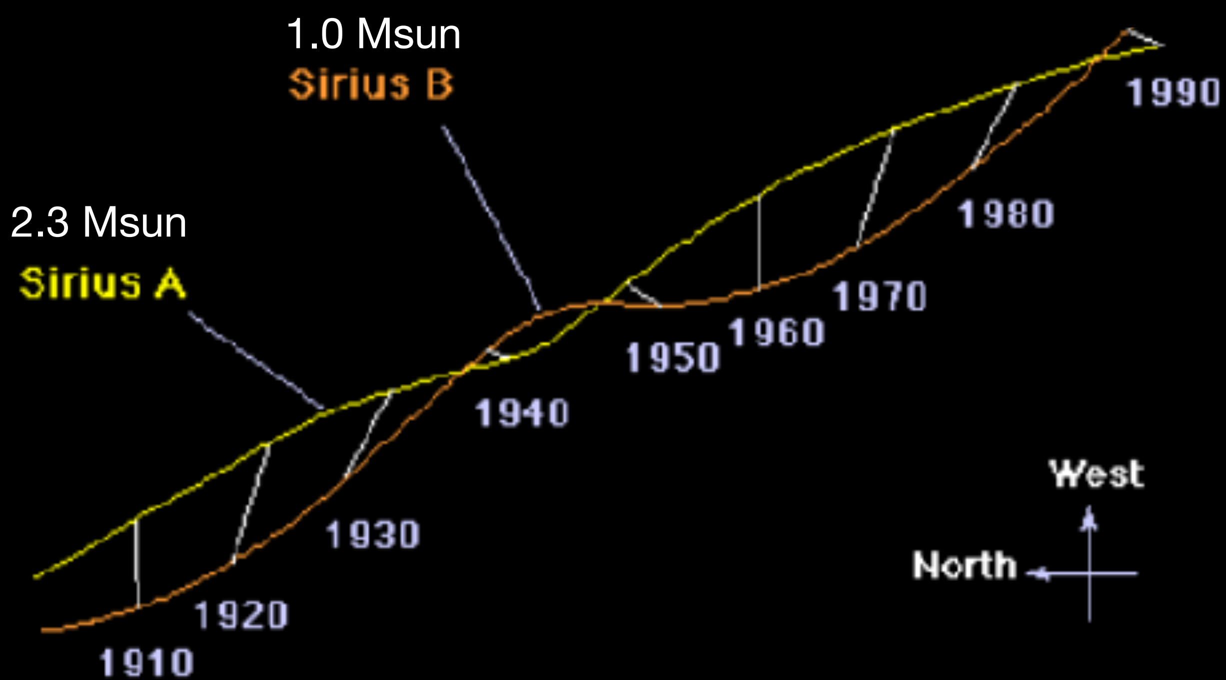
What is degeneracy?

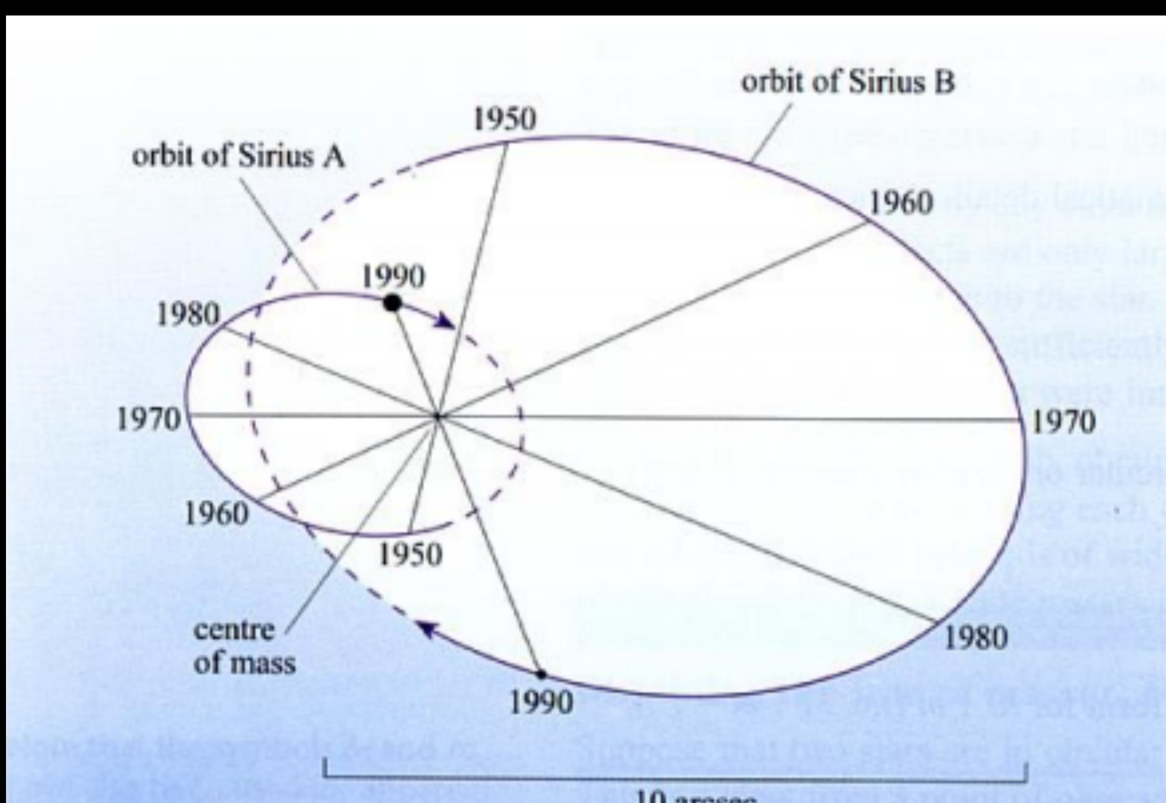
The story of degenerate gas started with the study of Sirius, the brightest star in the night sky



Sirius B - the "dark" companion of the Dog Star

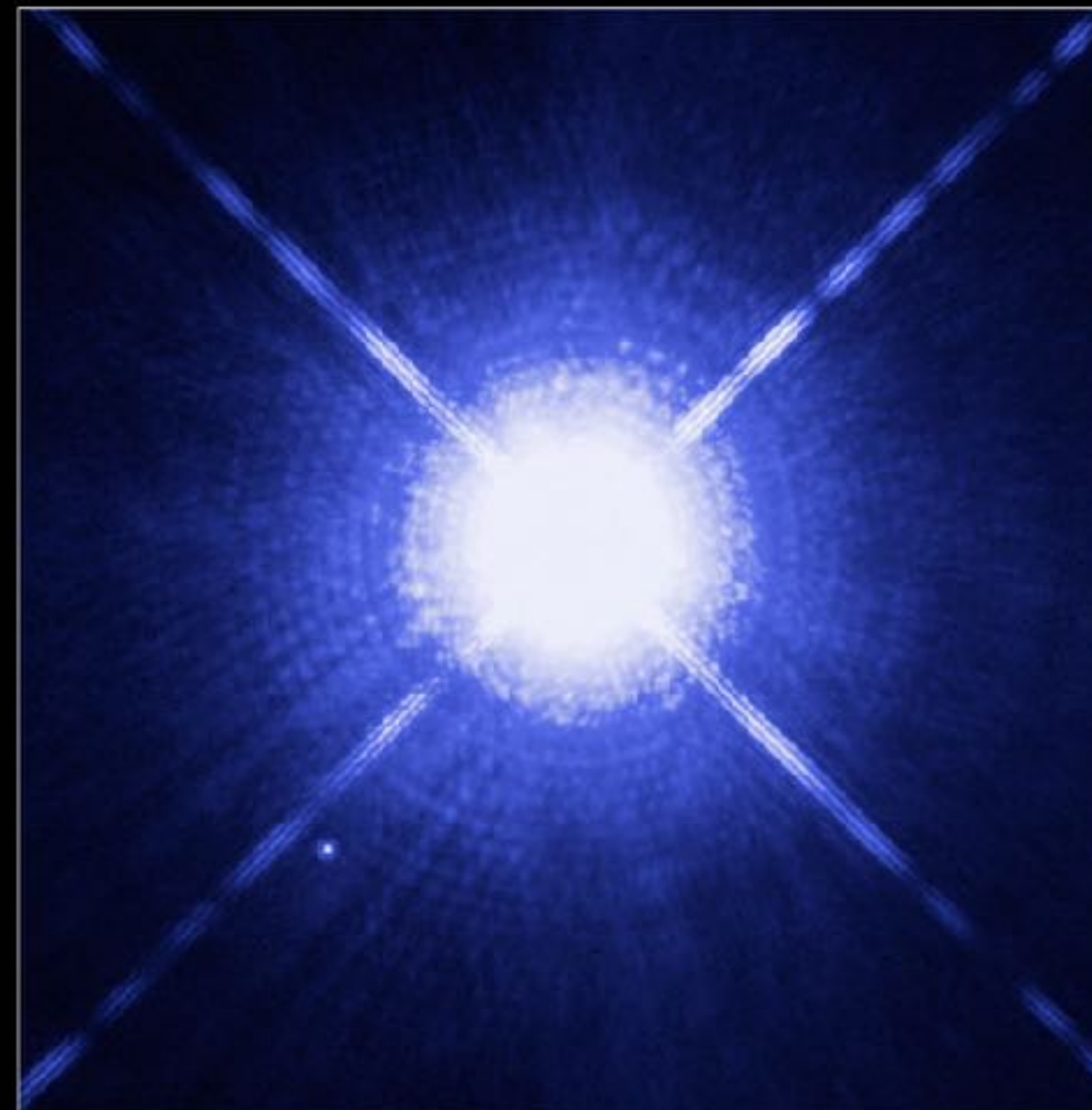
50 year orbit of the binary first inferred by Bessel in 1844
Bessel also measured the first stellar parallax (61 Cygni)





From the orbital motion, it was estimated that
A has 2.3 solar mass and B has 1.0 solar mass

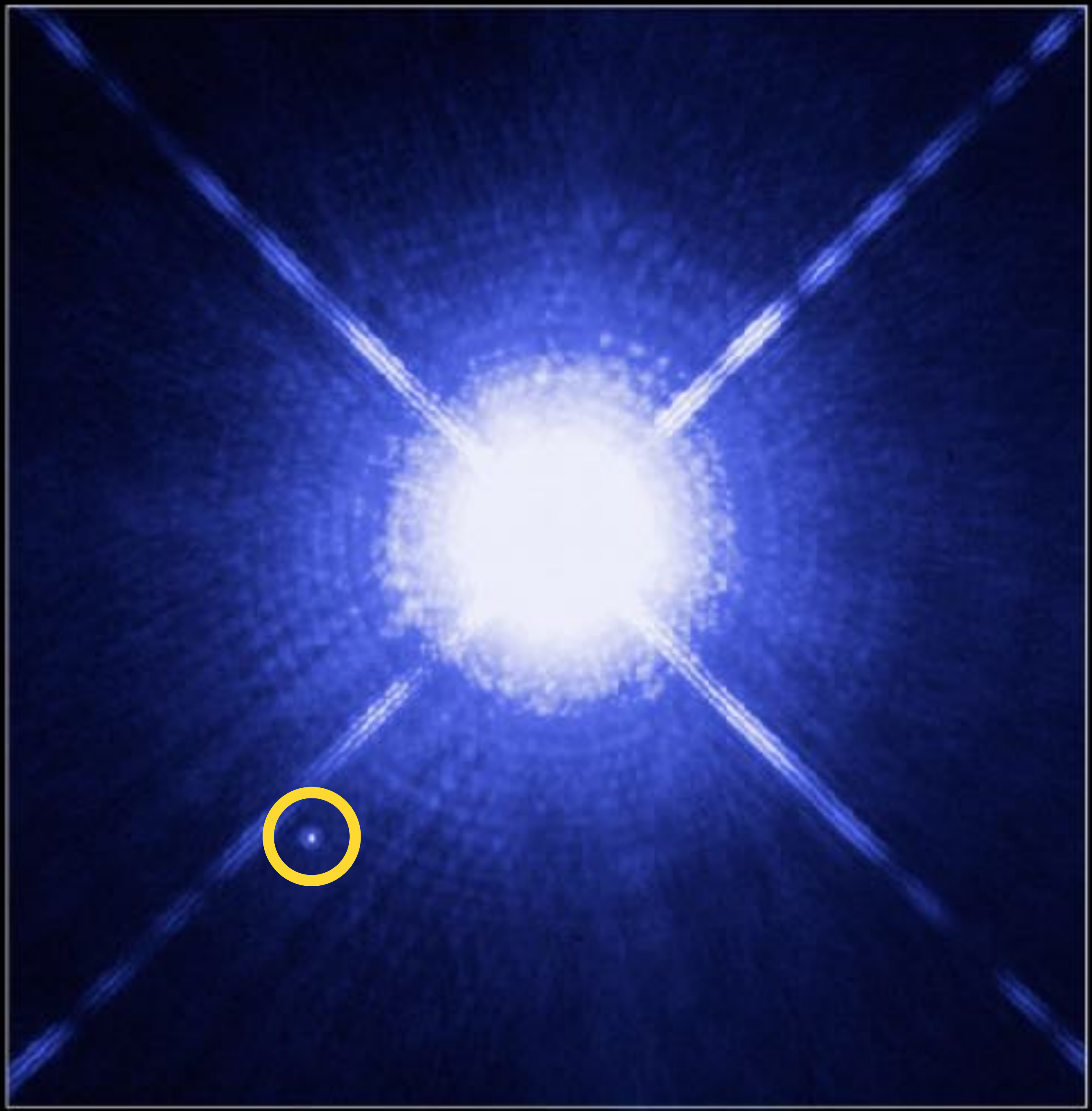
Sirius B is much fainter than Sirius A, is it surprising that if I tell you that it's much hotter (27000 K vs. 9900 K)?



Sirius A and Sirius B
Hubble Space Telescope • WFPC2

NASA, ESA, H. Bond (STScI), and M. Barstow (University of Leicester) STScI-PRC05-36a

Sirius B - the "dark" companion of the Dog Star



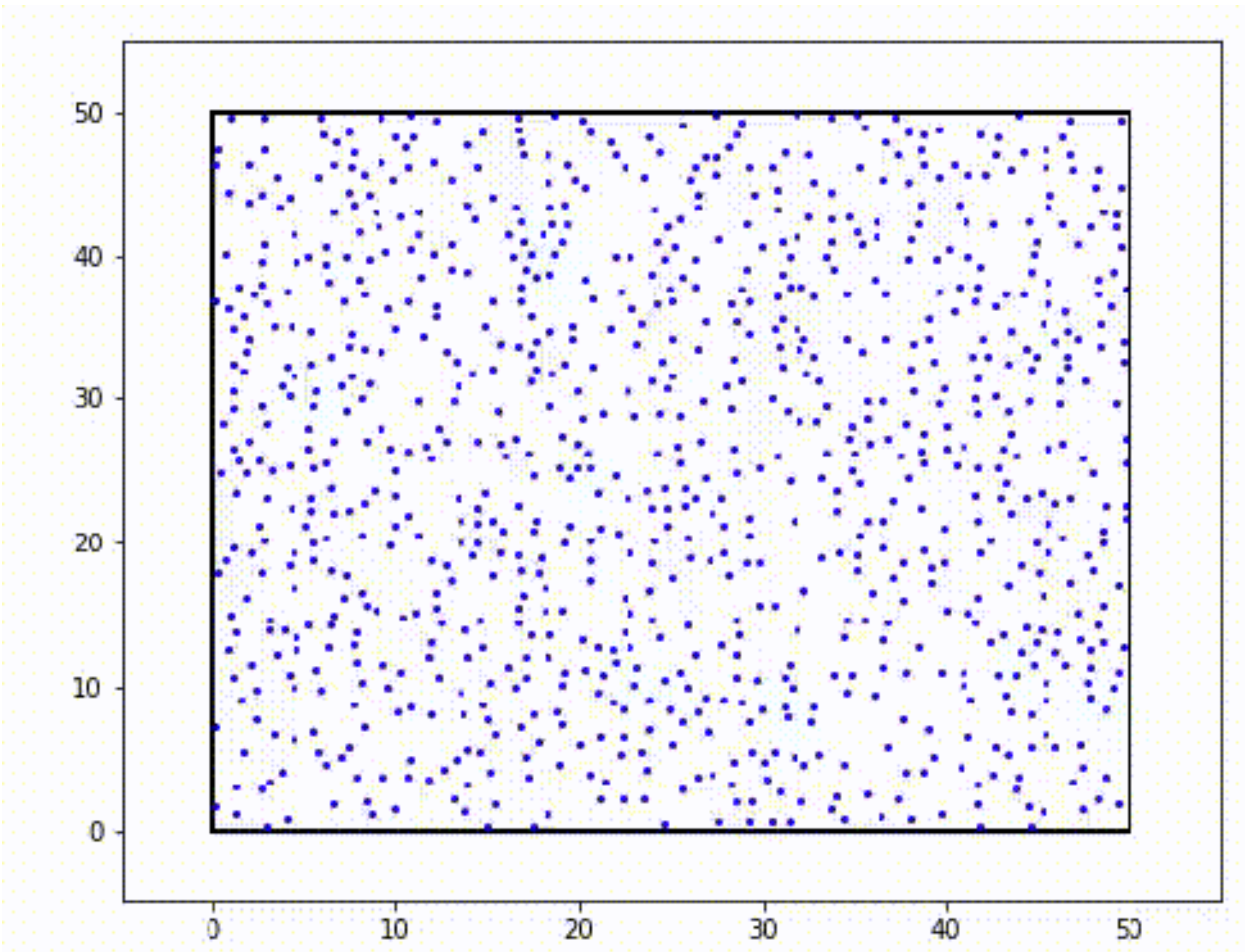
Sirius A and Sirius B
Hubble Space Telescope • WFPC2

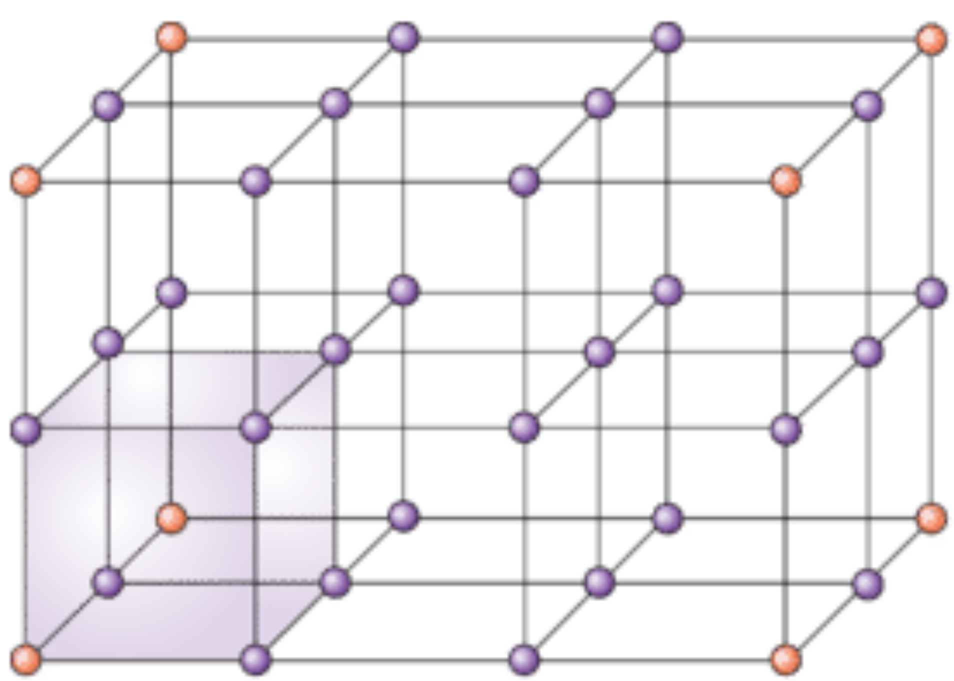
NASA, ESA, H. Bond (STScI), and M. Barstow (University of Leicester) STScI-PRC05-36a

- Inferred properties of Sirius B:
  - 1 Solar Mass
  - 0.03 Solar Luminosity
  - 27,000 K surface temperature
  - 5500 km radius (Earth-size)
- Sirius B represent a class of objects called **White Dwarfs (WDs)**

- The physical conditions of WDs are extreme:
  - extreme density ($\rho \approx$ 3e9 kg/m^3) ($n_e \sim$ 1e36 /m^3)
  - extreme surface gravity (HW)
  - extreme pressure at the center:
  - $P_c \propto GM^2/R^4$

Given a mass density, how to estimate the average distance between particles?





- Extremely high density of WDs:
  - mass density $\rho \approx$ 3e9 kg/m³
  - fully ionized carbon (12 & 6)
  - number density of electrons: $n_e \sim$ 1e36 /m³

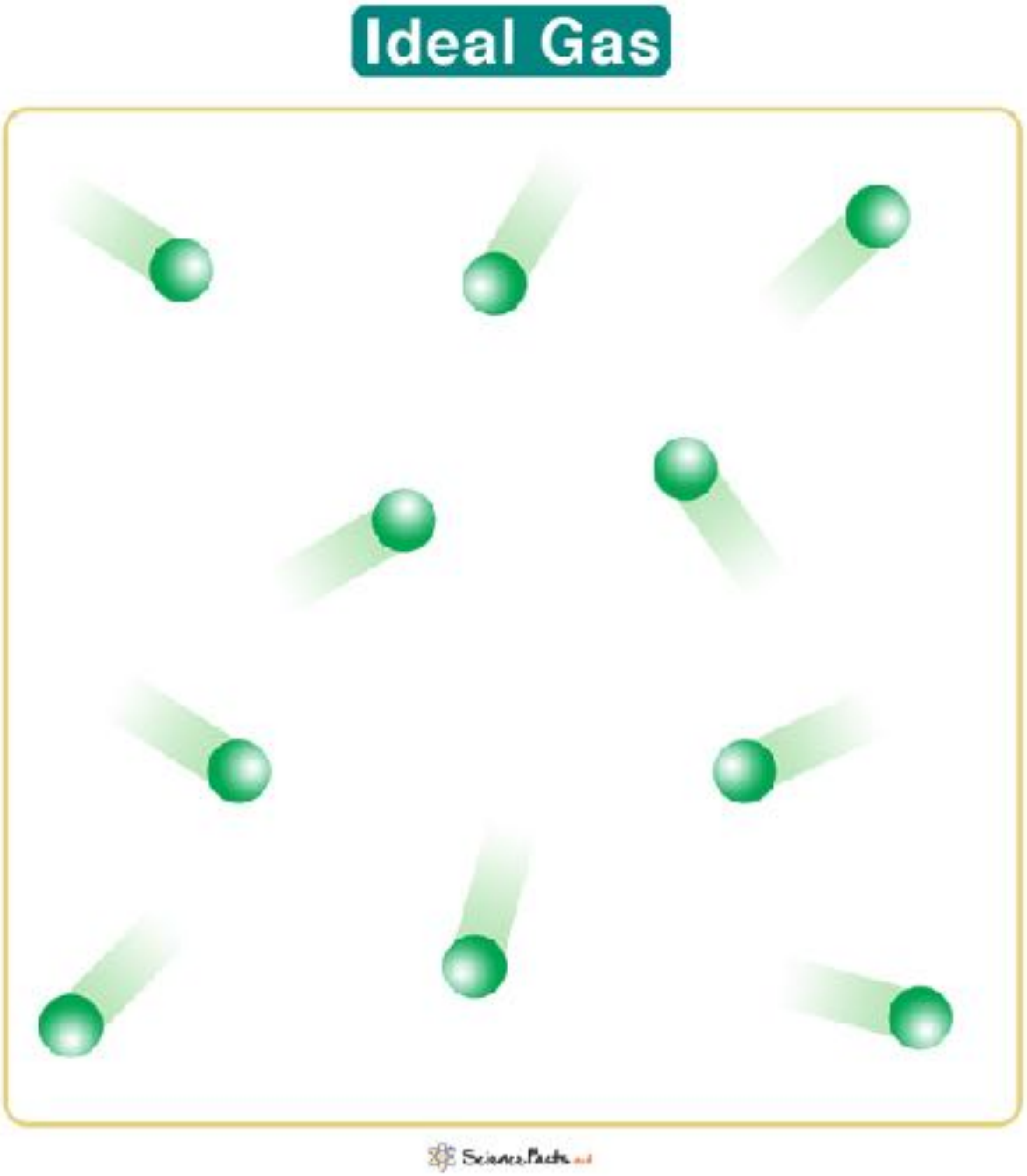
- What is the average distance between electrons? How does it compare with the size of an atom (~0.1 nm = 1e-10 m)?

- average distance: $\delta x = n_e^{-1/3} = 10^{-12} m$ (<< 0.1 nm, size of atom)

- Quantum Mechanics must be important in white dwarfs.

Pressure as kinetic energy density: Ideal Gas Case

- the pressure from ideal gas is 2/3 the **kinetic energy density**
  - Gven $P = nkT$, $K = mv^2/2 = p^2/(2m)$, $K = 3kT/2$
  - we have: $P = \frac{2}{3} n \frac{p^2}{2m}$



Deriving degenerate pressure using uncertainty principle

- **Heisenberg's uncertainty principle** (1927):

$$\Delta x \Delta p \approx h/2\pi$$

  the smaller the uncertainty in position, the larger the uncertainty in momentum.

- When packed very densely, the uncertainty of the electron's position cannot be larger than their actual separation:

$$\Delta x \lesssim n^{-1/3}$$

  because they are mutually exclusive given **Pauli exclusion principle** (1925): identical **fermions** can occupy the same quantum state.

- Combining the two and approximating $p \approx \min(\Delta p)$ (ground state):

$$p \approx h n^{1/3}/2\pi$$

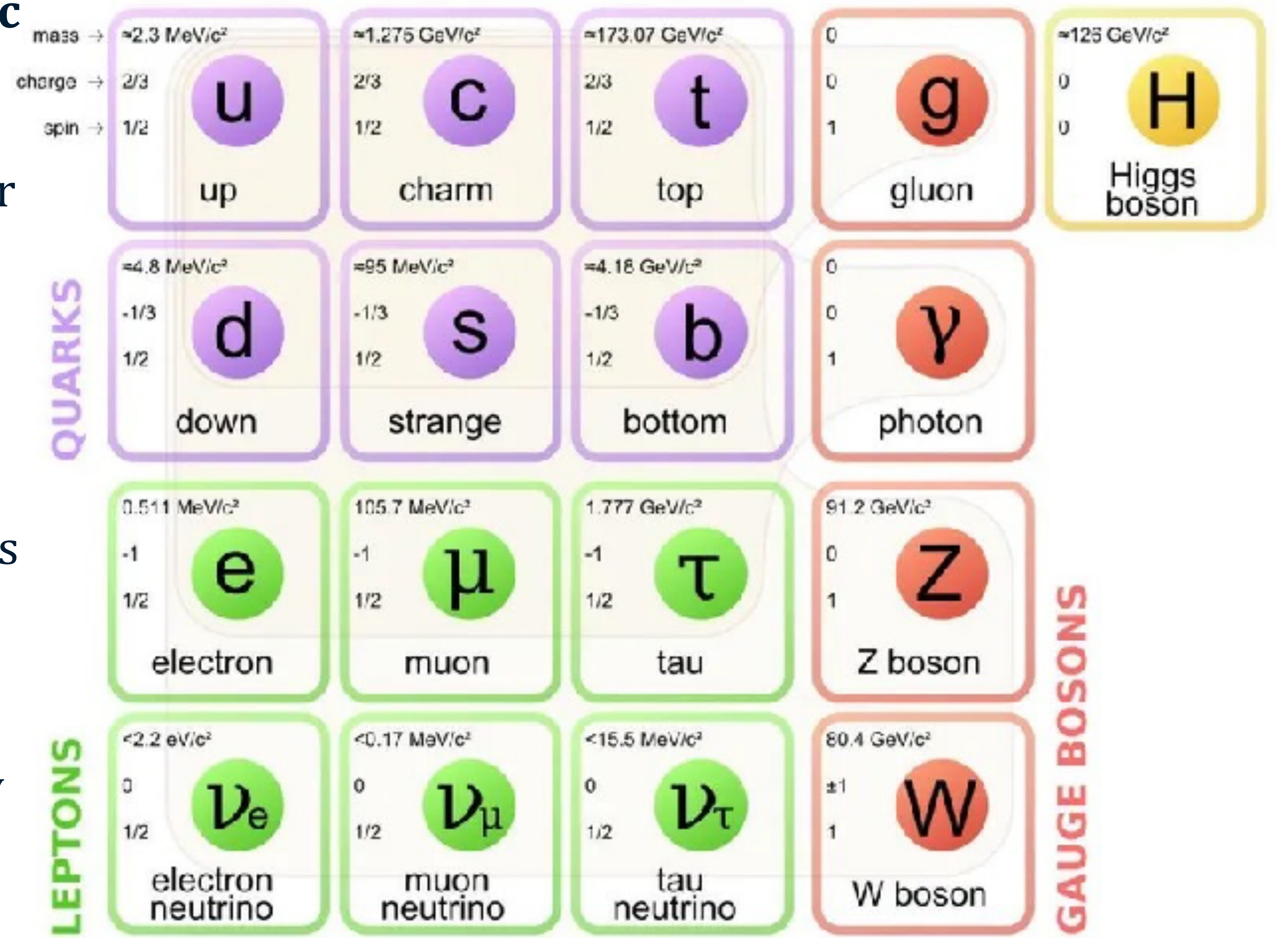
- Just like ideal gas, the pressure from degenerate gas is the kinetic energy density:

$$P_{\text{degen}} \approx n\frac{p^2}{m} = \frac{h^2}{4\pi^2}\frac{n^{5/3}}{m}$$

Fermions vs. Bosons

- **Fermions** are subatomic particles with **half-integer spin** following the **Fermi-Dirac distribution** and the **Pauli exclusion principle**, forcing them to form structured matter rather than condensing. They constitute all ordinary matter, including **electrons, protons, and neutrons**.

- **Bosons** are subatomic particles that act as **force carriers**, characterized by having an **integer spin** and following **Bose-Einstein statistics**. They mediate fundamental forces like electromagnetism (**photons**) and the strong force (**gluons**).



The Condition for Degeneracy: Pressure Comparison

- The pressure from non-relativistic degenerate gas is:

$$P_{\text{degen}} \approx \frac{h^2}{4\pi^2}\frac{n^{5/3}}{m}$$

- The pressure from ideal gas is:

$$P_{\text{ideal}} = \frac{2}{3}n\left(\frac{3}{2}kT\right) = nkT$$

- The condition for degeneracy is simply:

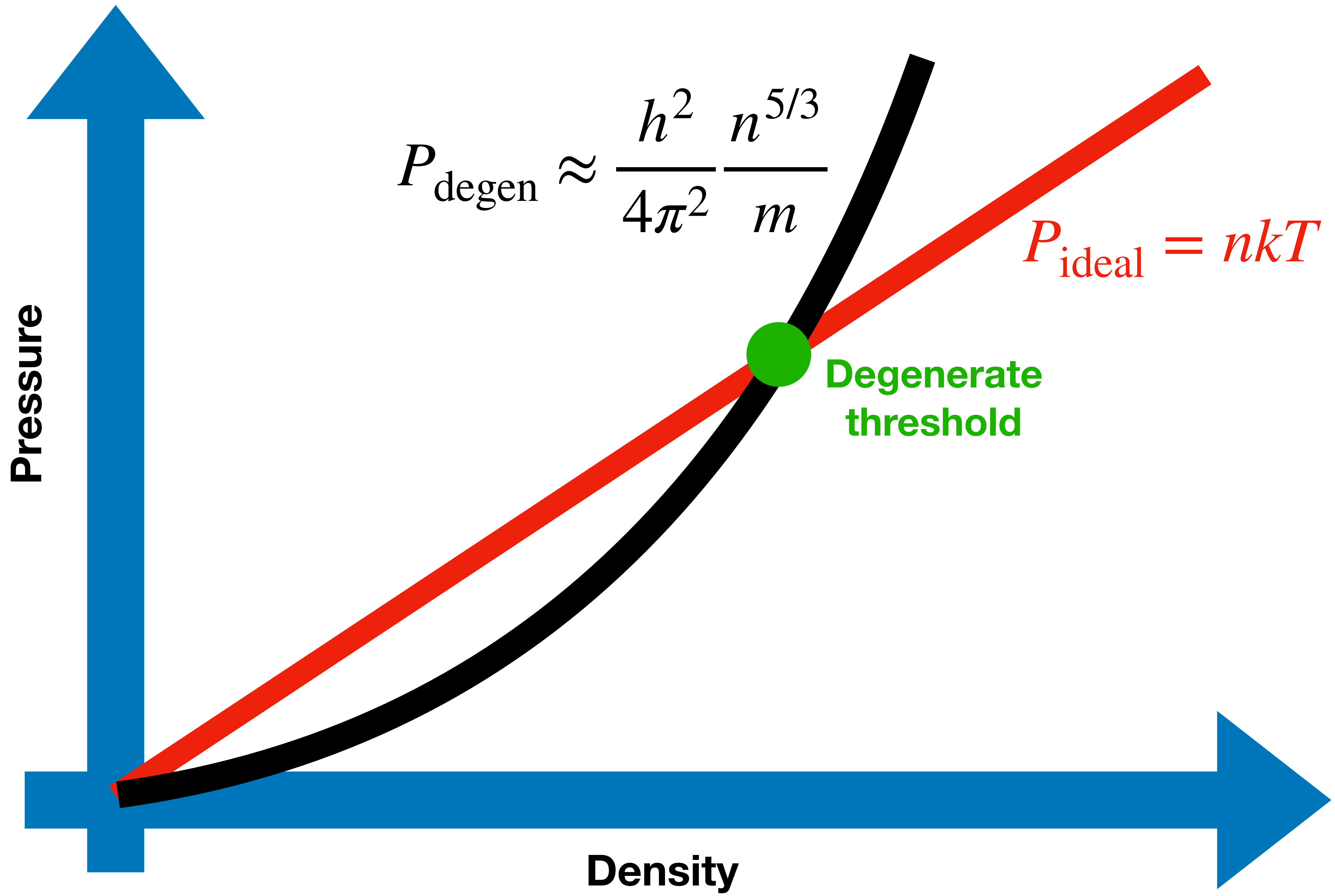
$$P_{\text{degen}} > P_{\text{ideal}}$$

which can be simplified as:

$$\frac{h^2}{4\pi^2}\frac{n^{2/3}}{m} > kT$$

What is degenerate pressure? How to understand it intuitively?

- The contraction of non-fusing cores packs a large amount of mass into a small volume. Each electron finds its position well constrained, which leads to large momentum and kinetic energy due to the **uncertainty principle**. **Pressure**, as the **density** of **kinetic energy**, increases rapidly as a result of increased **(1) number density** and **(2) kinetic energy per particle**. This pressure due to quantum mechanics is called **degenerate pressure**.



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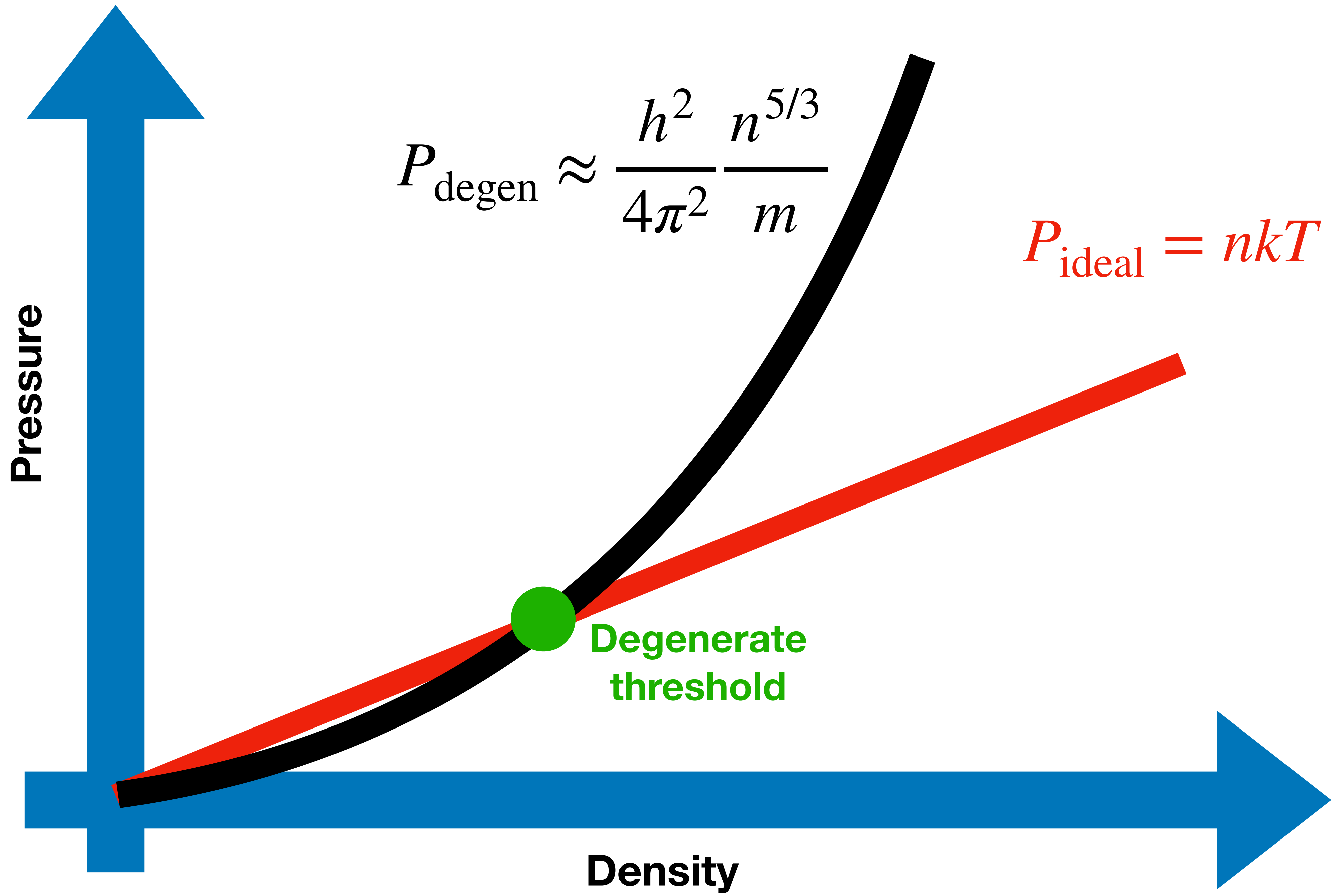


The Condition for Matter to Reach Degeneracy

- In a previous slide, we derived the condition for degeneracy:

$$\frac{h^2}{4\pi^2}\frac{n^{2/3}}{m} > kT$$

- The above condition can be satisfied when either:
  - **the temperature is very low**, or
  - **the density is very high**
- In the non-fusing core of a star, the density is extremely high, reaching degeneracy condition.
- *Now think about this: If* ***ions*** *and* ***electrons*** *share the same temperature in the core, which component will reach degeneracy first?*

Understanding Degenerate Condition using Fermi-Dirac Distribution

- **Fermions** follow **Fermi-Dirac distribution**, where the probability of a particle having an energy between *E* and *E+dE* is:

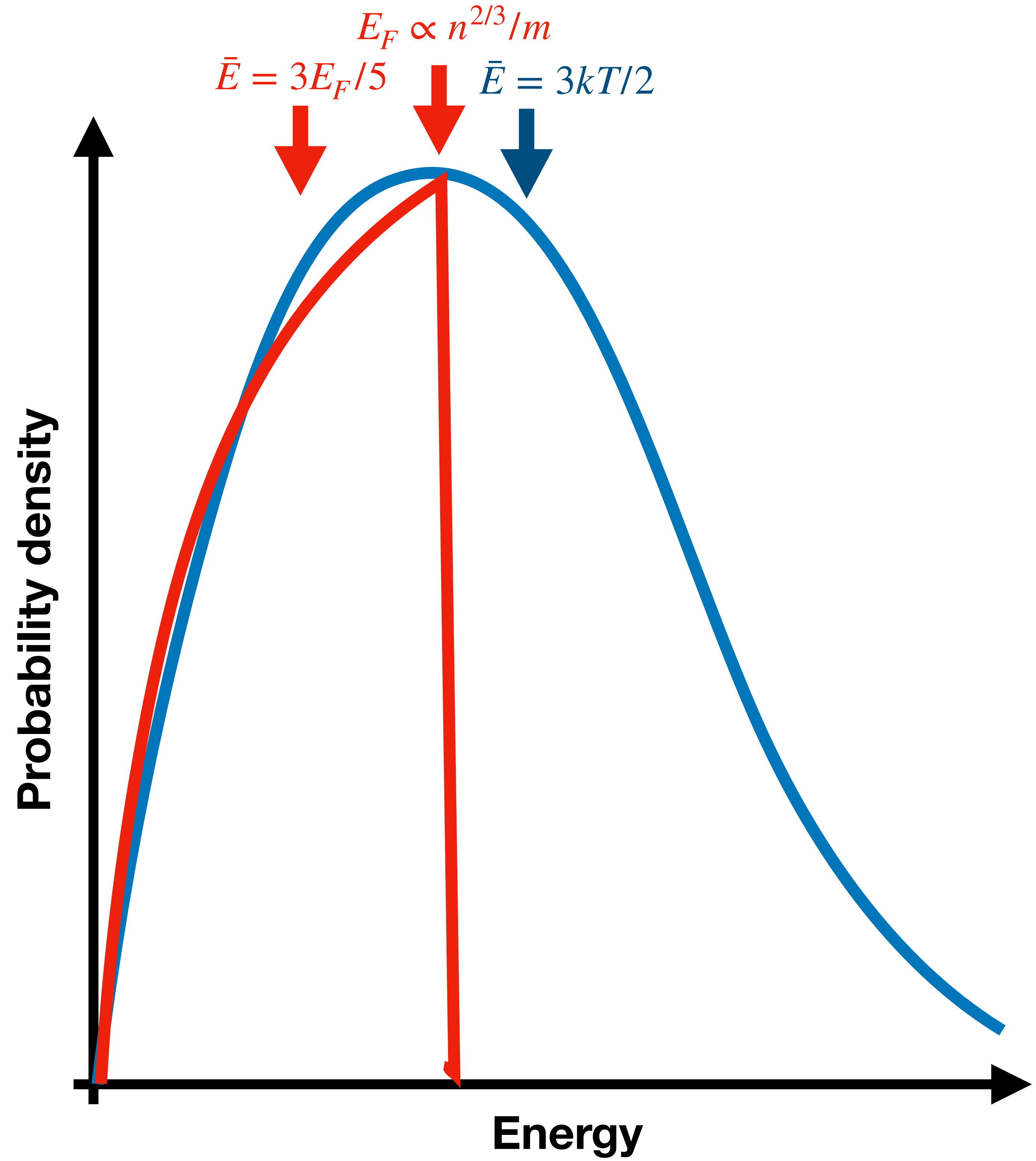
$$f_E dE \propto \frac{1}{e^{(E-E_F)/kT}+1} dE$$

where $E_F = \frac{h^2}{8m}\left(\frac{3n}{\pi}\right)^{2/3}$

- when $kT >> E_F$, it is indistinguishable from the classic Maxwell-Boltzmann distribution: $f_E dE \propto \frac{1}{e^{E/kT}} dE$

  The *mean kinetic energy* is $3kT/2$, as a result, pressure depends on **both density and temperature**

- when $kT << E_F$, all energy states greater than the Fermi energy become **unoccupied**. As a result, the *mean kinetic energy* is dictated by Fermi energy: $3E_F/5$, and **pressure depends only on density.**



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- The condition for degeneracy is simply:

$$P_{\text{degen}} > P_{\text{ideal}}$$

which can be simplified as:

$$\frac{h^2}{4\pi^2}\frac{n^{2/3}}{m} > kT$$

or equivalently (expressed using Fermi energy):

$$kT \lesssim 0.2\ E_{\text{F}} \quad \text{since} \quad E_F = \frac{h^2}{8m}\left(\frac{3n}{\pi}\right)^{2/3}$$

Mass-Radius Relation
&
the Chandrasekhar Limit

Core Pressure Scaling Relation from Hydrostatic Equilibrium

We start with the fundamental equation of hydrostatic equilibrium, which describes the pressure gradient needed to counteract gravity at any radius r inside the star:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Substitute these scales into the hydrostatic equation:

$$\frac{P_c}{R} \propto \frac{G \cdot M \cdot \left(\frac{M}{R^3}\right)}{R^2}$$

Simplify the right side:

$$\frac{P_c}{R} \propto \frac{GM^2}{R^5}$$

Understanding the Mass-Size relation of white dwarfs with degenerate pressure and hydrostatic equilibrium

- The pressure from non-relativistic degenerate gas is:

$$P_{\text{degen}} = \frac{h^2}{4\pi^2}\frac{n^{5/3}}{m} \propto \left(\frac{M}{R^3 \mu m_H}\right)^{5/3}$$

- Hydrostatic equilibrium provides an estimate of the central pressure:

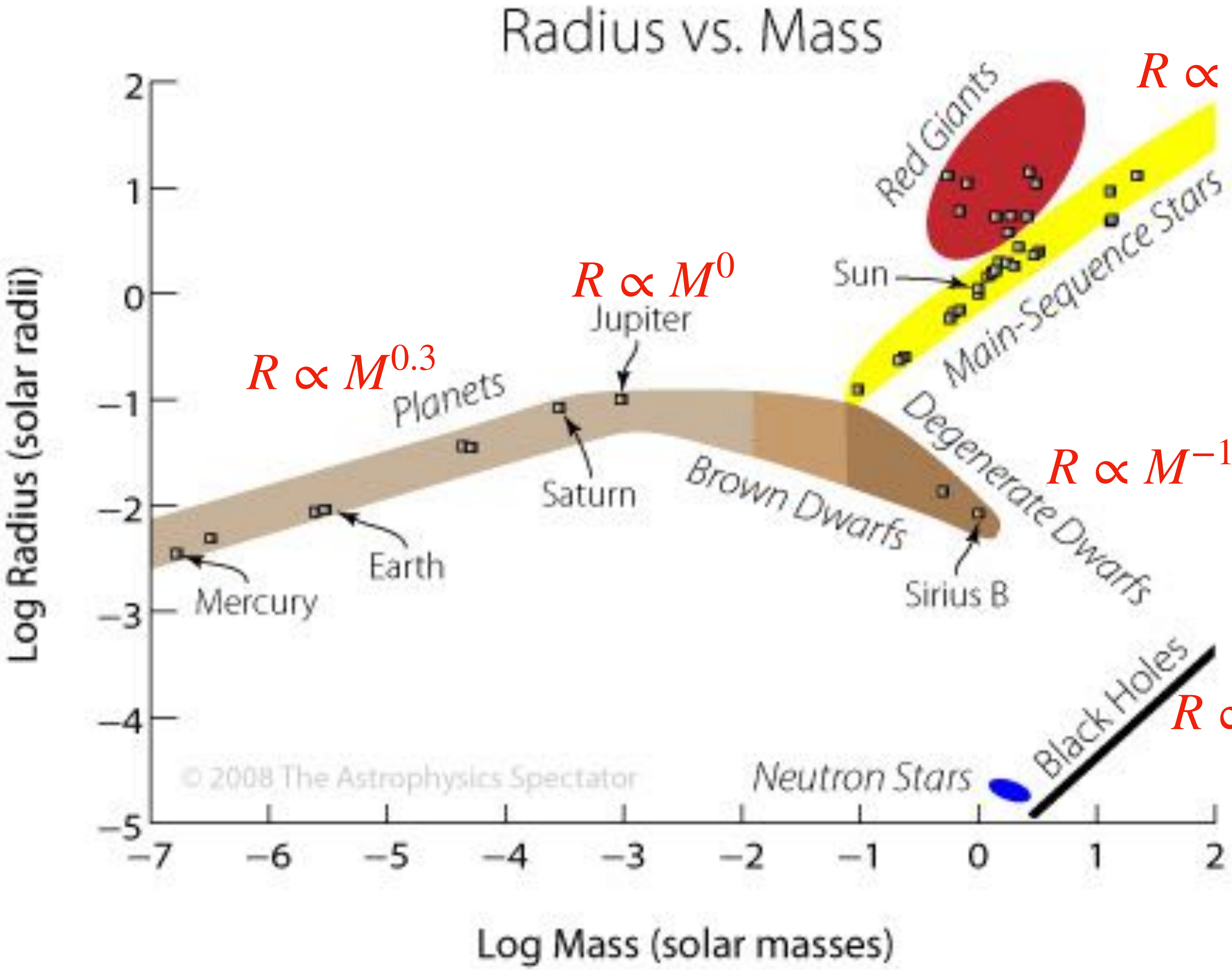
$$P_{\text{c}} \propto \frac{GM^2}{R^4}$$

- If degenerate gas provide the central pressure, we can equate the two and solve for the Mass-Radius relation:

$$\frac{M^{5/3}}{R^5} \propto \frac{M^2}{R^4} \Rightarrow R \propto M^{-1/3}$$

- This is in contrast to main sequence stars, where **$R \sim M^{0.7}$**

Mass-Radius Relations from Planets, Stars, to Black Holes



$R \propto M^{0.8}$

$R \propto M^{0}$

$R \propto M^{0.3}$

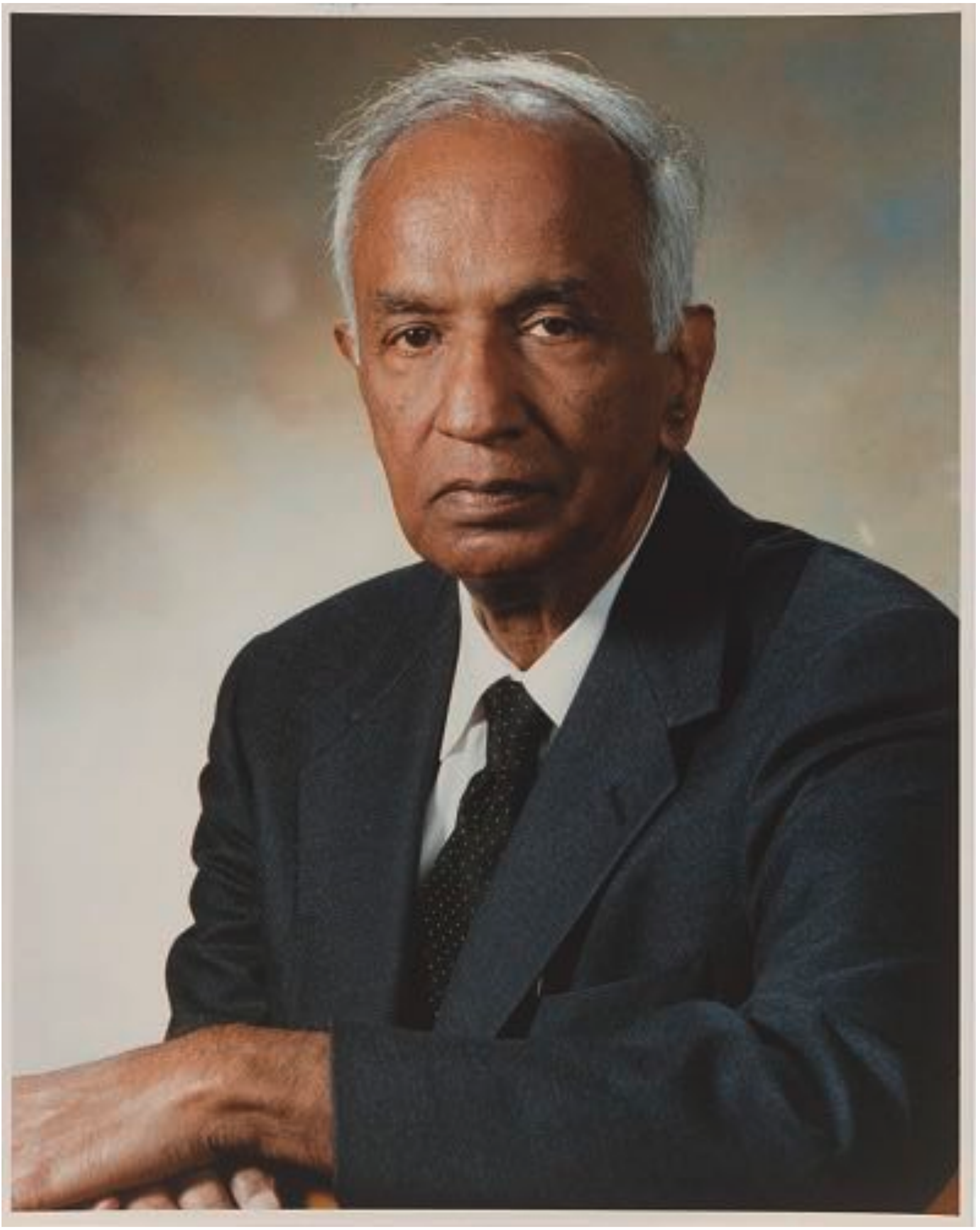
$R \propto M^{-1/3}$

$R \propto M$

Subrahmanyan Chandrasekhar (1910-1995)

Subrahmanyan Chandrasekhar was an Indian-born American astrophysicist who, with William A. Fowler, won the 1983 Nobel Prize for Physics.

He discovered the **mass limit** of white dwarfs **at age 19 during a sea voyage from India to England** while en route to begin graduate studies at Cambridge.



The Chandrasekhar Limit of White Dwarfs

- Degeneracy pressure is not infinitely powerful, eventually it breaks when degenerate particles become **relativistic ($v \to c$)**
- For relativistic particles: $E^2 = p^2c^2 + (mc^2)^2 \approx p^2c^2$
- So the energy density is: $u = n(pc) = nc\dfrac{h}{2\pi\delta x} \approx hcn^{4/3}/2\pi$
- The pressure from relativistic degenerate electron gas is:

$$P_{\text{degen}} = \frac{1}{3}u = \frac{hc}{6\pi}n^{4/3} \propto \left(\frac{M}{R^3}\frac{Z}{Am_p}\right)^{4/3}, \text{ where } A = 12, Z = 6$$

- Hydrostatic equilibrium provides an estimate of the central pressure:

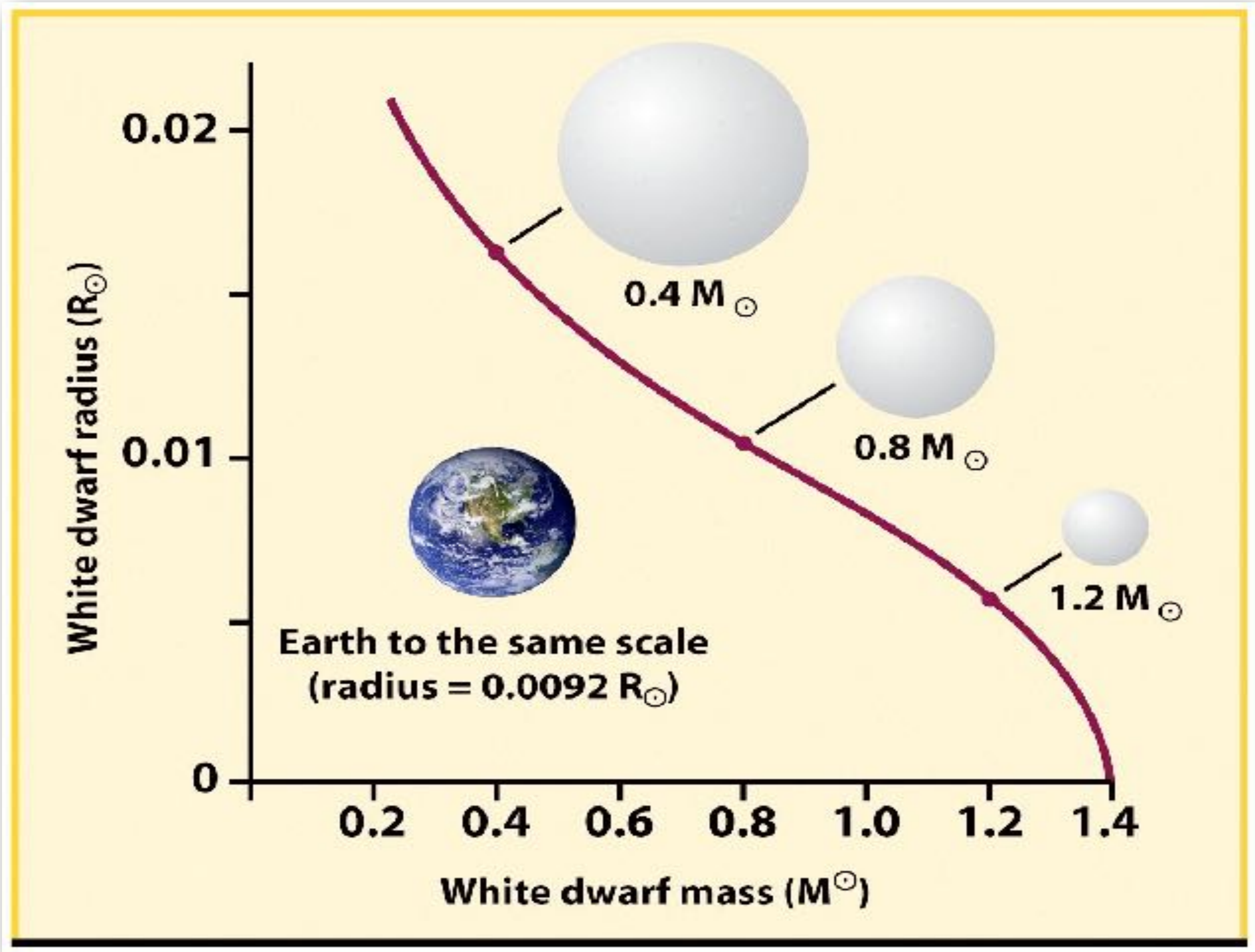
$$P_{\text{c}} \propto G\frac{M^2}{R^4}$$

- If degenerate gas provide the central pressure, we can equate the two, which leads **NOT** to a Mass-Radius relation, but **a mass of the star**:

$$M = \left(\frac{h^3c^3}{8\pi^3G^3m_p^4}\right)^{1/2} \approx 1.4M_\odot$$

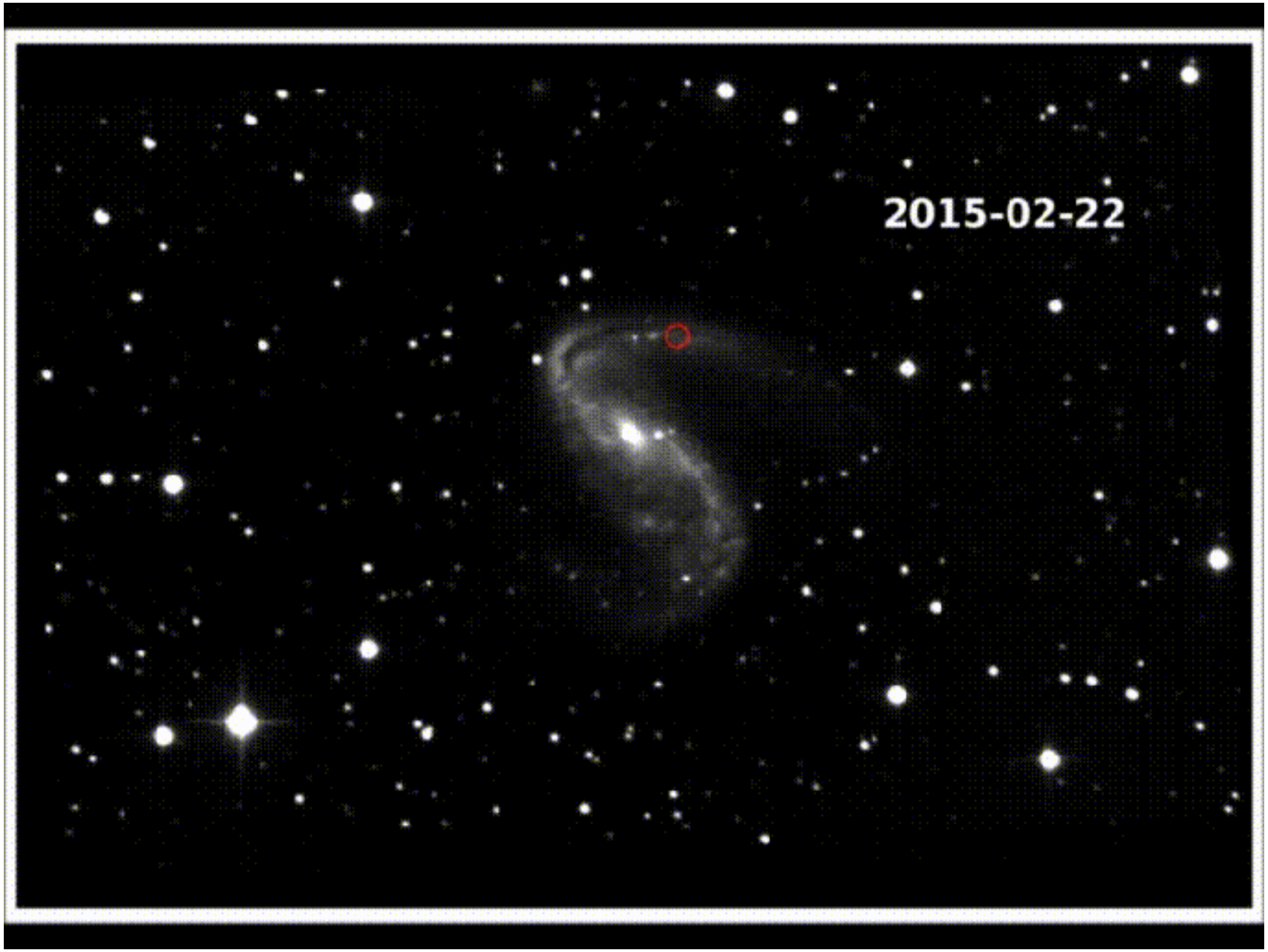
The Chandrasekhar Limit of White Dwarfs

- Degeneracy pressure at **relativistic limit** ($p \approx hn^{1/3}/2\pi \rightarrow mc \Rightarrow P \propto n^{4/3}$ **)** places a limit on the **maximum mass** of the white dwarf, the **1.4 Solar Mass (the Chandrasekhar limit)**
- The collapse of a WD at 1.4 Solar Mass results in **uncontrollable** Carbon fusion (**Type Ia SNe**)



The Standard Candle Method — Type Ia SNe

- **Type Ia supernovae (SNe Ia)** have been used as standard candles to measure cosmological distances to other galaxies.
- They work as standard candles because presumably the white dwarfs have to reach **1.44 solar mass (the Chandrasekhar mass)** to trigger the thermonuclear explosion, reaching a peak absolute magnitude of $\mathbf{M_V = -19}$.



Type Ia Supernova is a thermonuclear explosion!

X-ray Image of the Remnant of Tycho's Supernova: SN 1572

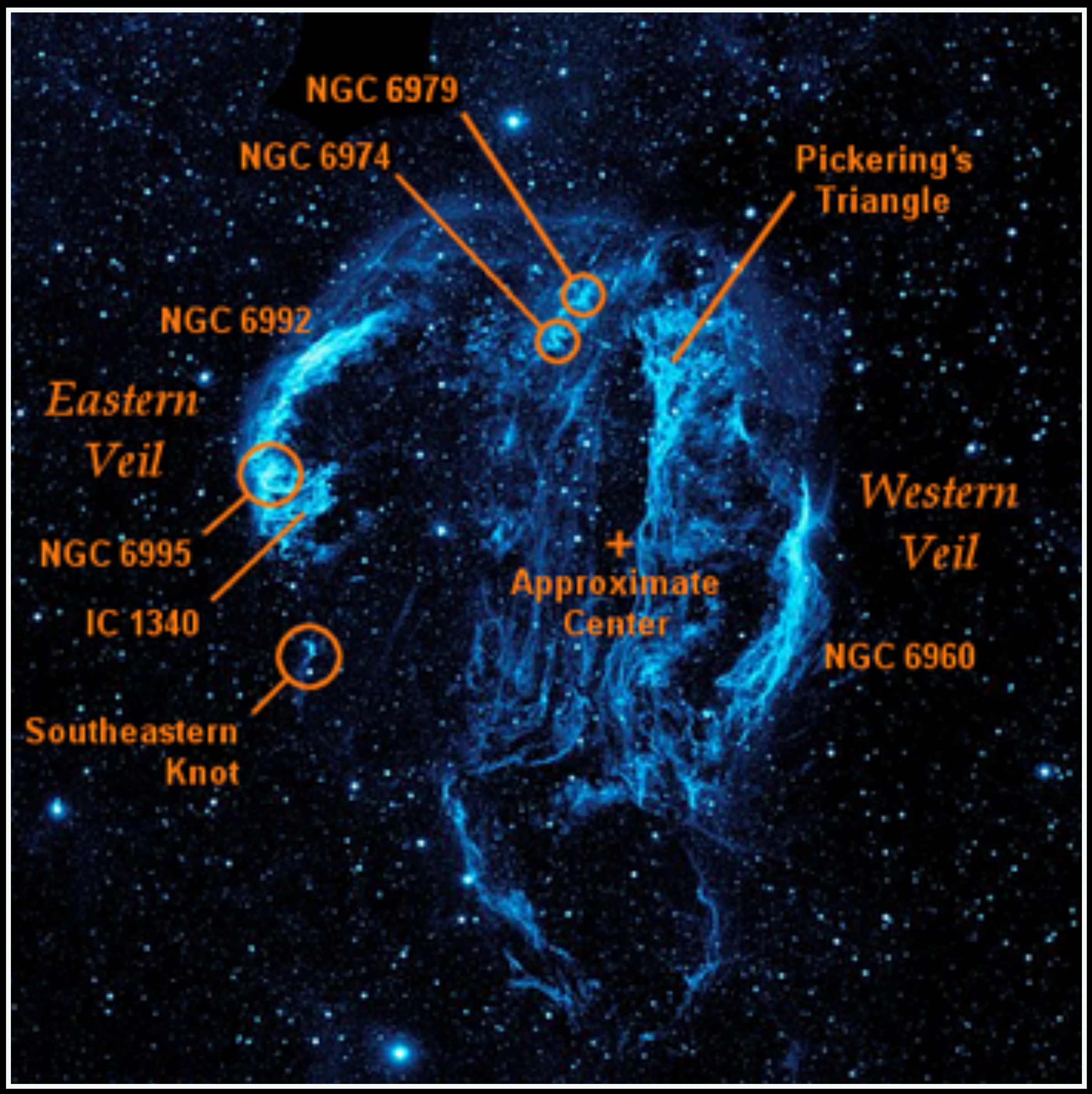
Sonic boom from a volcanic explosion

Sonic boom from a supersonic fighter jet

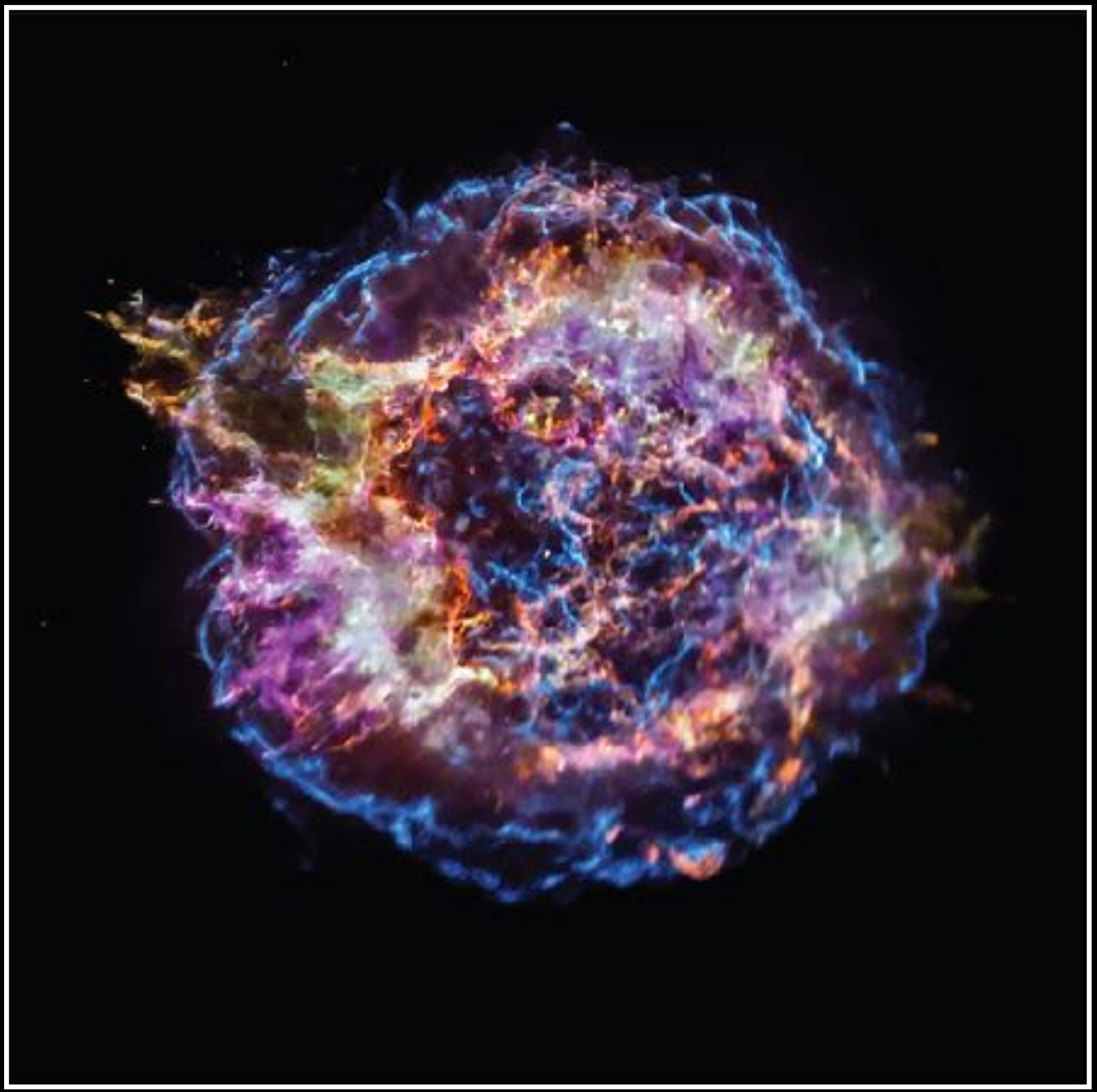
How to Measure the Age of a Supernova Remnant?

When we see a SN remnant, how could we measure the age of the remnant and thus estimate when the SN exploded?

Veil Nebula



Cassiopeia A



Detecting the Expansion of SN remnants

How to associate a remnant with a supernova in the past?

Expansion of the Crab Nebula

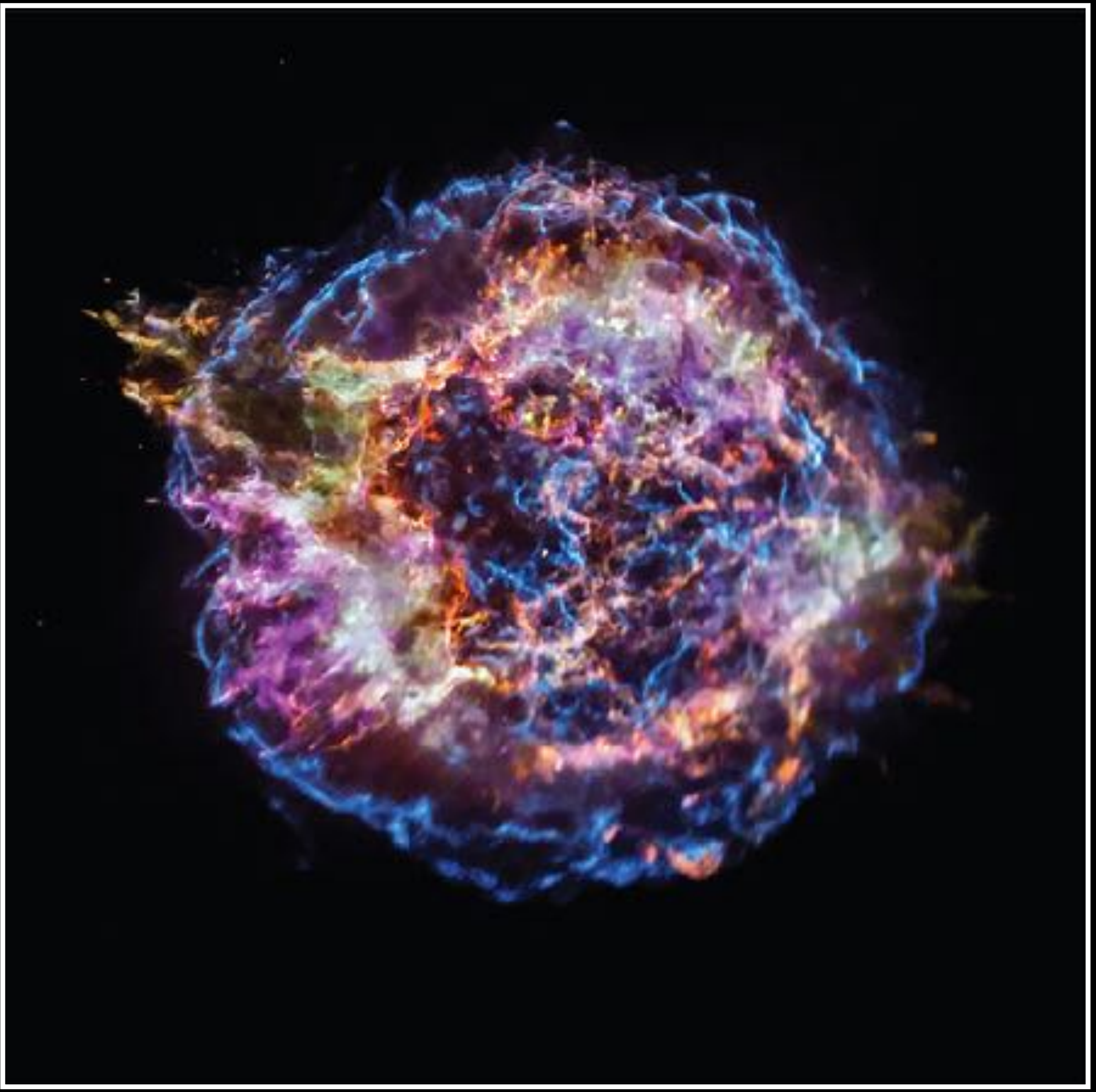
Years 1999 and 2012

Estimating Age from Angular Expansion Rate (assuming constant angular expansion rate)

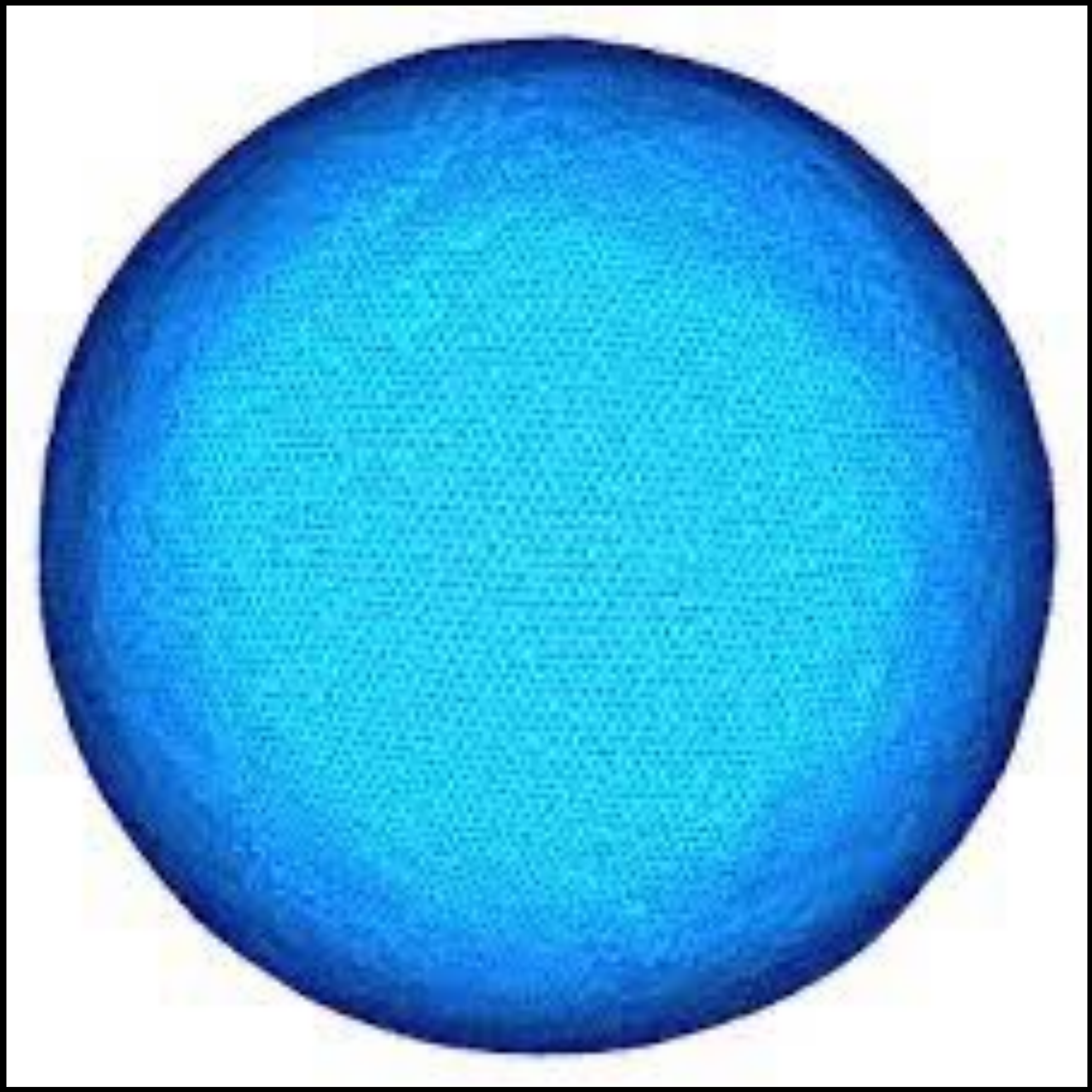
Age = Angular Size / Angular Expansion Rate

$$t = \theta/\dot{\theta}$$

Data: Cassiopeia A

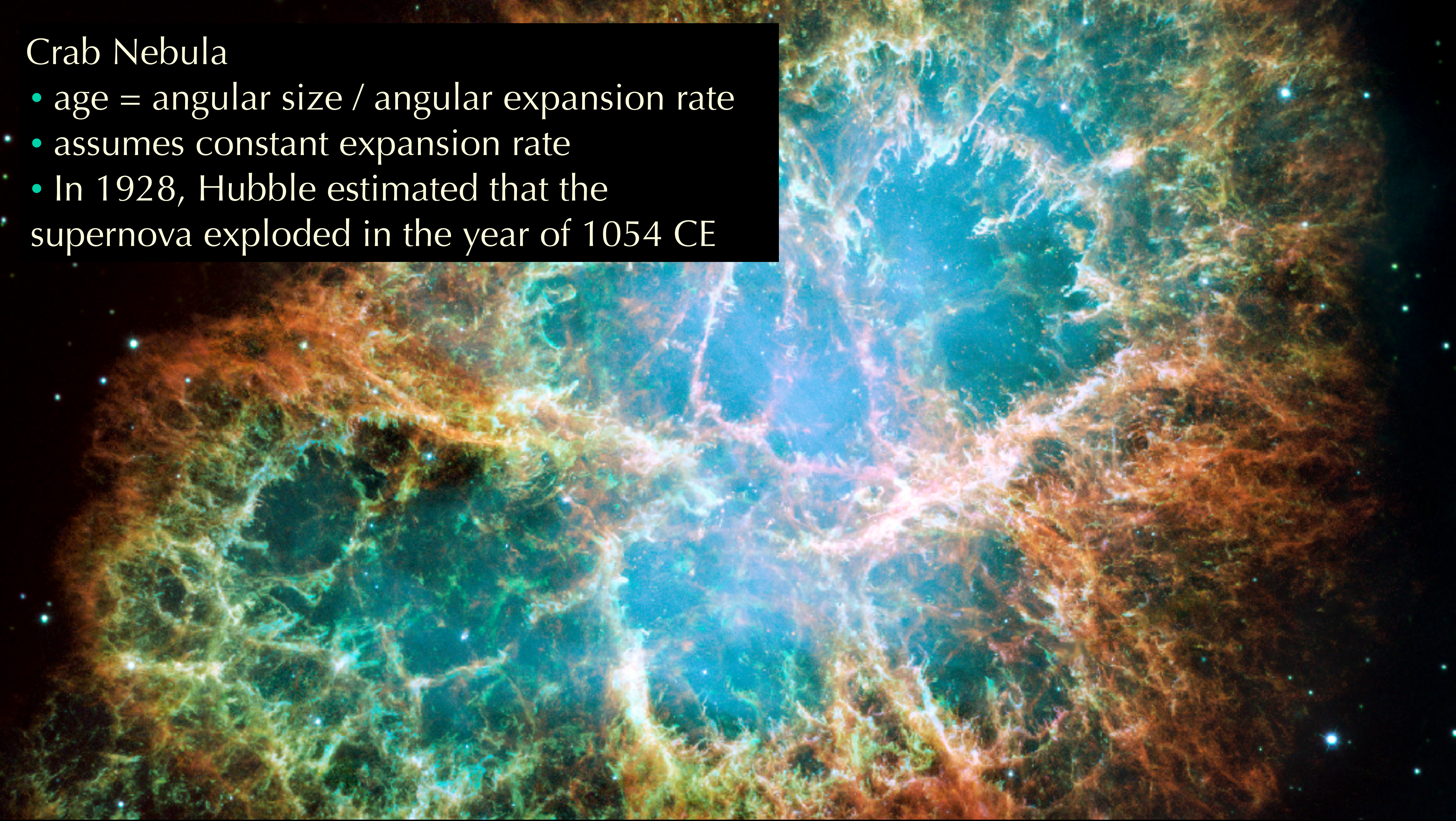


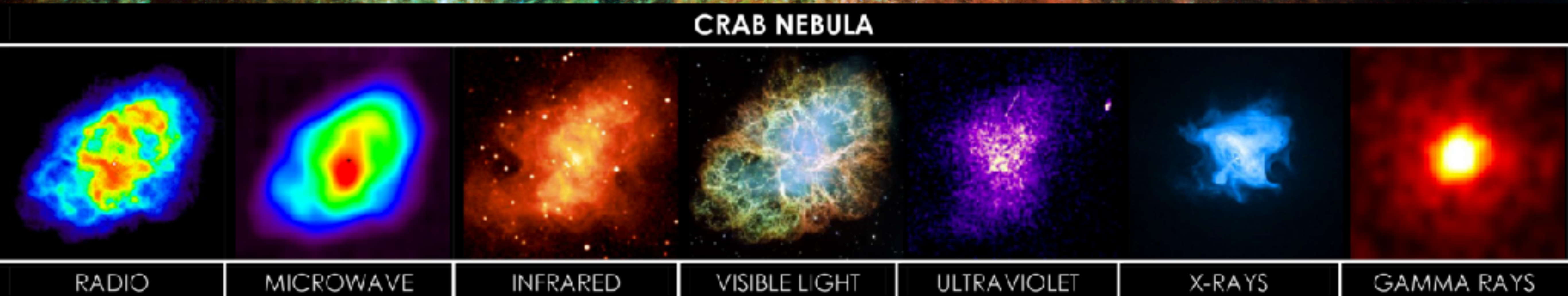
Model: spherical expansion of a shell



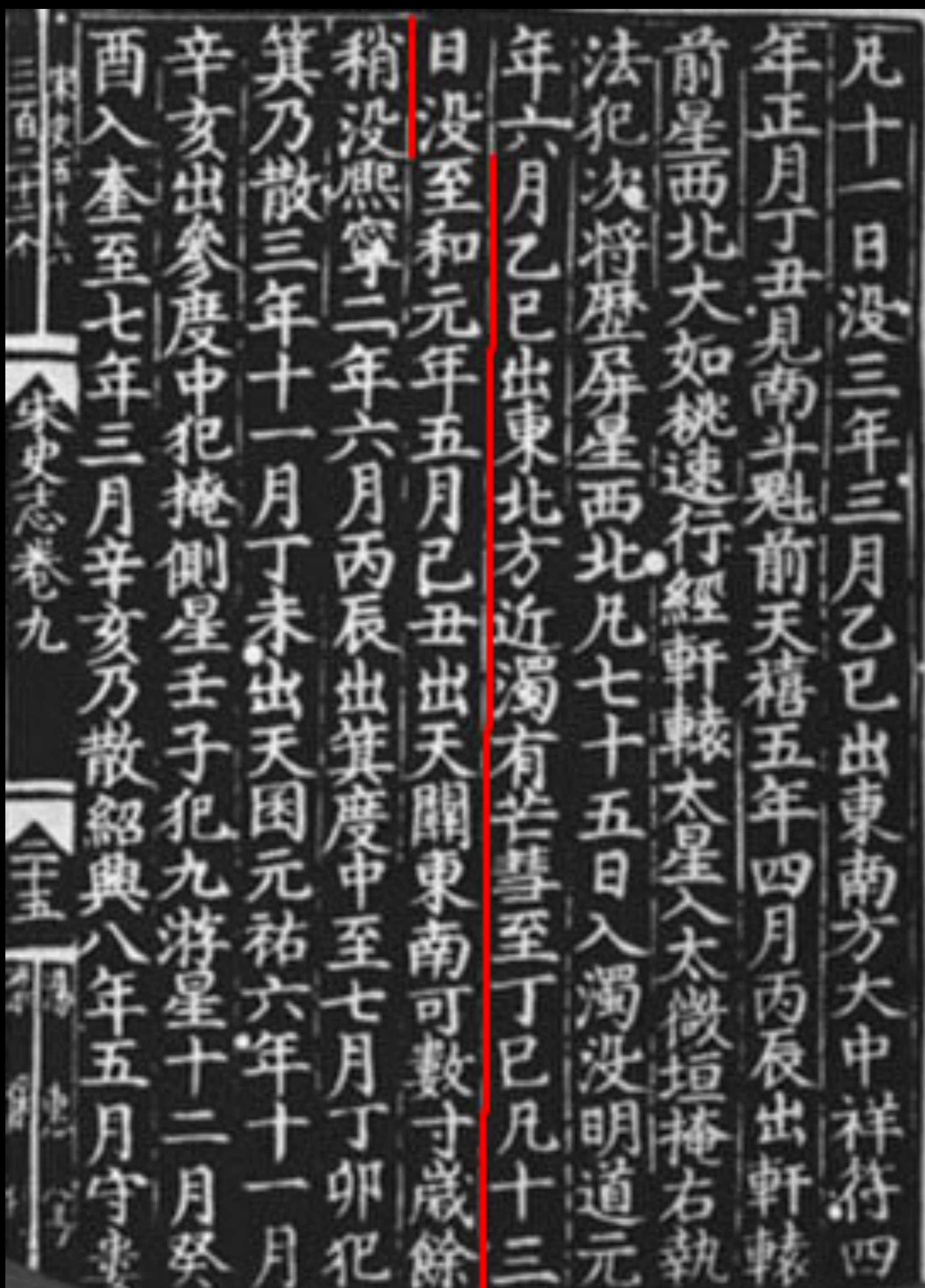
Crab Nebula

- age = angular size / angular expansion rate
- assumes constant expansion rate
- In 1928, Hubble estimated that the supernova exploded in the year of 1054 CE





- The supernova was seen by the entire world in July 1054 CE.
- Peak magnitude between -7 and -4.5 (brighter than Venus)
- The event was documented by astronomers in Song Dynasty
- There is also some drawing evidence in Native American ruins in New Mexico, Chaco Canyon


凡十一日沒三年三月乙巳出東南方大中祥符四
年正月丁丑見南斗魁前天禧五年四月丙辰出軒轅
前星西北大如桃速行經軒轅太星入太微垣掩右執
法犯次將歷屏星西北凡七十五日入濁沒明道元
年六月乙巳出東北方近濁有芒彗至丁巳凡十三
日沒至和元年五月己丑出天關東南可數寸歲餘
稍沒熙寧二年六月丙辰出箕度中至七月丁卯犯
箕乃散三年十一月丁未出天囷元祐六年十一月
辛亥出參度中犯掩側星壬子犯九游星十二月癸
酉入奎至七年三月辛亥乃散紹興八年五月守婁

宋史志卷九

How to Measure the Distance to a Supernova Remnant?

the “Expansion Parallax” method

Determining Distance from “Expansion Parallax”

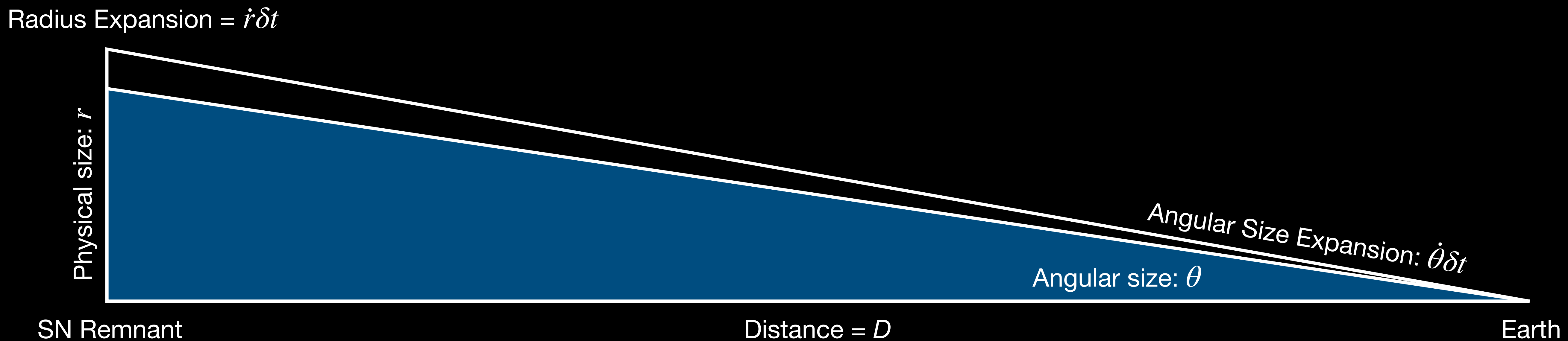
Physical Size vs. Angular Size: $r = D\theta$

Physical Expansion Rate vs. Angular Expansion Rate:

$$\dot{r}\delta t = D\dot{\theta}\delta t \rightarrow \dot{r} = D\dot{\theta}$$

Distance = Physical Expansion Rate / Angular Expansion Rate

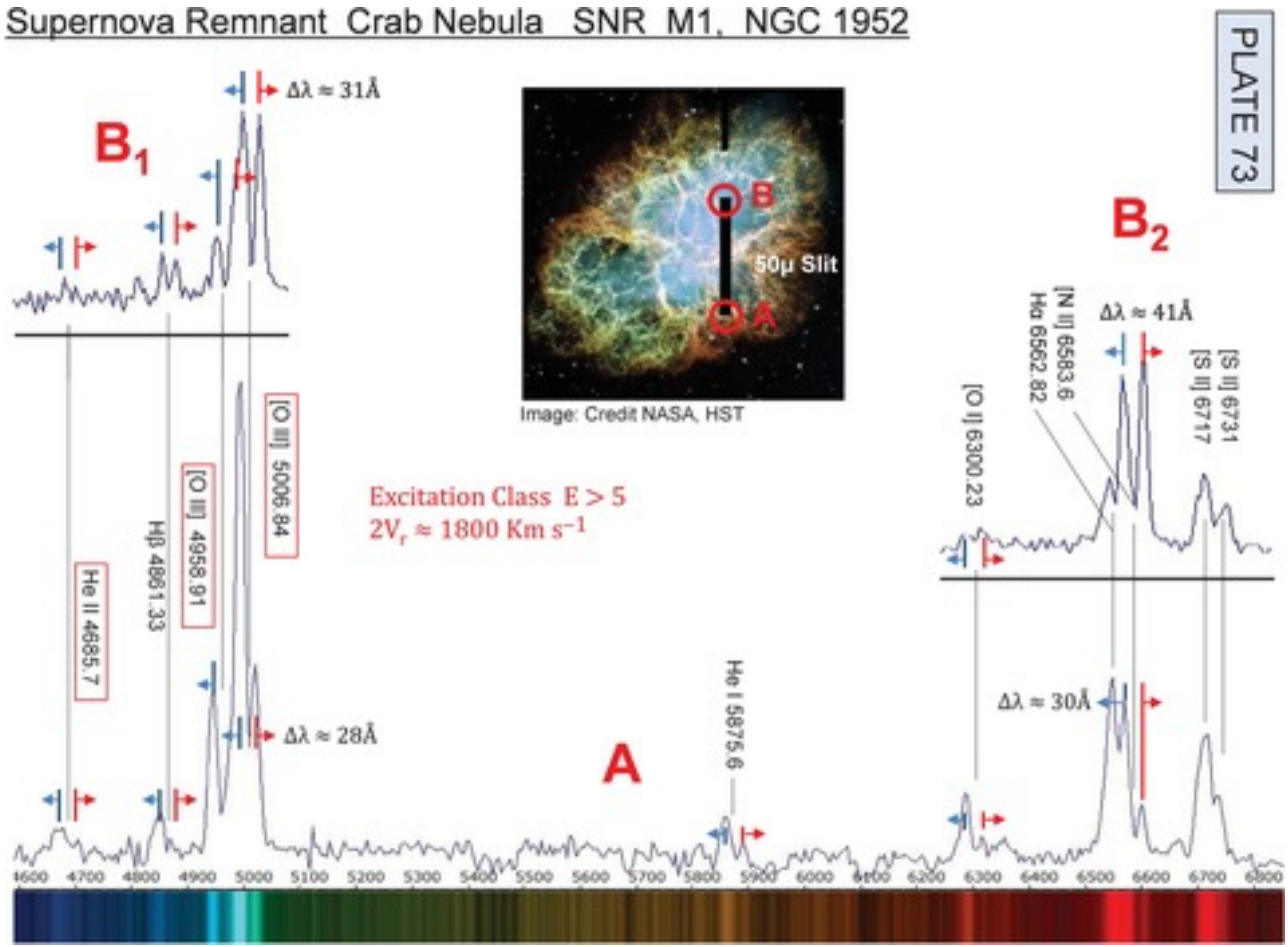
$$D = \dot{r}/\dot{\theta}$$



e.g., $\dot{\theta} = 0.01''/yr$, and $\dot{r} = 250\ \mathrm{km/s}$, what is the distance?

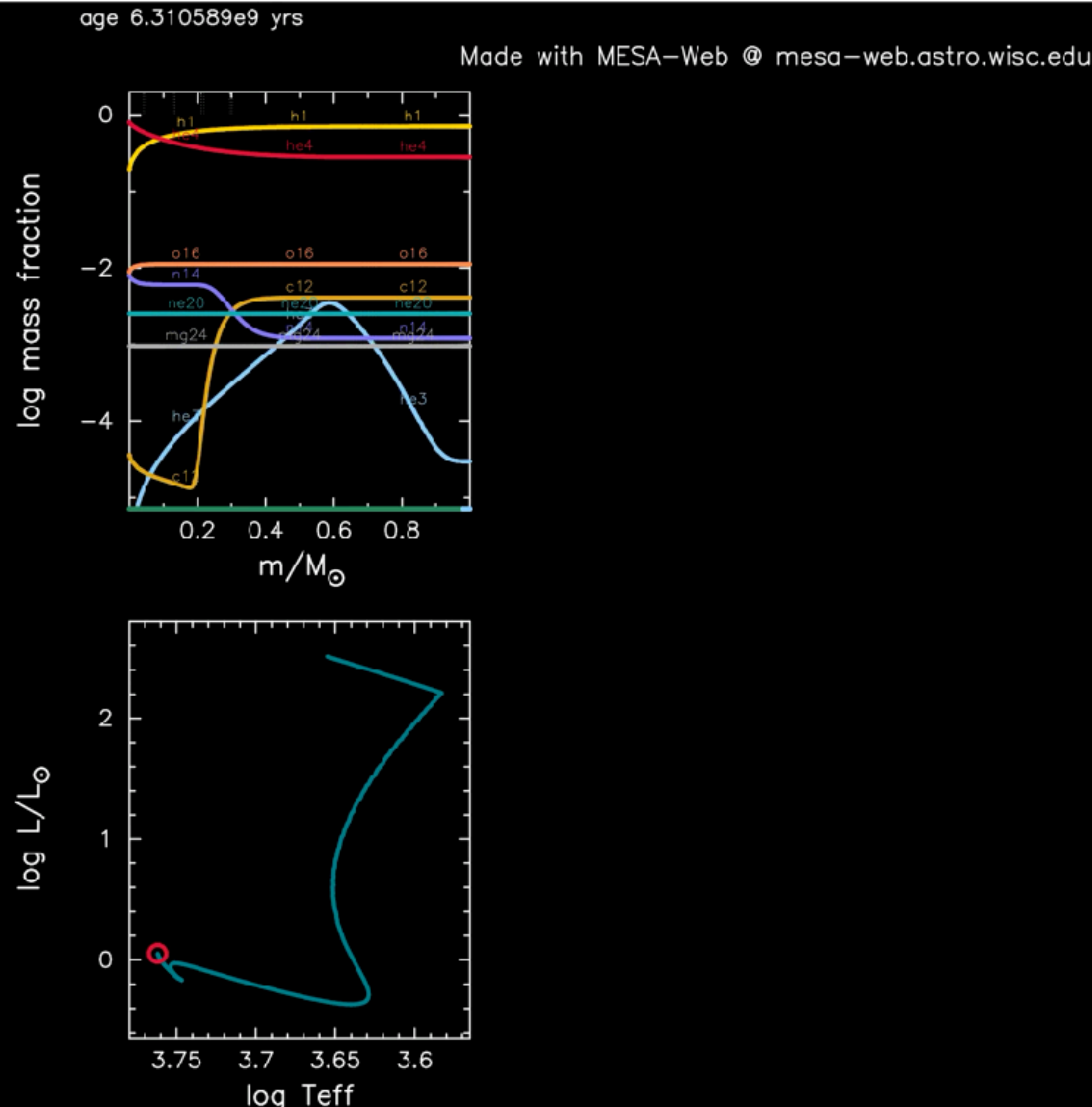
But how do we measure the physical expansion rate $\dot{r}$ in km/s?

Physical Expansion Rate from Doppler Shifts: Each Line Splits into Multiples



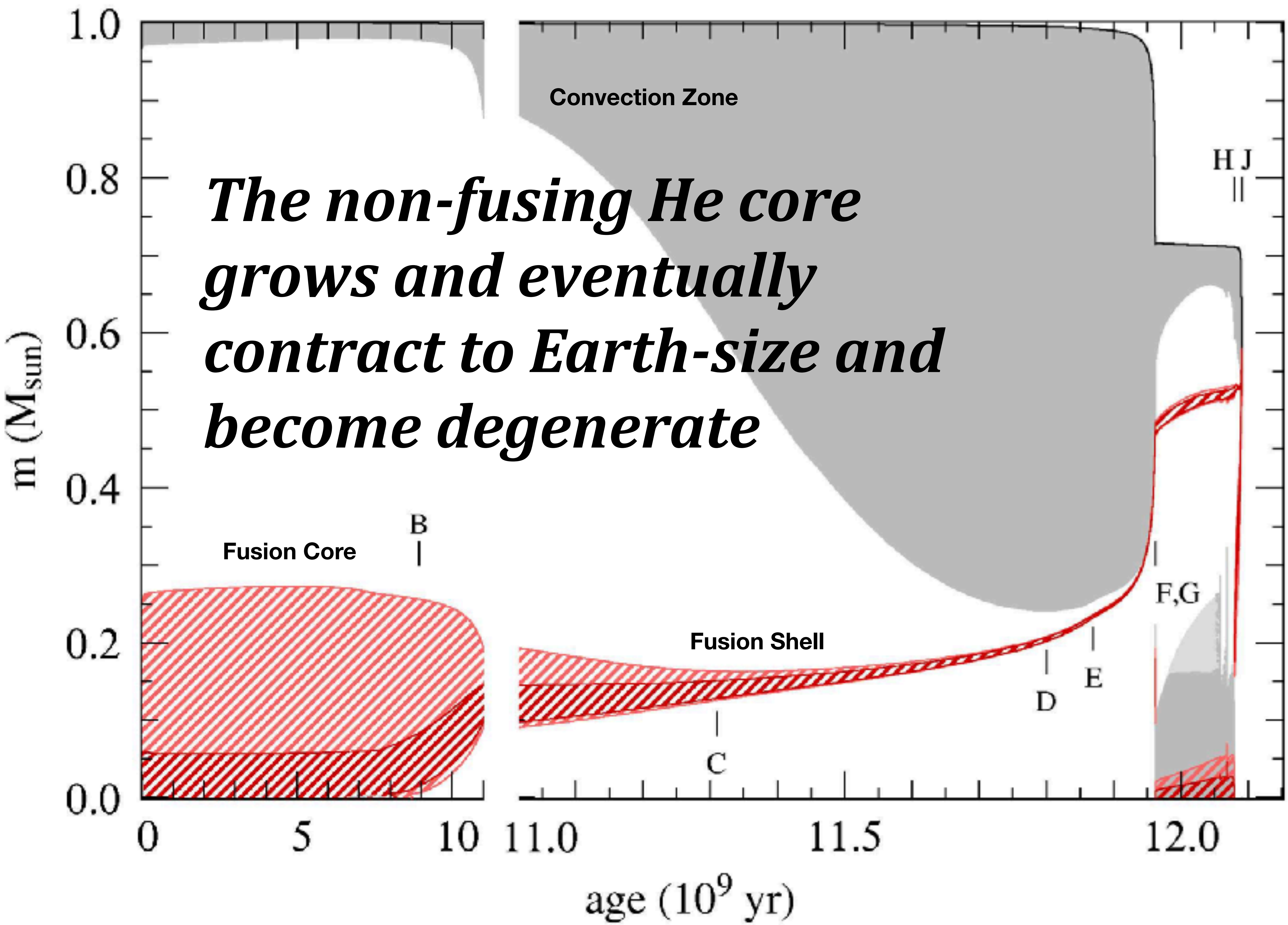
The Onset of Post-Main Sequence Evolution:

developing a degenerate He core



age 6.310589e9 yrs
Made with MESA-Web @ mesa-web.astro.wisc.edu
log mass fraction
m/M⊙
h1
he4
o16
n14
c12
ne20
mg24
he3
0
−2
−4
0.2
0.4
0.6
0.8
log L/L⊙
log Teff
2
1
0
3.75
3.7
3.65
3.6

Kippenhahn Diagram of a Star with an Initial Mass of 1.0 Solar Mass



The non-fusing He core grows and eventually contract to Earth-size and become degenerate

Schonberg-Chandrasekhar Limit

- In 1942, **Schonberg** and **Chandrasekhar** found that when the non-fusing Helium core reaches ~8% of the total mass of the star, the core will contract because under isothermal condition, its pressure can no longer support the envelope:

$$\left(\frac{M_{\text{core}}}{M}\right)_{SC} = 0.37\left(\frac{\mu_{\text{env}}}{\mu_{\text{core}}}\right)^2$$

- The mass-ratio limit above is derived in the following way:
  - **Virial theorem** (self-gravitating) **with external pressure** at the core-envelop boundary to calculate the maximum pressure that the core can support:

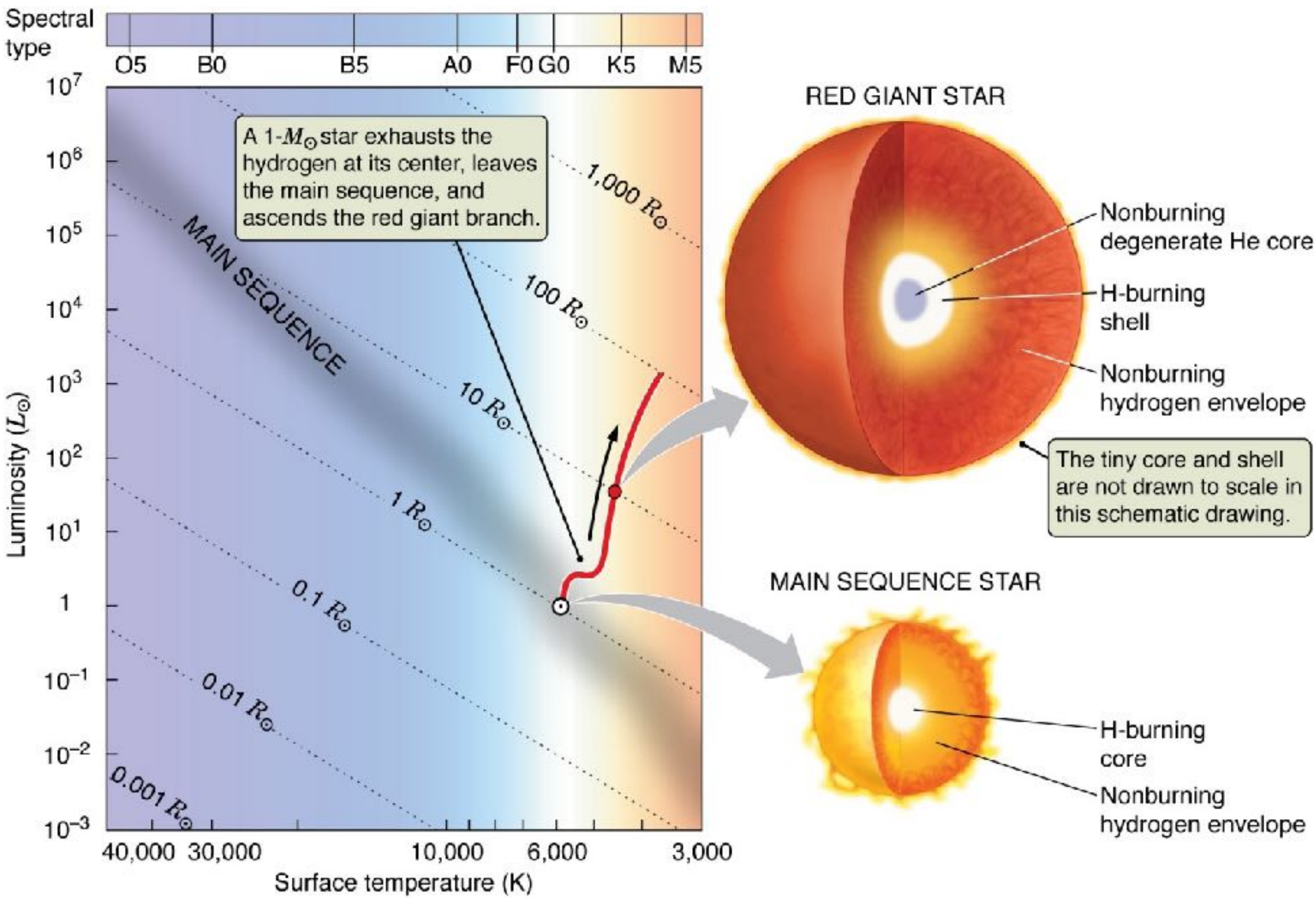
$$P_{\text{core,max}} = \frac{A}{G^3 M_{\text{core}}^2}\left(\frac{kT_{\text{core}}}{\mu_{\text{core}} m_H}\right)^4$$

  - **Hydrostatic equilibrium** to calculate the actual pressure from the envelope plus the ideal gas law:

$$P_{\text{env}} \approx \frac{G}{4\pi R^4}(M^2 - M_{\text{core}}^2) \approx \frac{B}{G^3 M^2}\left(\frac{kT_{\text{boundary}}}{\mu_{\text{env}} m_H}\right)^4$$

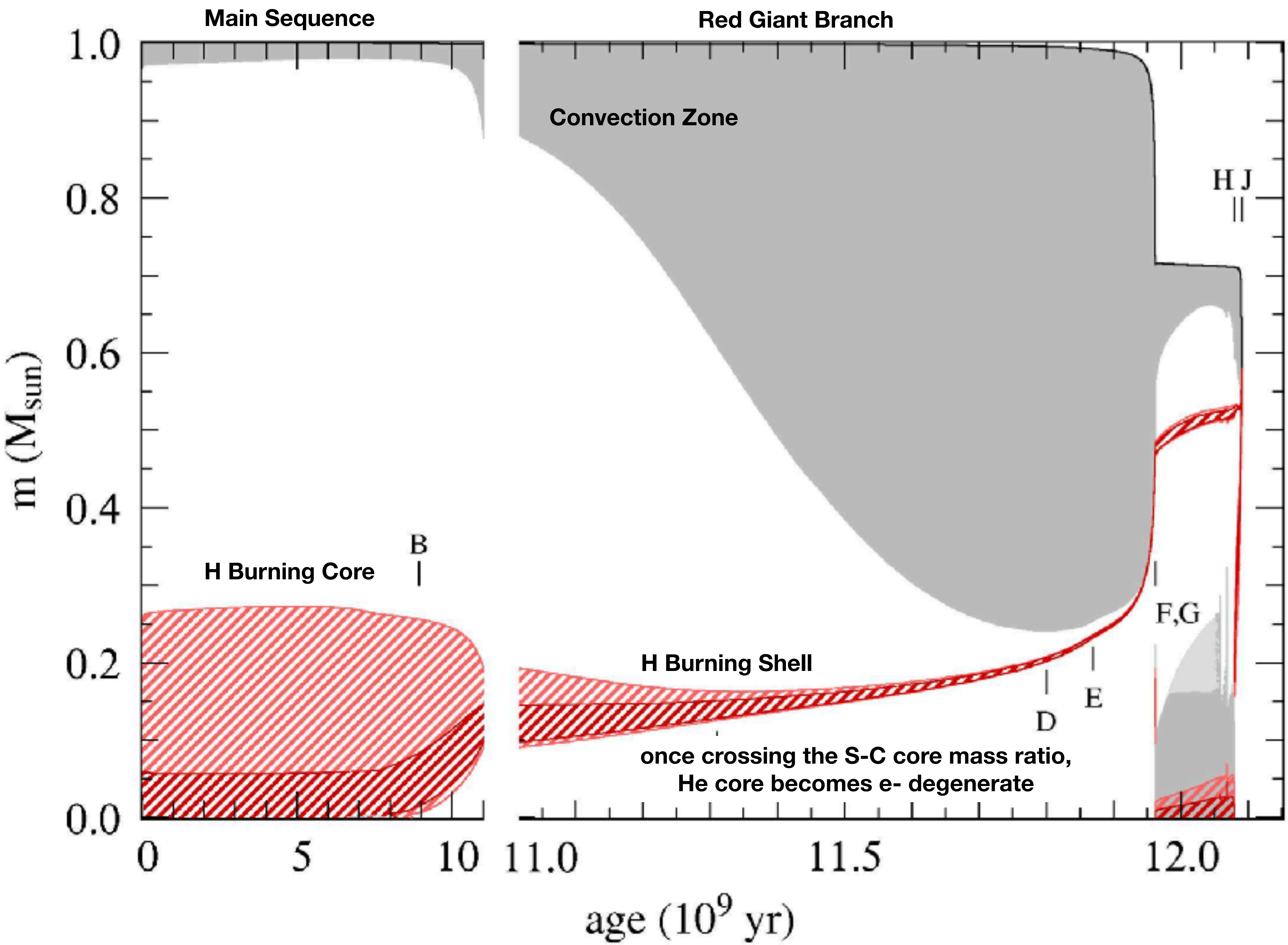
  - Given that at the boundary, the core and the envelope have the same temperature, the condition for collapse ($P_{\text{env}} > P_{\text{core,max}}$) gives us a maximum core-mass-total-mass ratio that is *inversely* proportional to the square of the ratio of the mean molecular mass.

Violation of the SC limit causes the core to contract and become degenerate

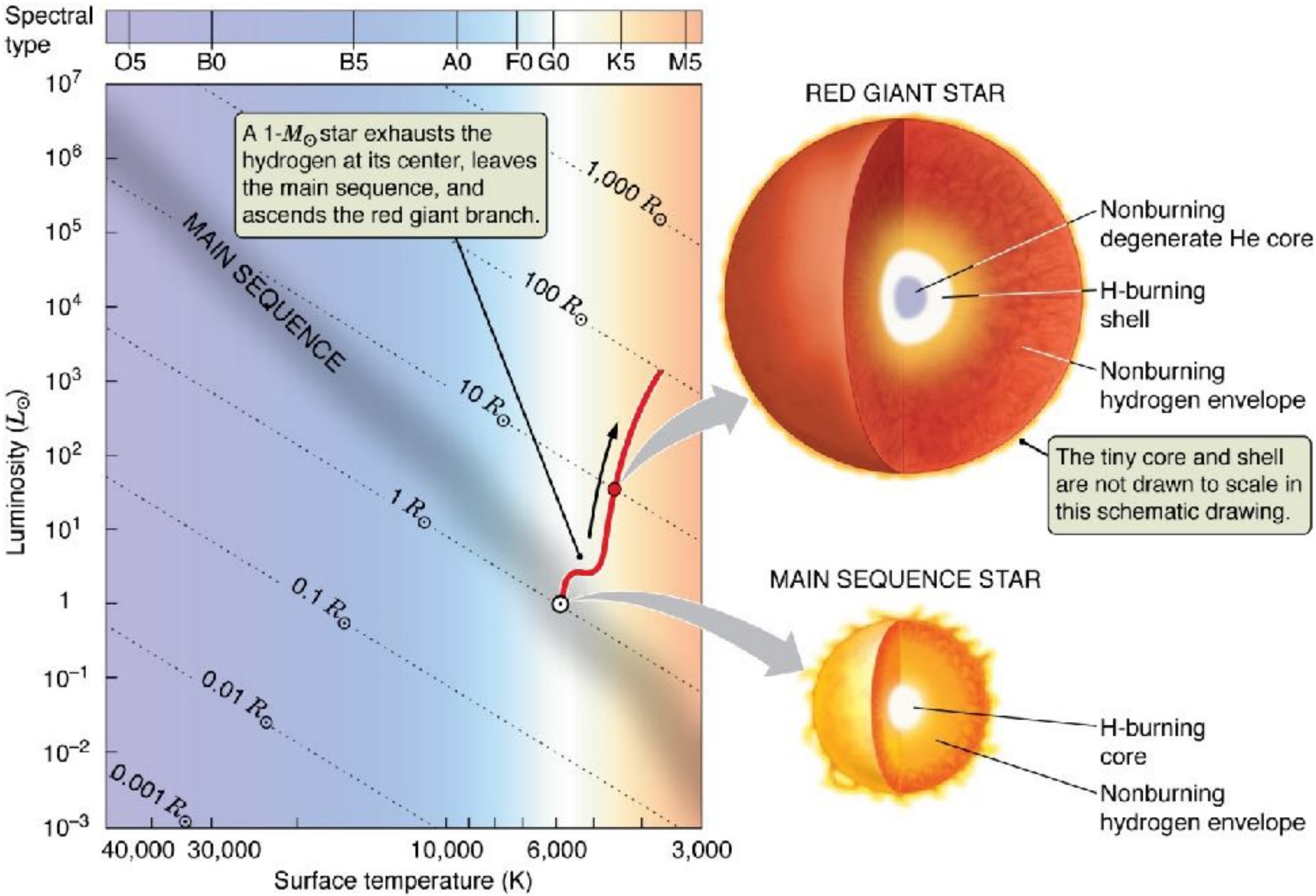


Red Giant Branch

Kippenhahn Diagram of a Star with an Initial Mass of 1.0 Solar Mass

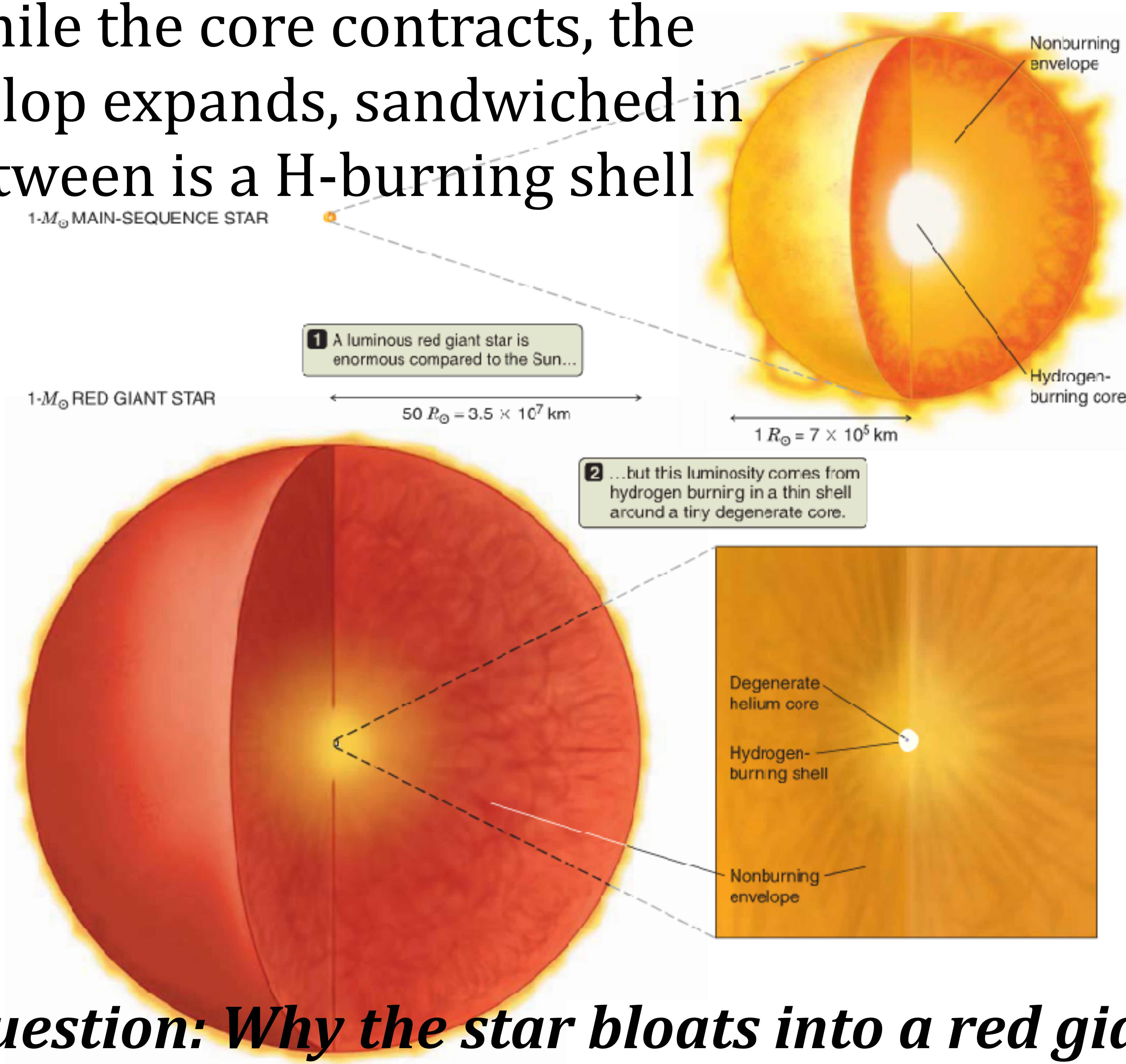


The Red Giant Branch:
H-burning shell + degenerate He core



Sunrise in 7 Billion Years

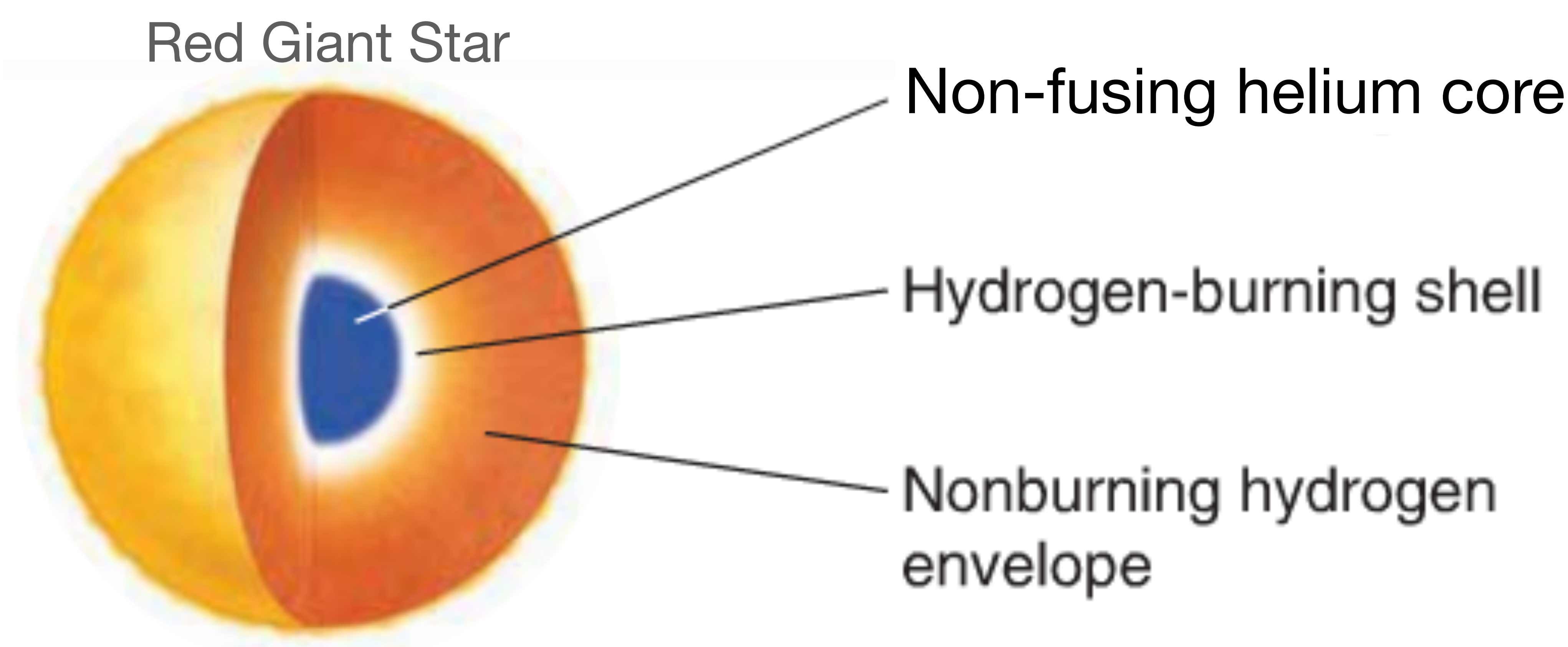
While the core contracts, the envelop expands, sandwiched in between is a H-burning shell



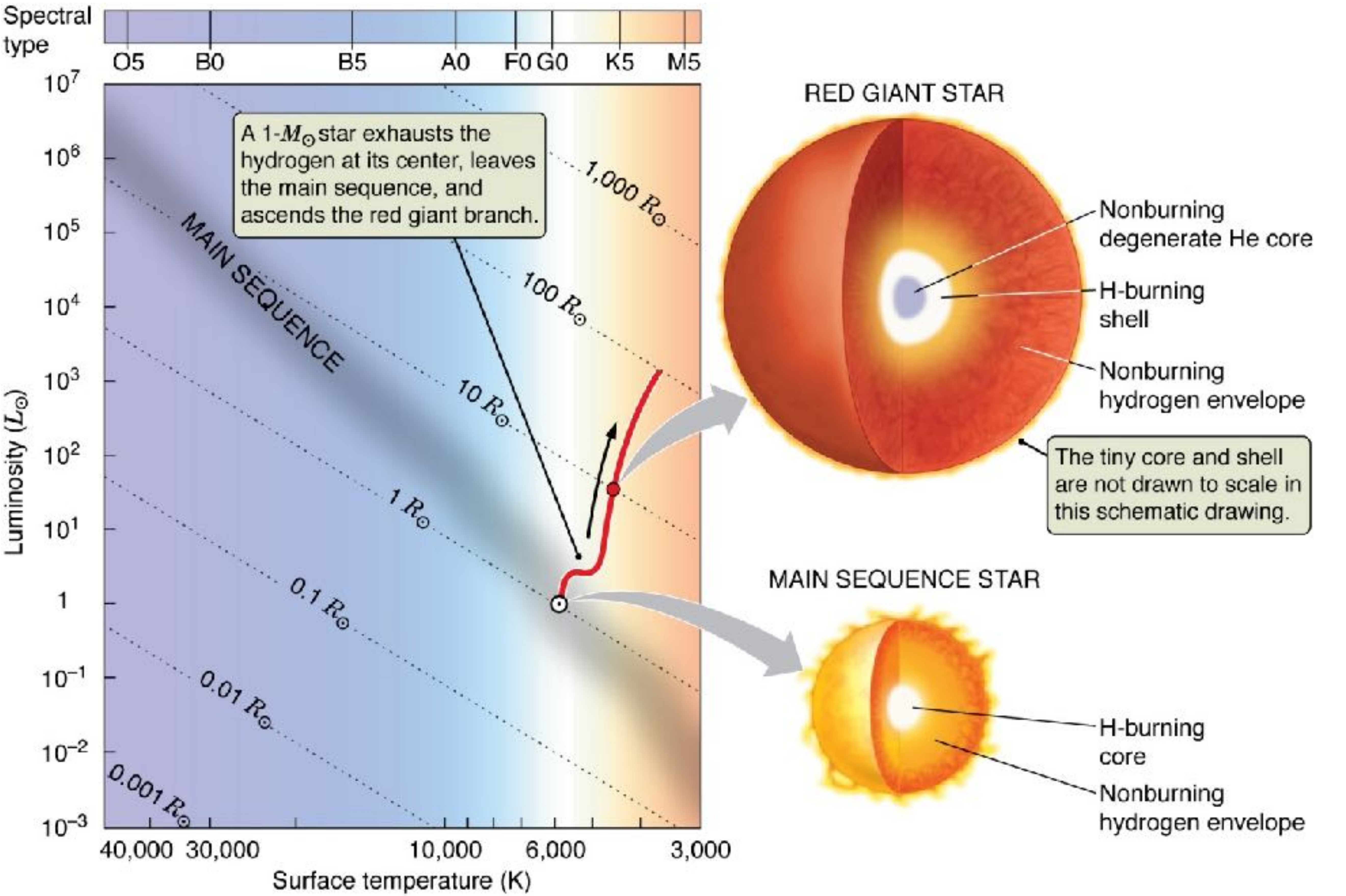
Question: Why the star bloats into a red giant?

Mirror Principle: when interior contracts, exterior expands

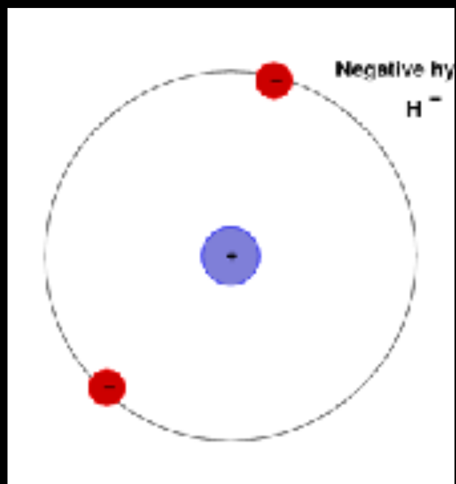
- While the **gravitational thermostat** works well to control the **burning core**'s temperature, it cannot control the temperature of a **burning shell**
- When fusion stops in the **core**, it **contracts**. Gravitational potential energy **heats up the core** and it conducts its heat to the surrounding shell.
- As the **shell's temperature rises**, its fusion reaction rate increases rapidly
- To avoid a thermonuclear runaway, the shell must decrease its temperature by dumping its energy to the **non-burning envelope**, causing the star to expand to a giant.



How Temperature stays roughly constant while Luminosity increases thousands of times?



The H^- thermostat controls the atmosphere temperature of red giant stars



Surface temperature decreases

H^- increases

Atmosphere becomes more opaque

Surface temperature increases

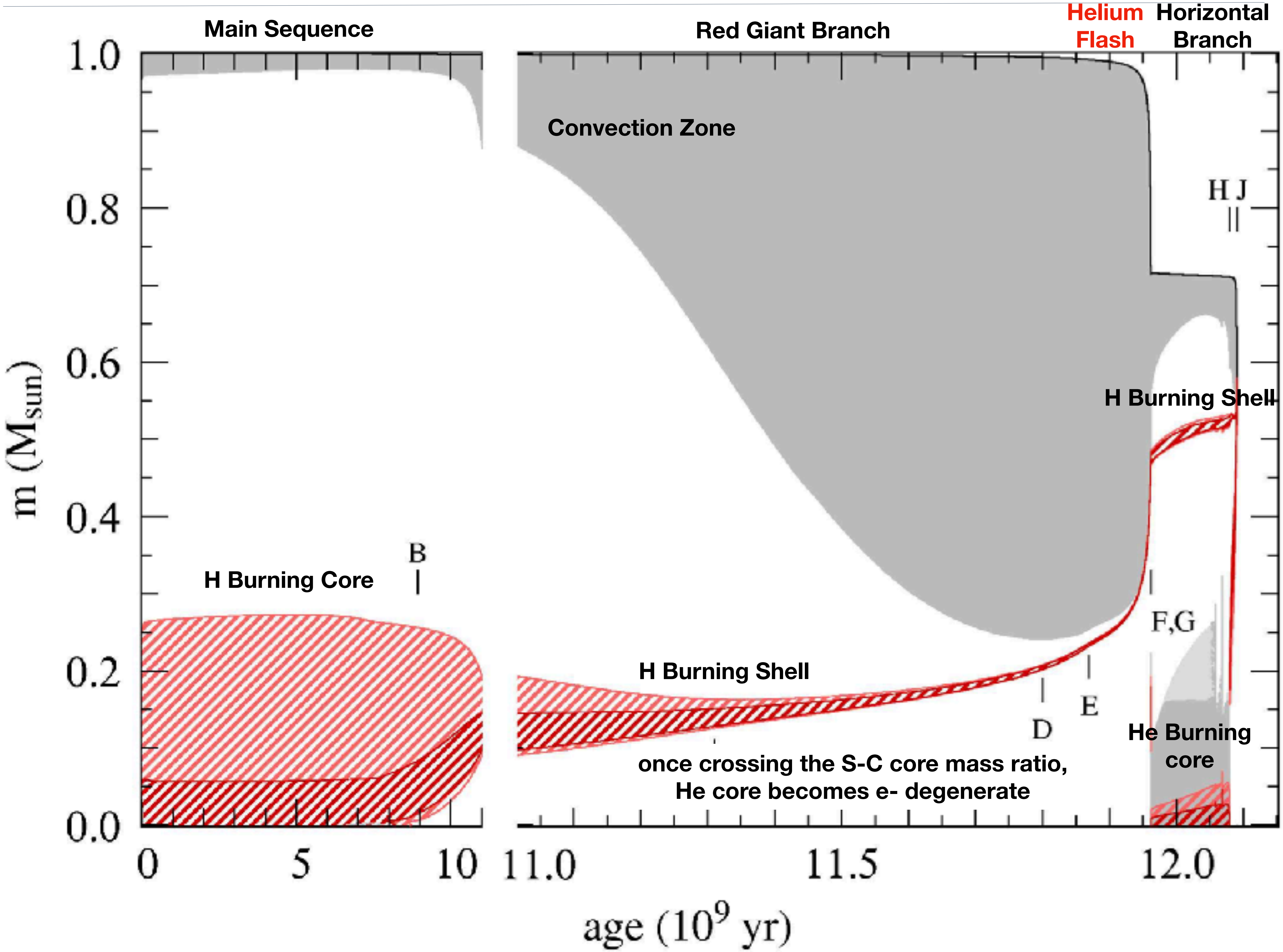
H^- decreases

Atmosphere becomes more transparent

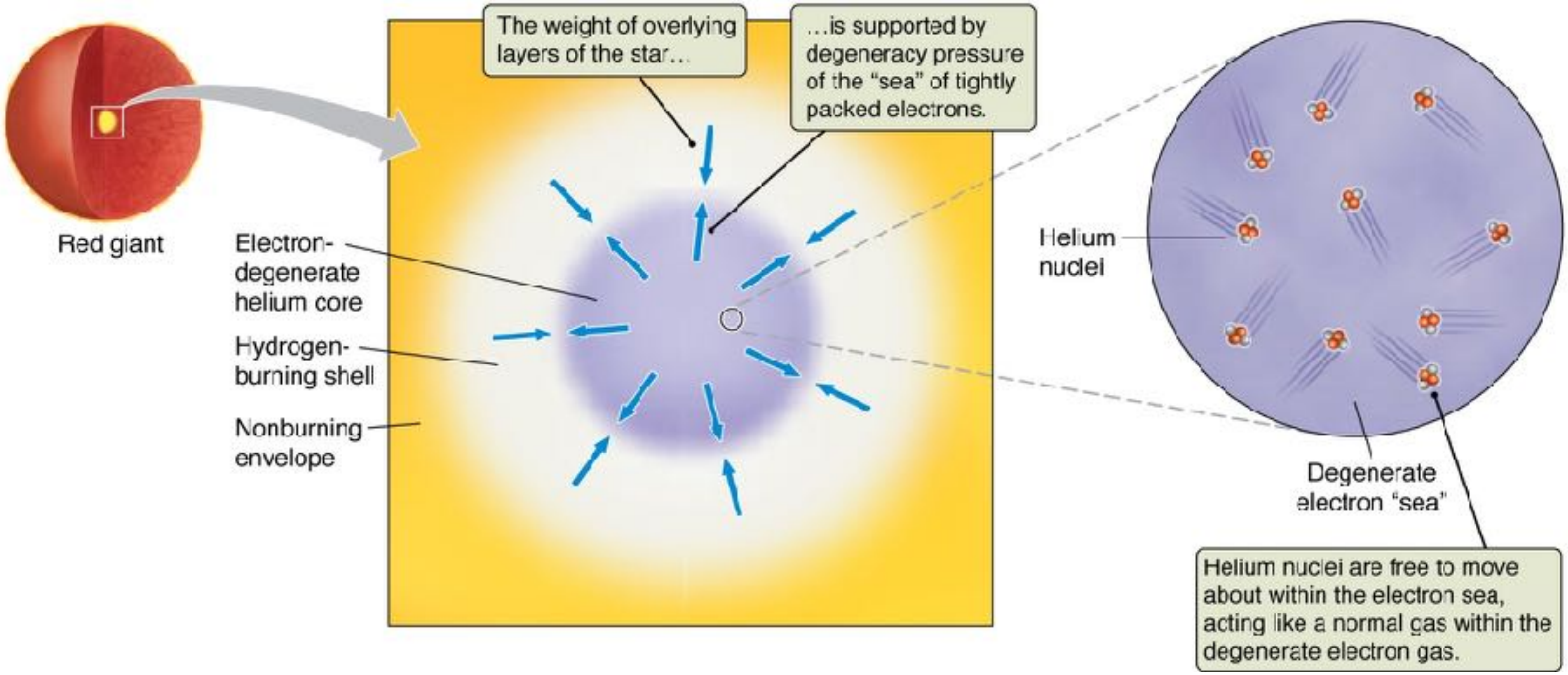
Surface temperature decreases

Helium Flash:
a nuclear explosion inside a star

Kippenhahn Diagram of a Star with an Initial Mass of 1.0 Solar Mass



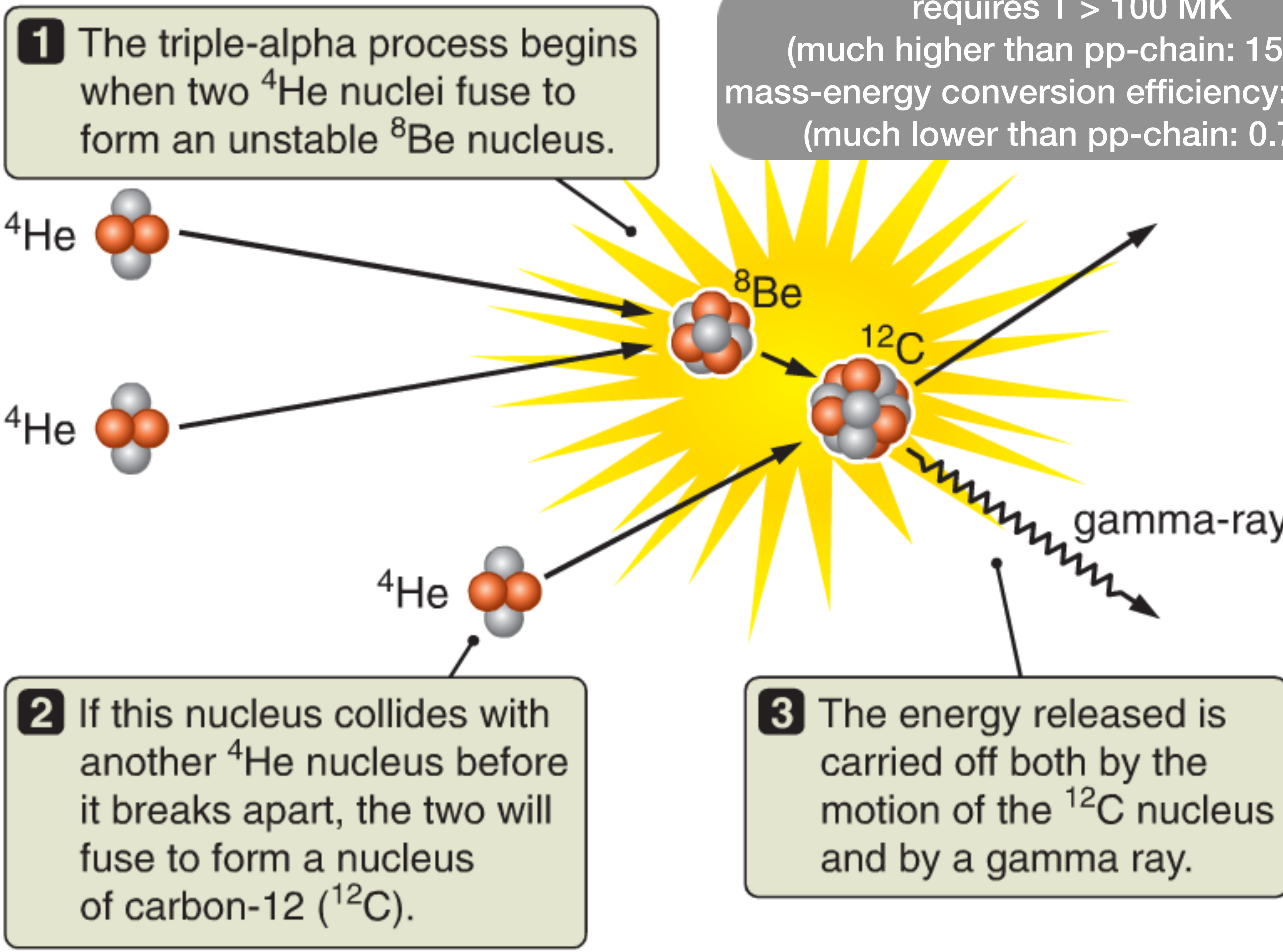
Helium Flash: The End of the Red Giant Phase



- Hydrogen shell-burning adds Helium to the core, making it heavier.
- Core pressure doesn't increase past the electron-degenerate limit, but **ion temperature is allowed to increase without bound**.
- …**until T reaches ~ 100 million K, at which time Helium can fuse.**

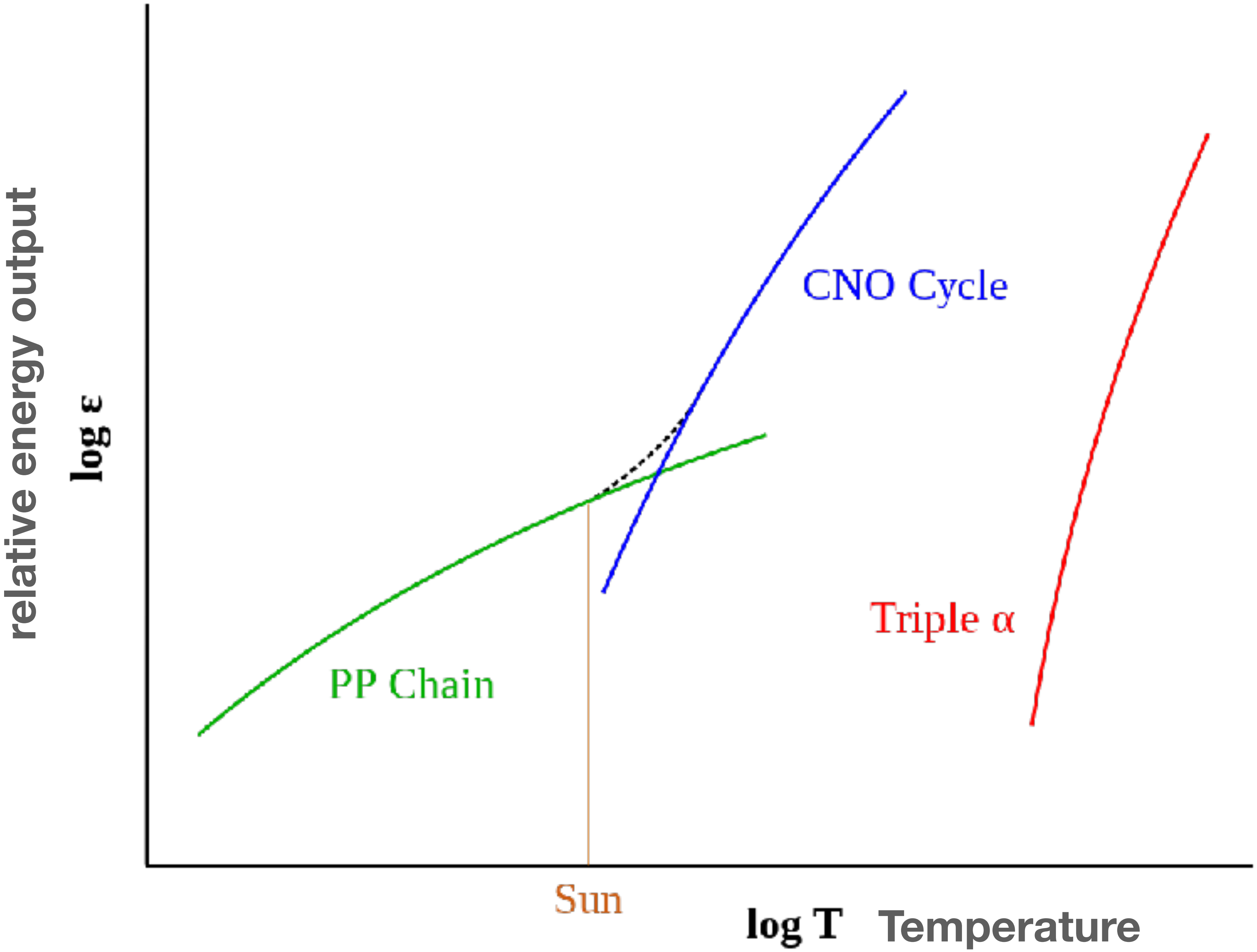
Helium burning – the triple-alpha process

requires T > 100 MK
(much higher than pp-chain: 15 MK)
mass-energy conversion efficiency: 0.065%
(much lower than pp-chain: 0.7%)



Fusion Reaction Rate Strongly Depends on Temperature

- For PP chain, reaction rate ~ T^4, for CNO cycle, rate ~ T^{20}, Triple Alpha cycle, rate ~ T^{40}



Helium Flash: An Underground Nuclear Explosion!

Question: Why the onset of He burning in a degenerate core makes a bomb?
Why the nuclear fusion cannot be controlled as in the core of the Sun?

When the gravitational thermostat is out of order

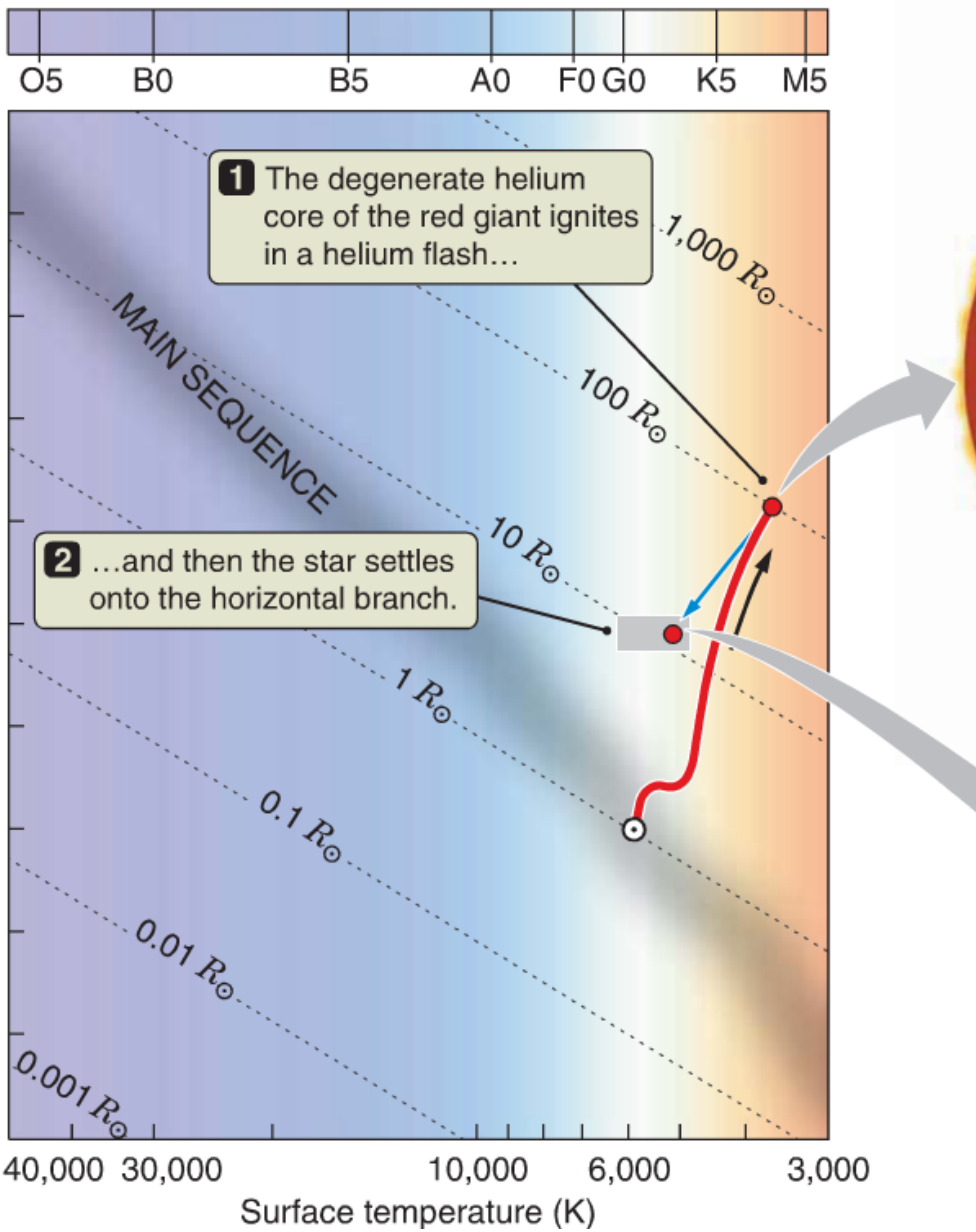
non-degenerate core

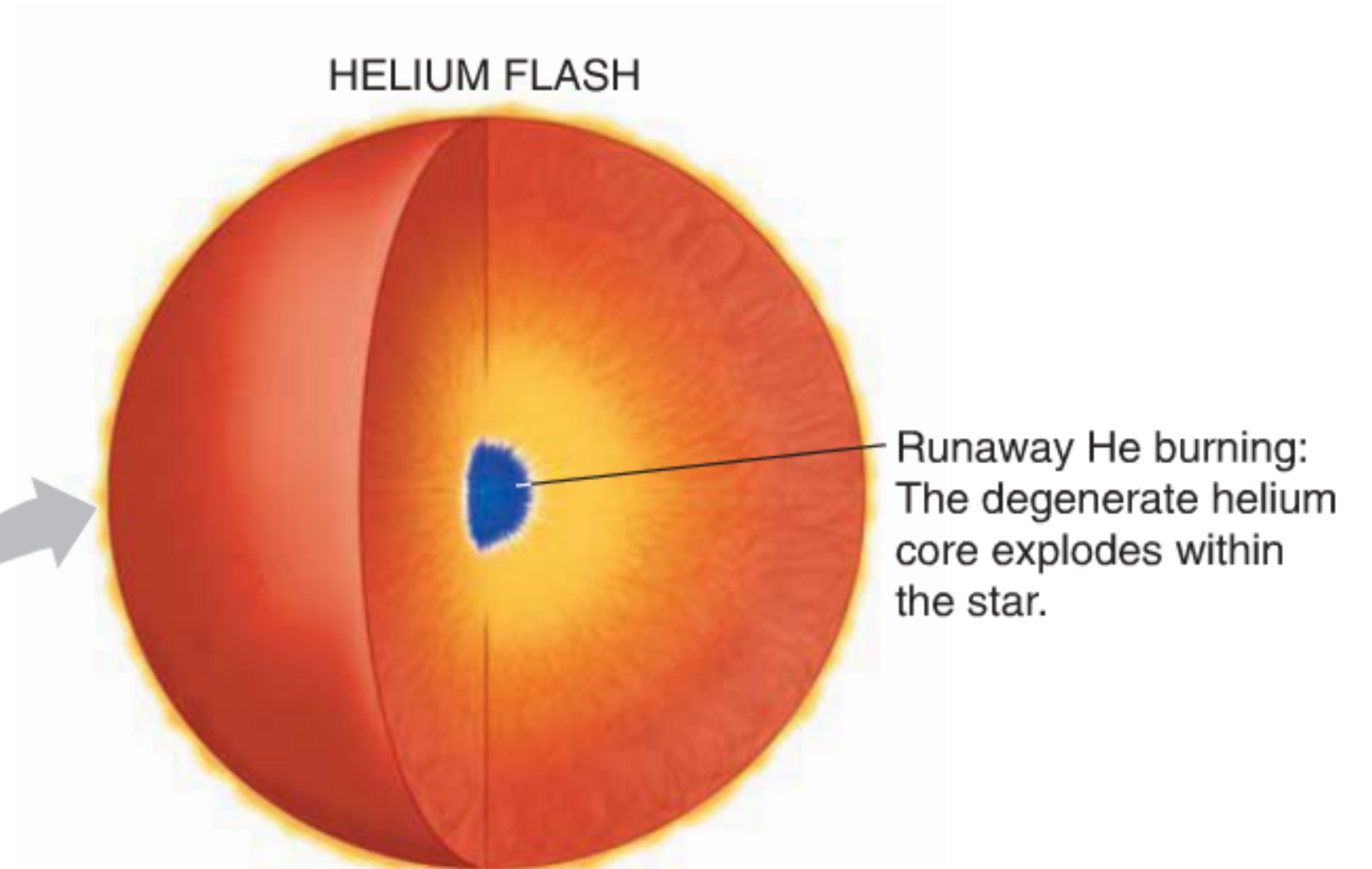
- H fusion rate increases
- T & P increases
- Core expands, work against Gravity
- T decreases
- H fusion rate decreases
- Steady H Burning

electron-degenerate core

- He fusion rate increases
- T(He^{2+}) increases, P(e-) do not change
- Core doesn't expand
- T(He^{2+}) still high
- He fusion rate increases
- He Flash

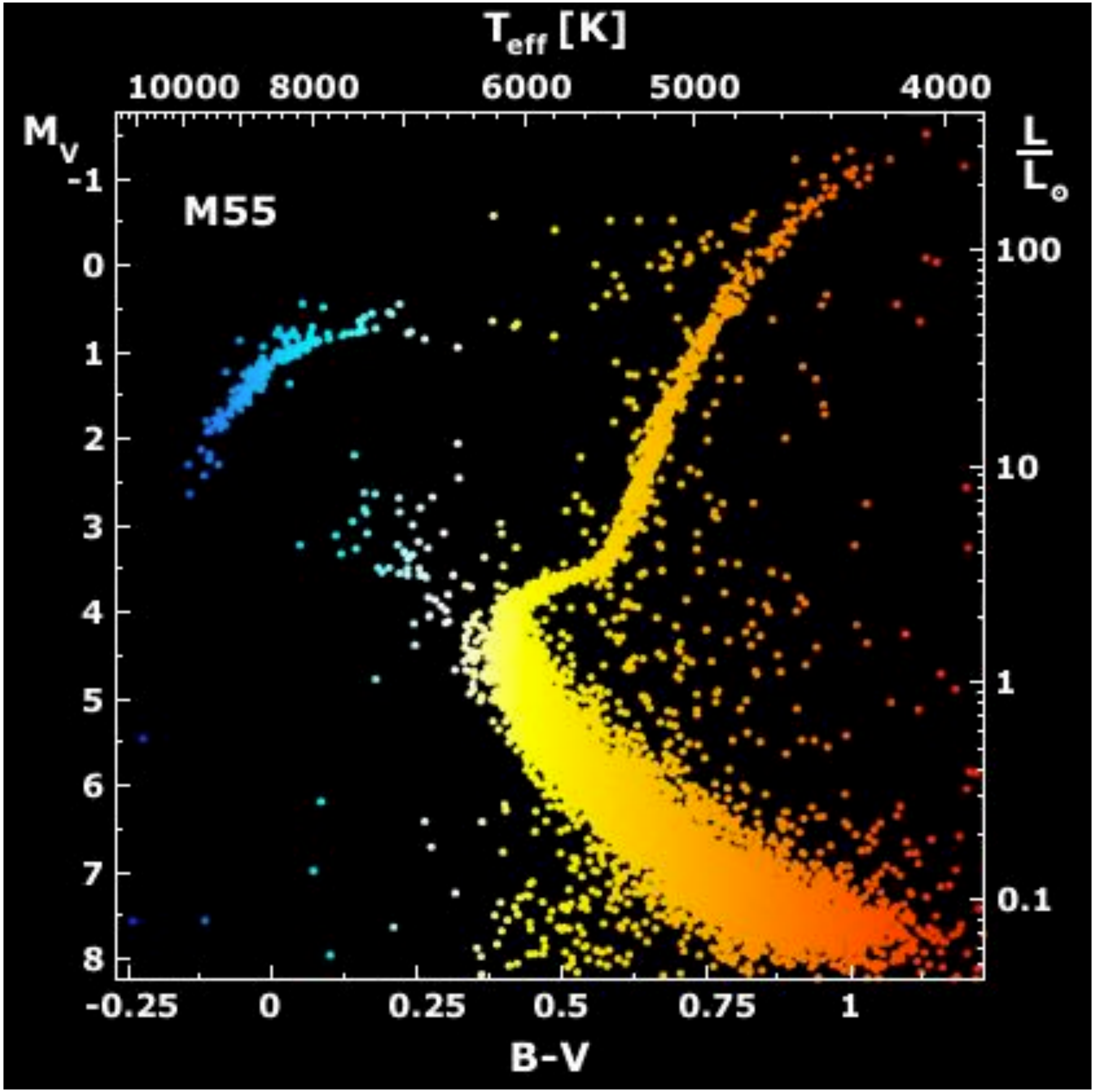
Helium flash - a thermonuclear runaway



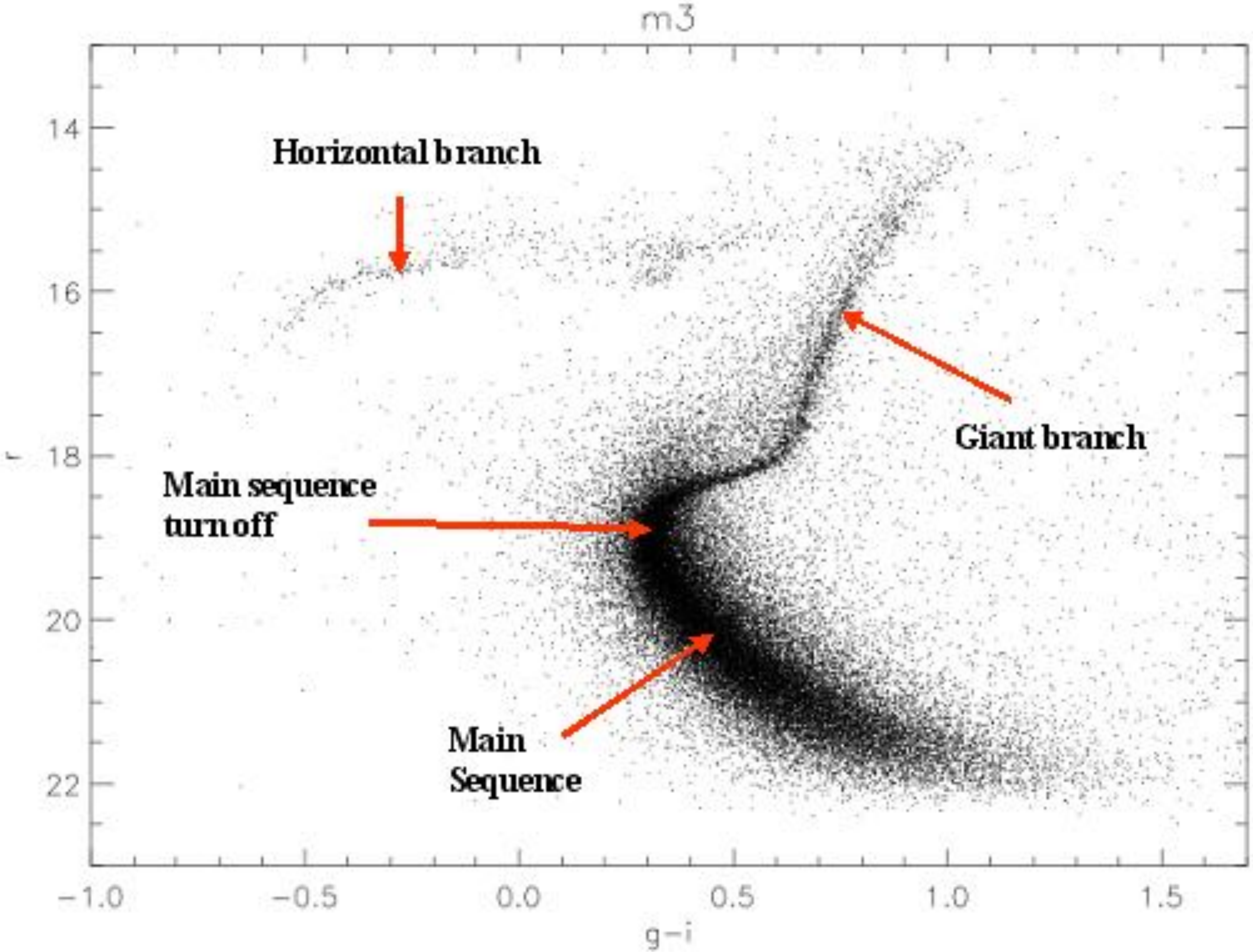


- a thermonuclear **runaway**
- The explosion lasts only **a few hours** thanks to the **heavy envelop**, which helped the star to **regain stability while the core loses degeneracy**

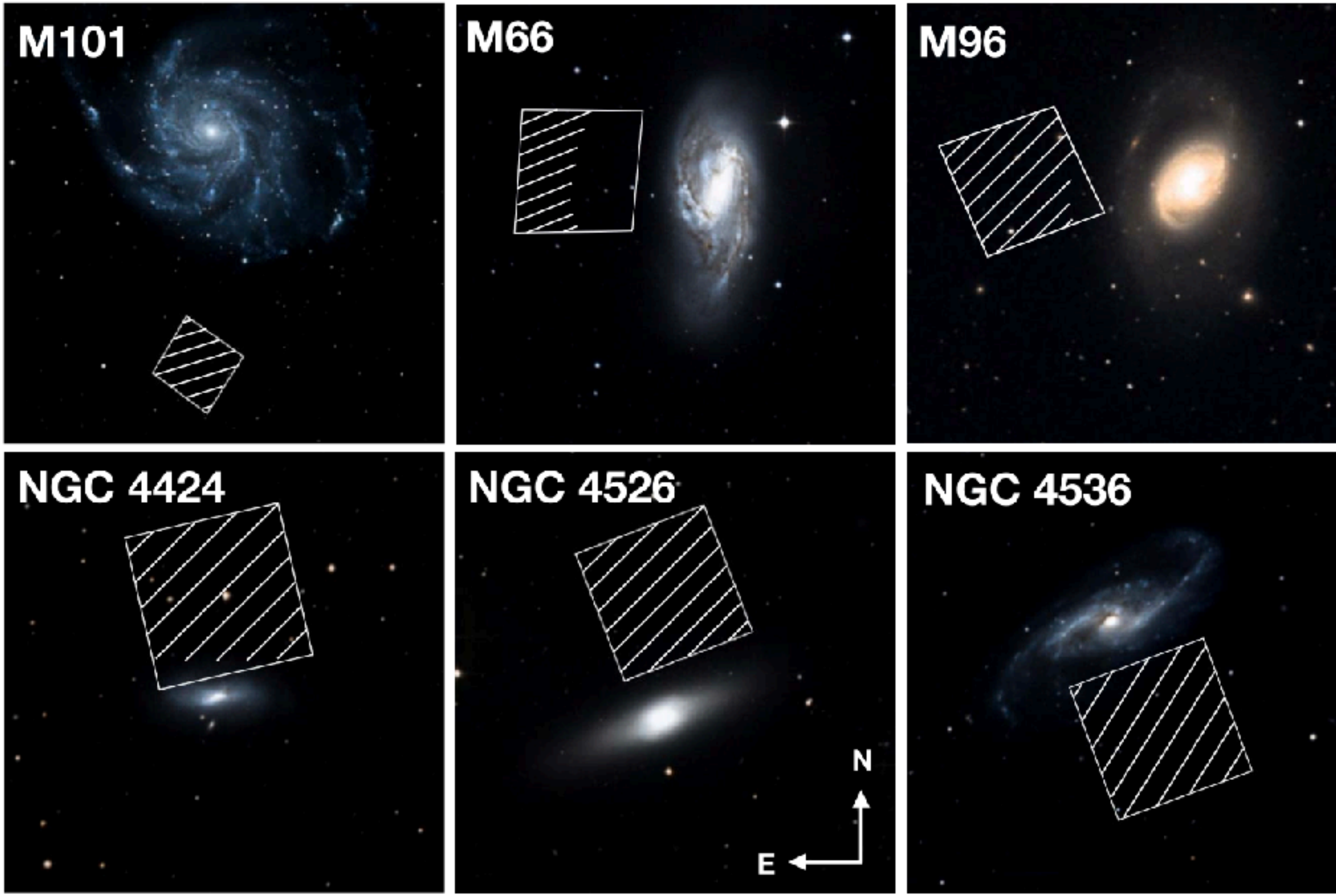
Helium Flash & Tip of the Red Giant Branch (TRGB)



Helium Flash & Tip of the Red Giant Branch (TRGB)

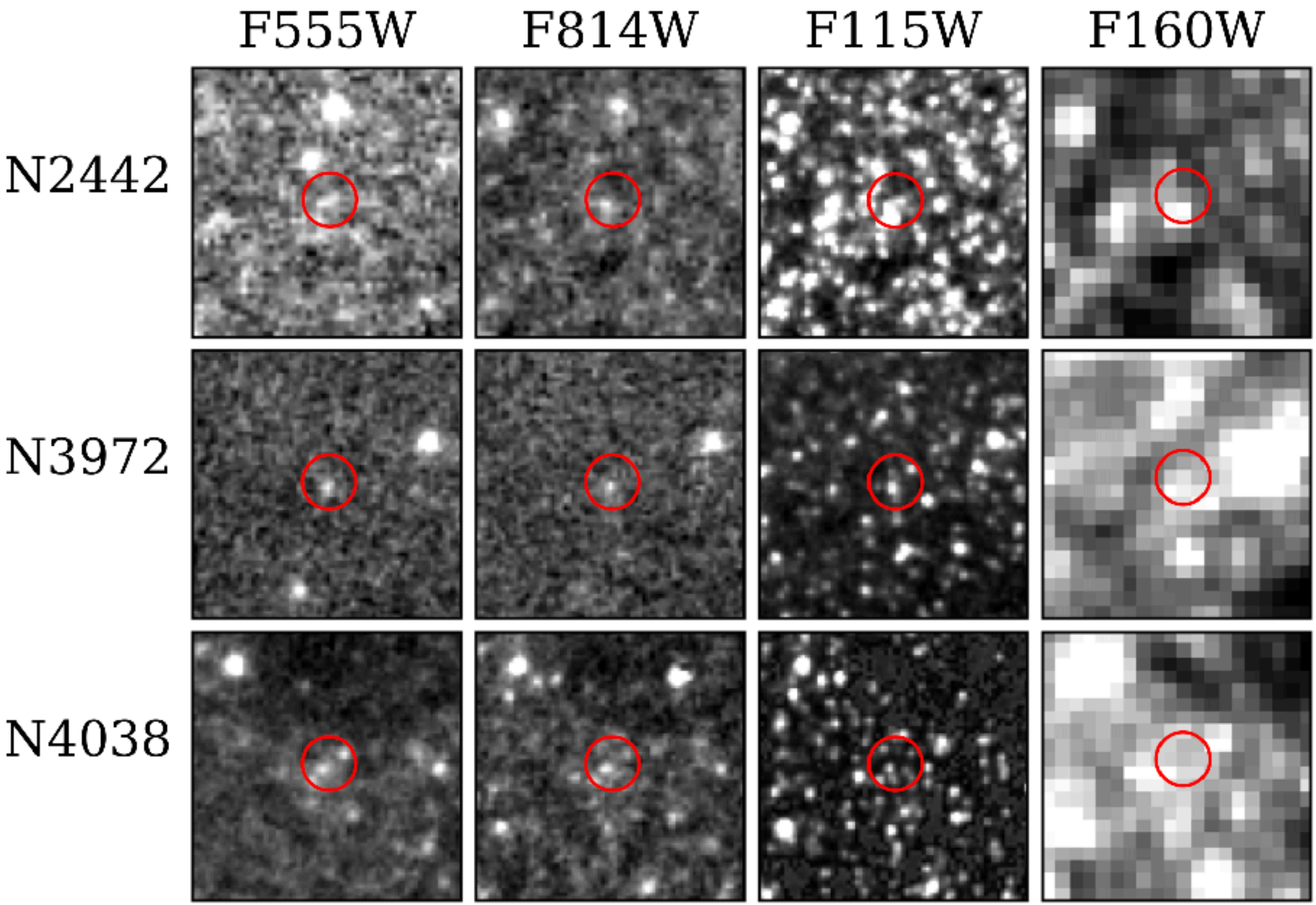


TRGB of Halo Stars in External Galaxies



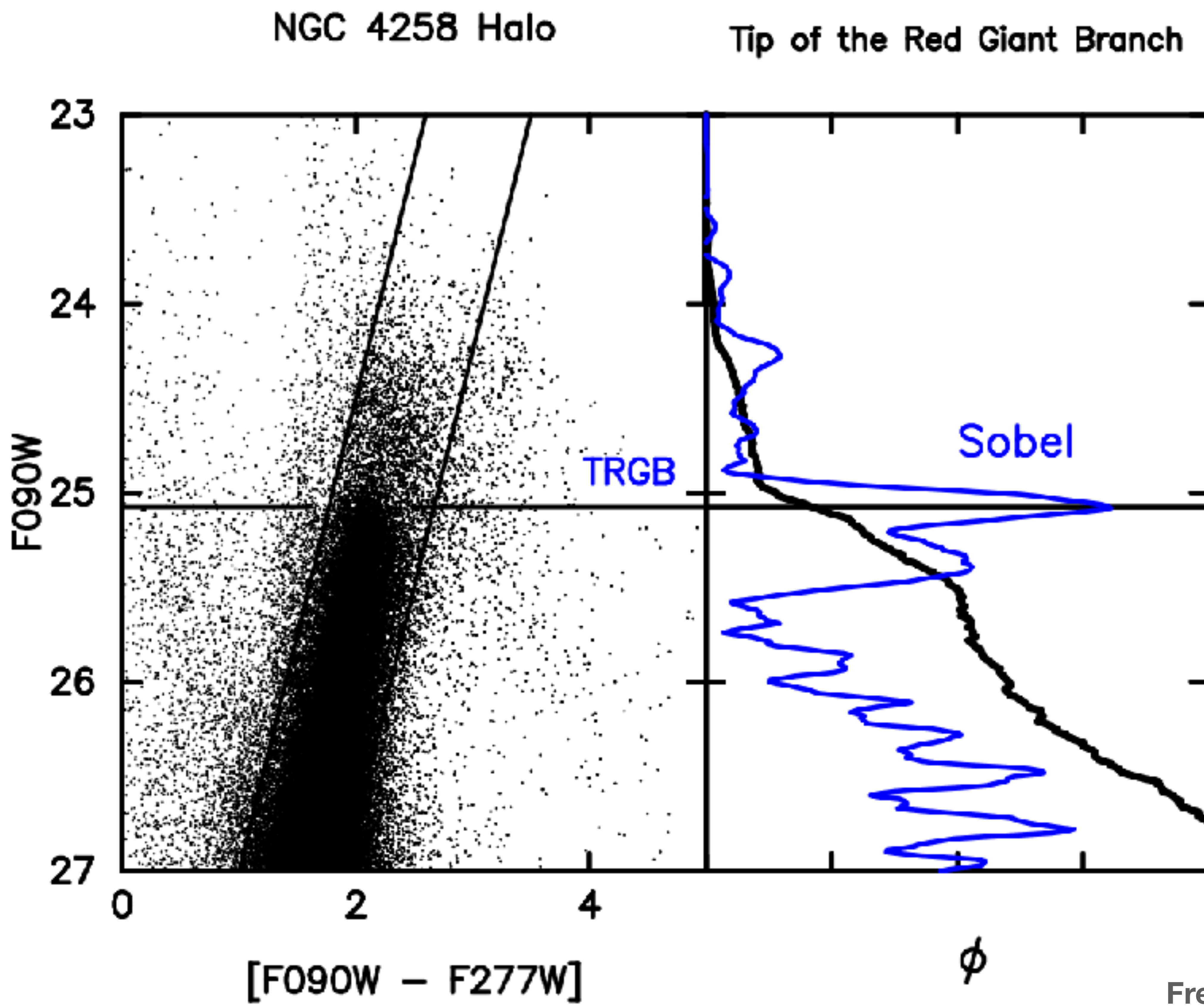
Freedman+2019

Crowded Field Photometry even at HST Resolution



Example Cepheids in external galaxies

TRGB of Halo Stars in External Galaxies



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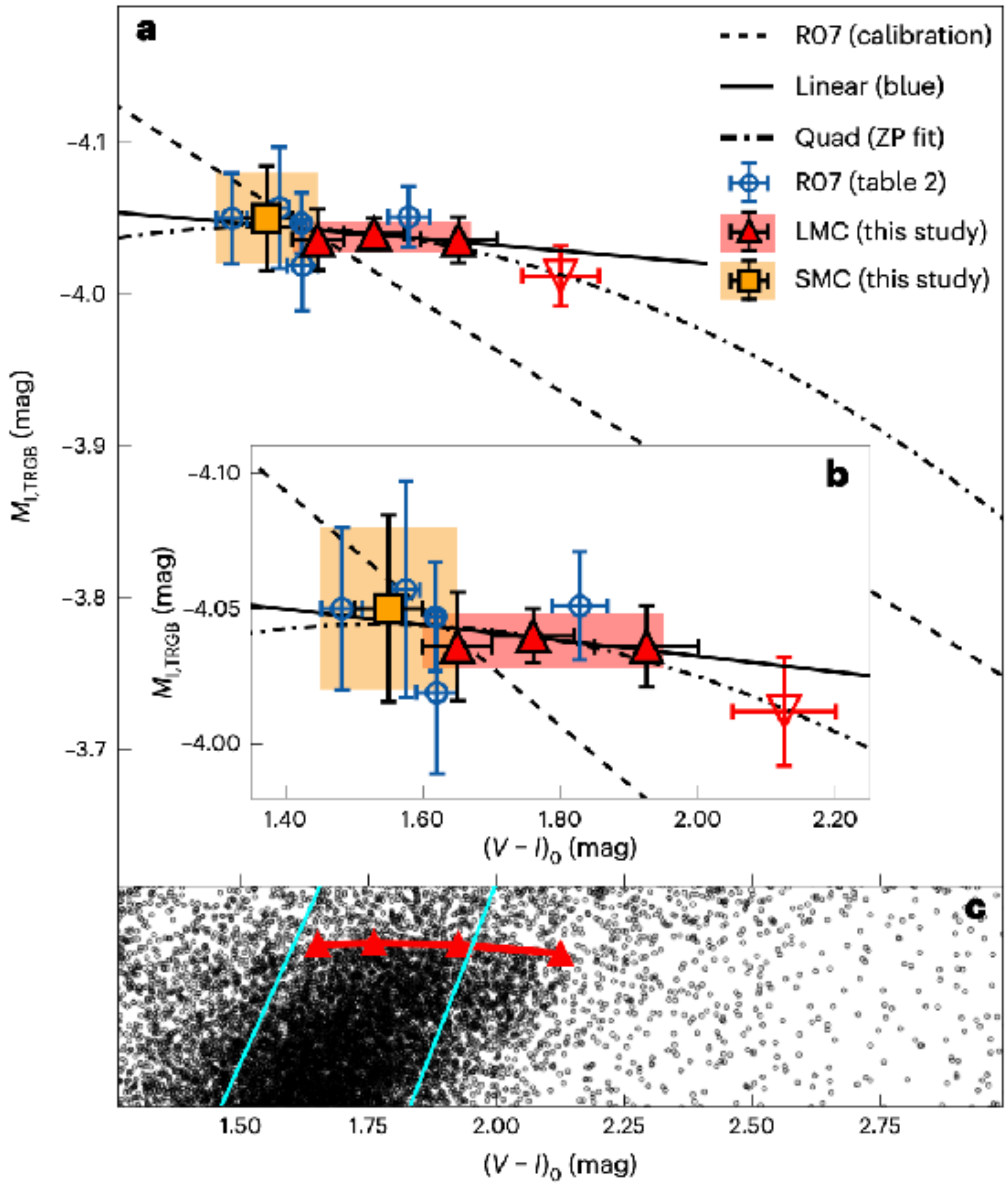
Tip of the Red Giant Branch (TRGB): Another Standard Candle



Hoyt 2023

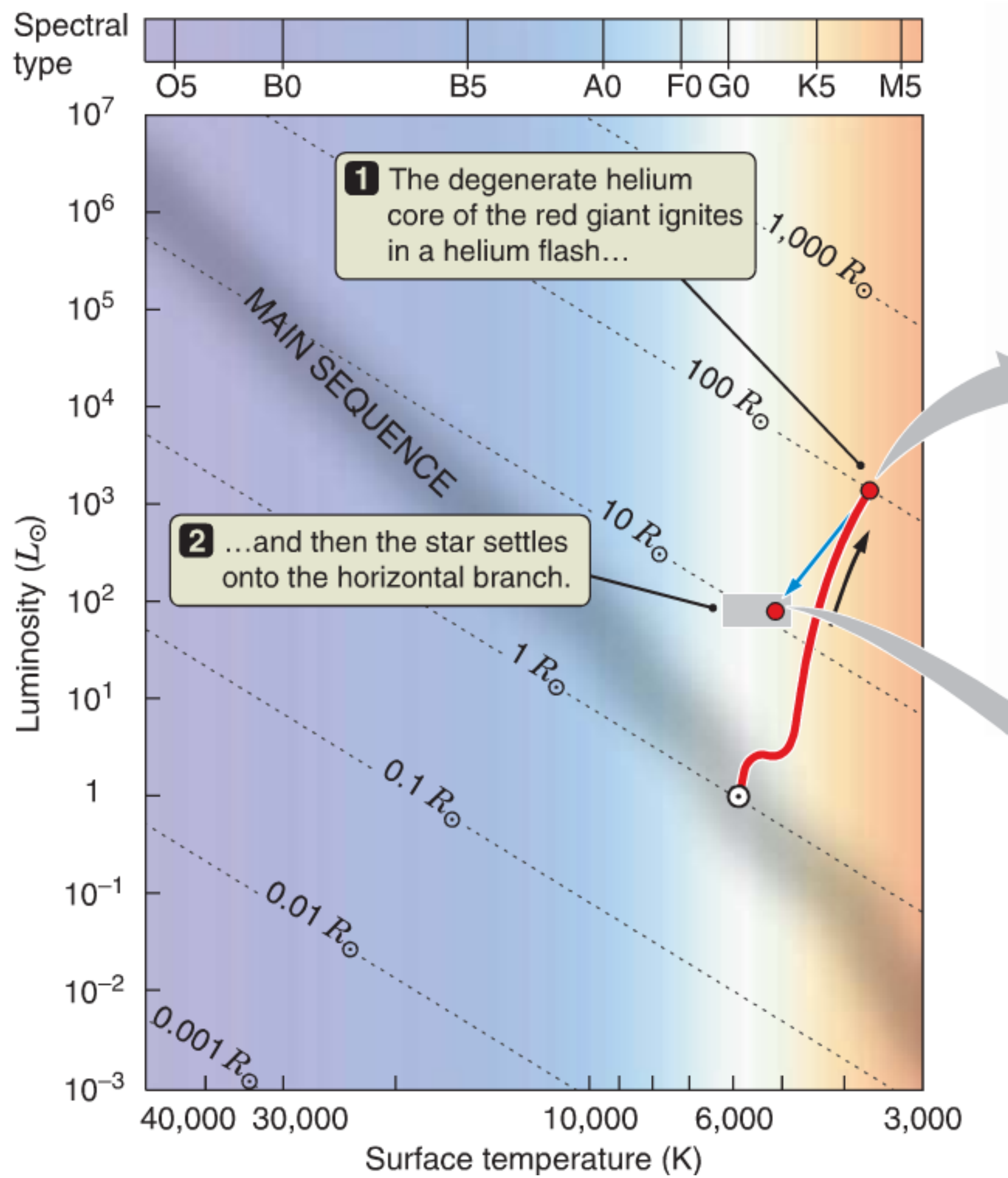
Horizontal Branch

a new main sequence

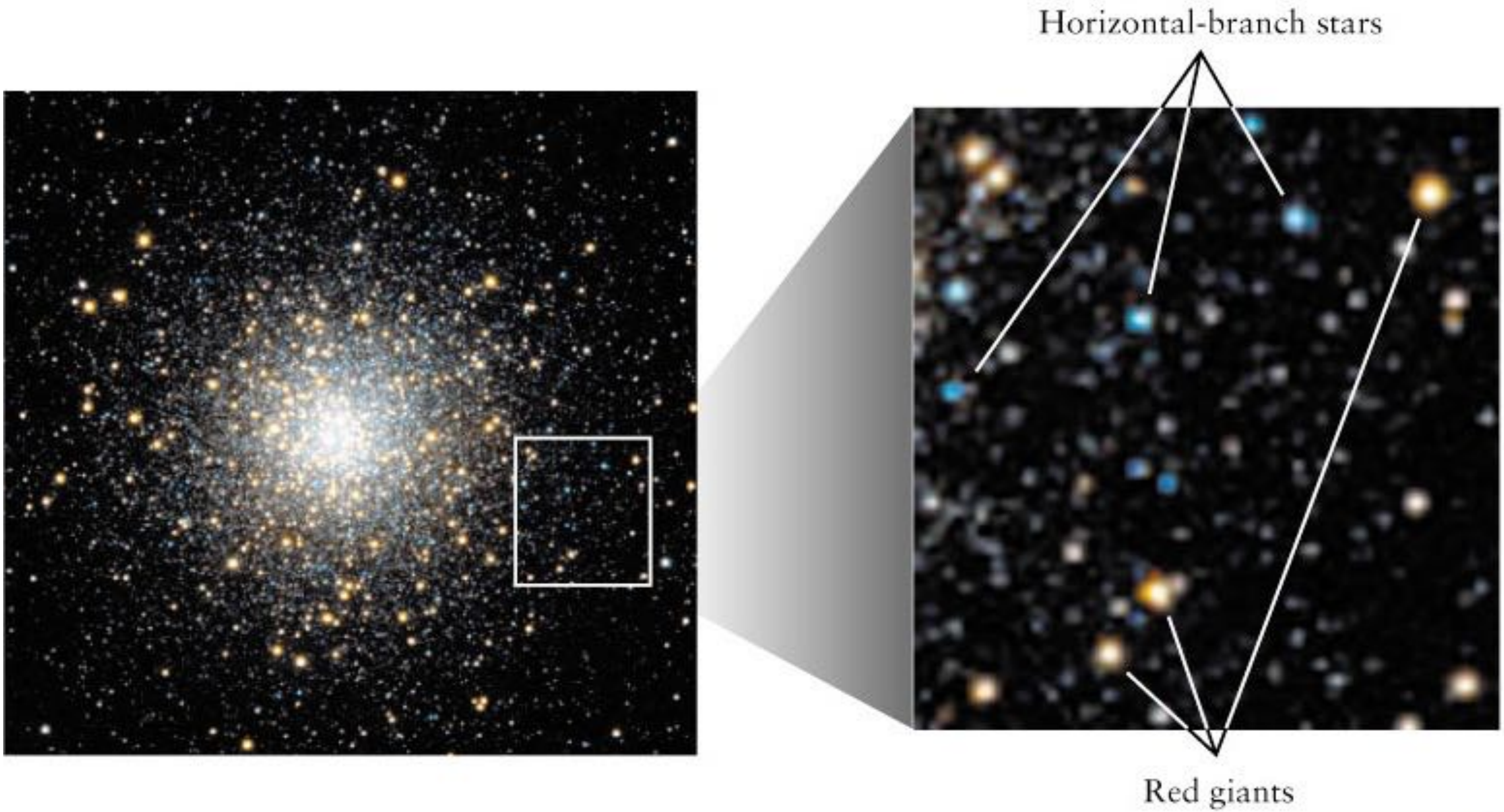
After Helium flash

- Settles onto ***horizontal branch***
- Stable Helium core burning, similar to a main sequence star
- H burning continues in shell surrounding the He burning core

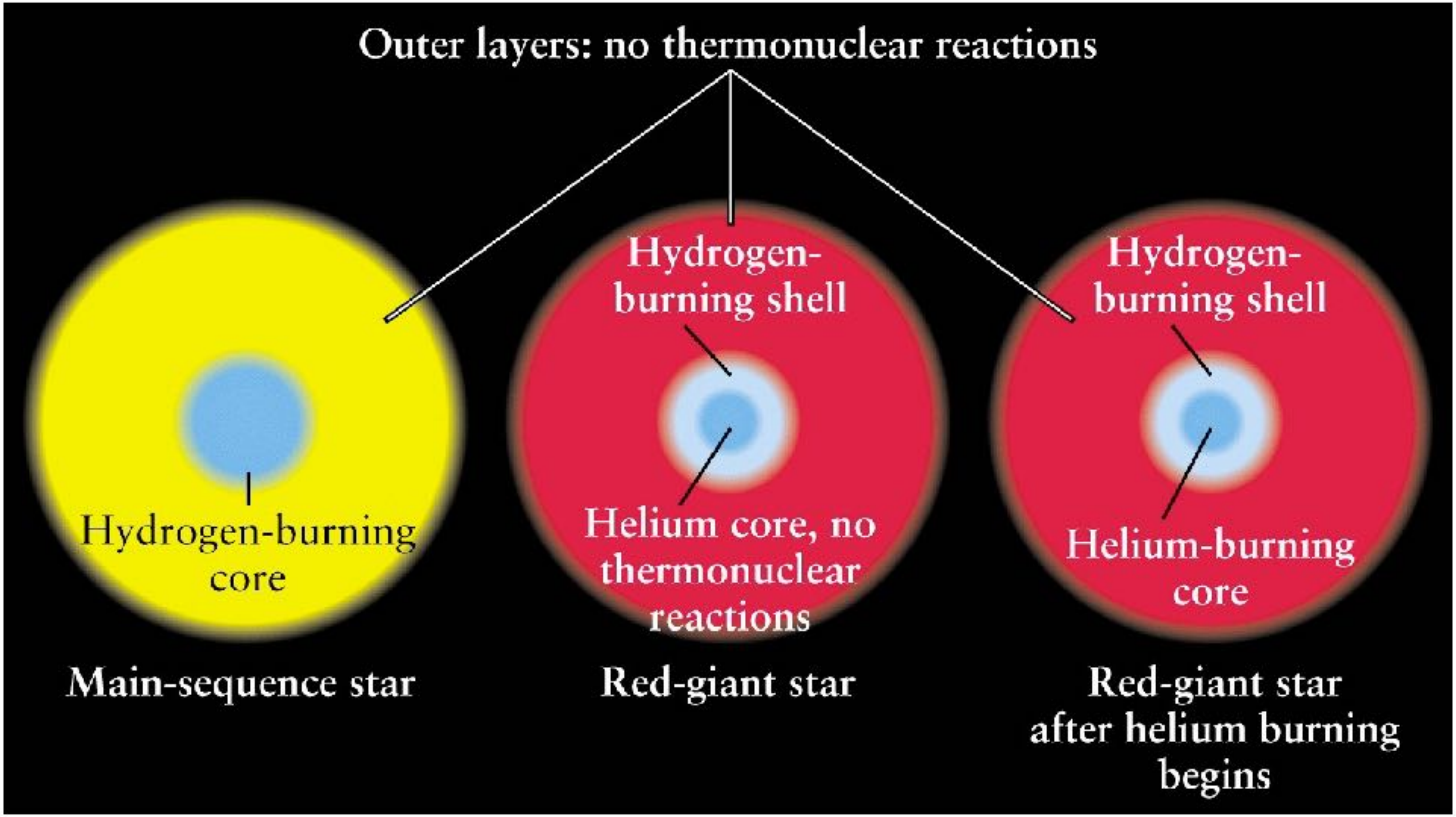
- Main Sequence – 10 Gyr
- Red Giant – 1 Gyr
- He flash - a few hours
- He flash to HB – 100 Kyrs
- HB - 100 Myrs, a new “MS”



Horizontal branch stars are hotter (bluer) than RGs, and are evolving horizontally to left on HR diagram



Main Sequence → Before He flash: Red Giant → After He flash: Horizontal Branch



Post-Main-Sequence Evolution

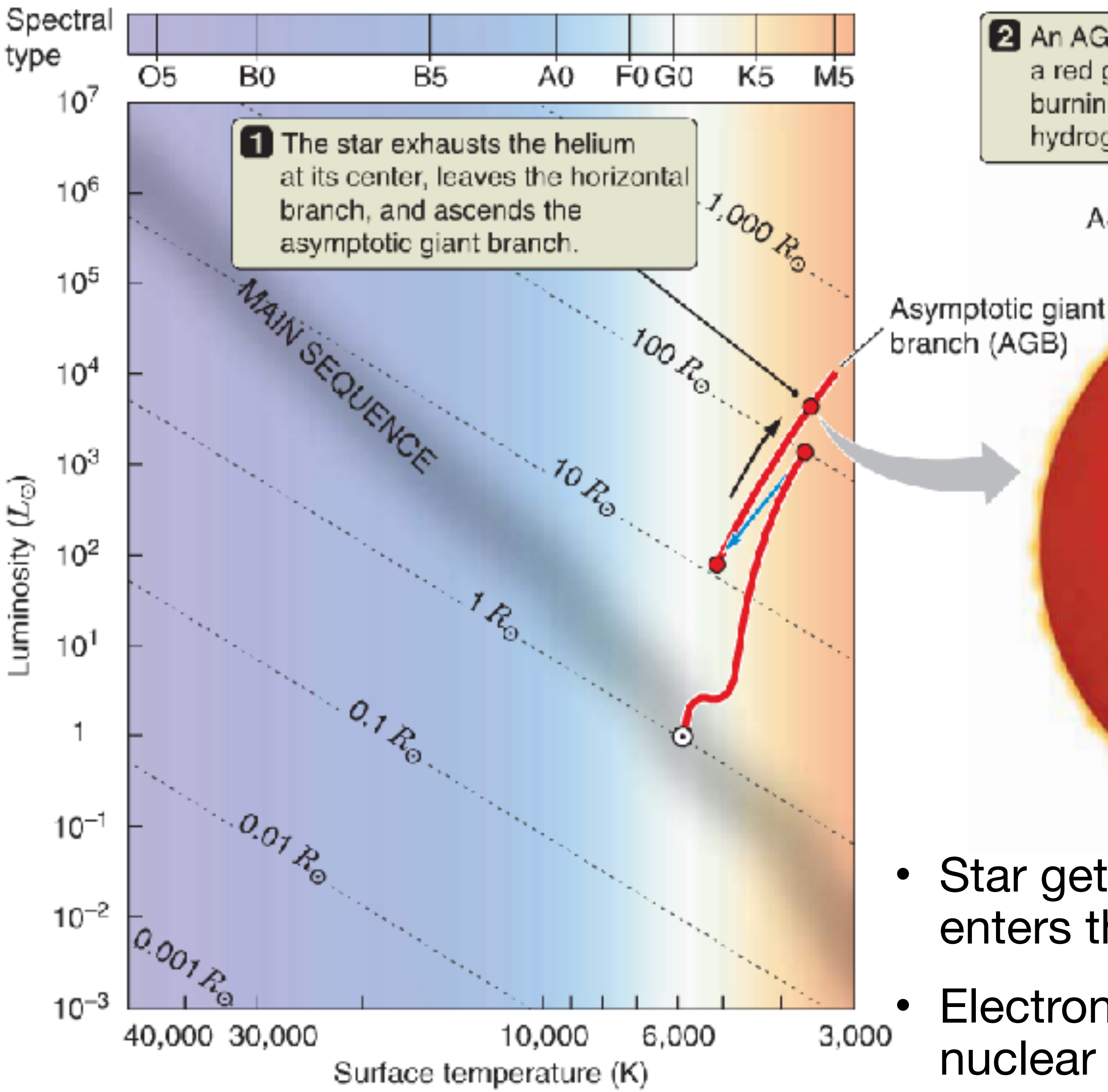
- non-fusing Helium core surrounded by H-burning shell
- At **SC-limit**, Helium-core becomes **e- degenerate**
- **Red giant:** H^- controls surface T as L increases
- **Helium Flash:** uncontrolled Helium-burning in the e-degenerate core (thermonuclear runaway)
- **Horizontal branch:** core expands and become non-degenerate, allowing steady Helium burning

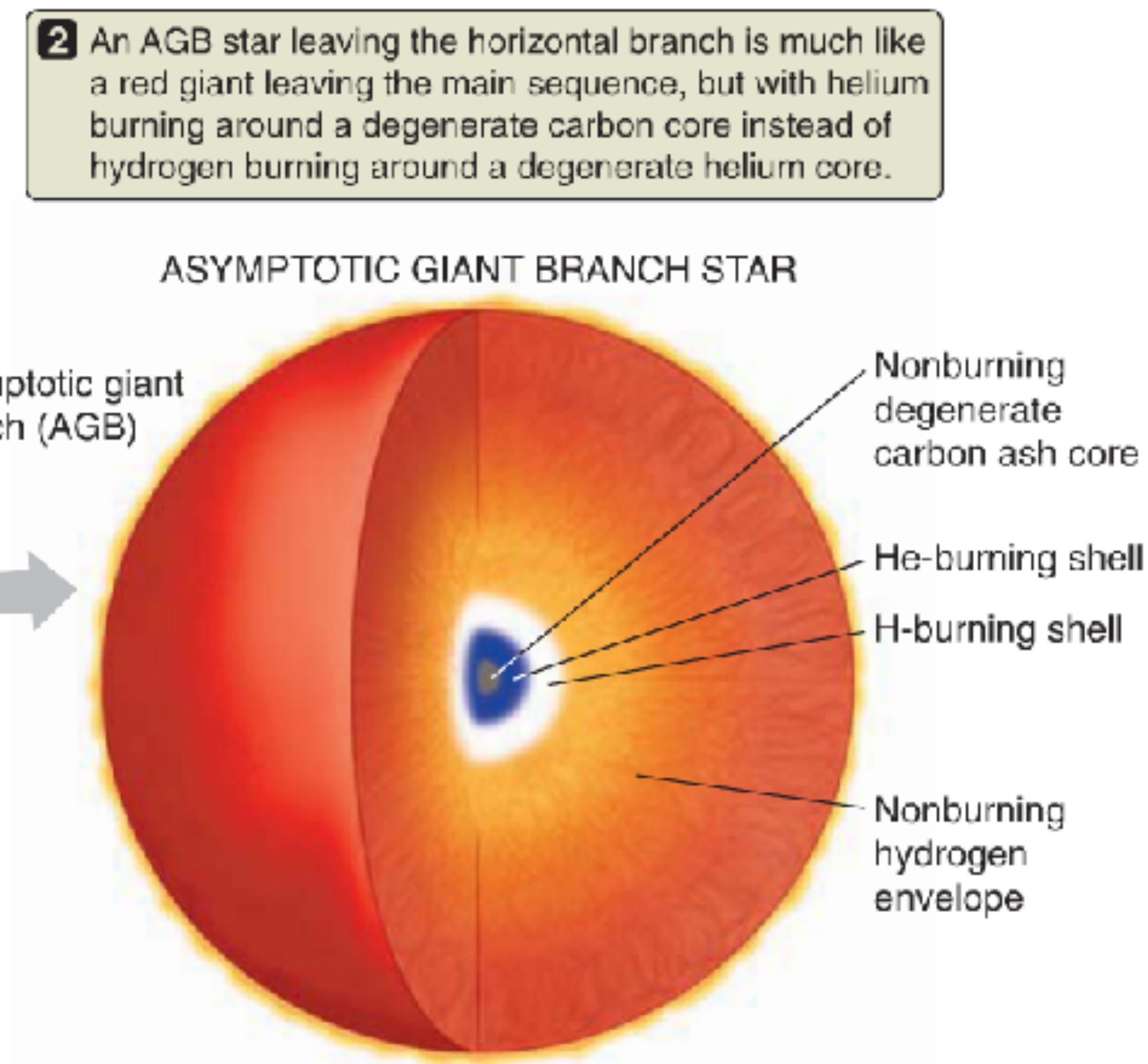
Eventually, Helium will be exhausted at the center of a Horizontal Branch star, and a non-fusing Carbon core will form under a Helium-burning shell? This marks the end of the horizontal branch phase. Based on what you learned about the post-MS evolution, **deduce the post-HB evolution of the star**.

Post-Horizontal Branch Evolution

The Asymptotic Giant Branch (AGB), Post-AGB phase: Planetary Nebula, White Dwarf

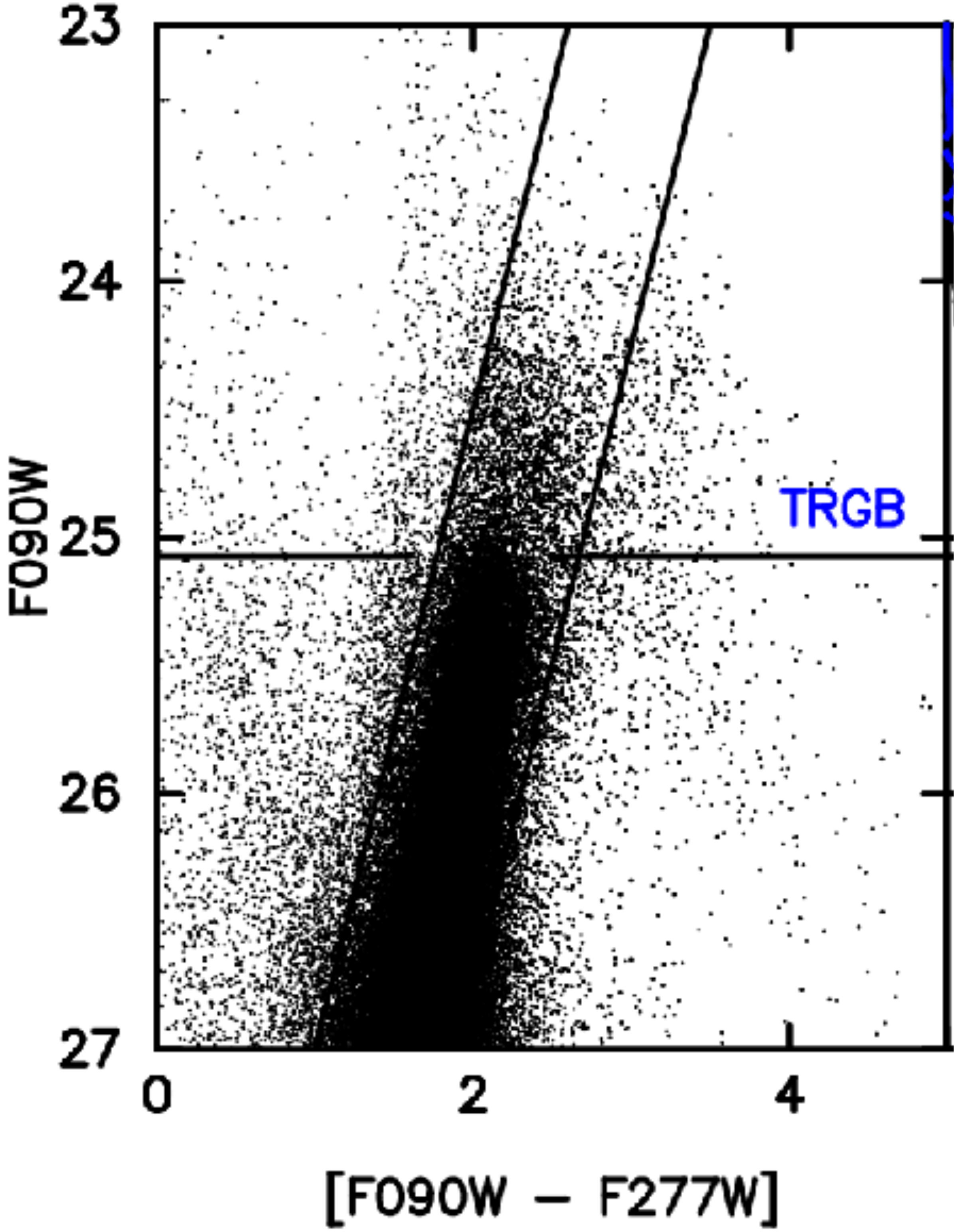
When Helium is depleted in the core, the star evolves into the Asymptotic Giant Branch (AGB)

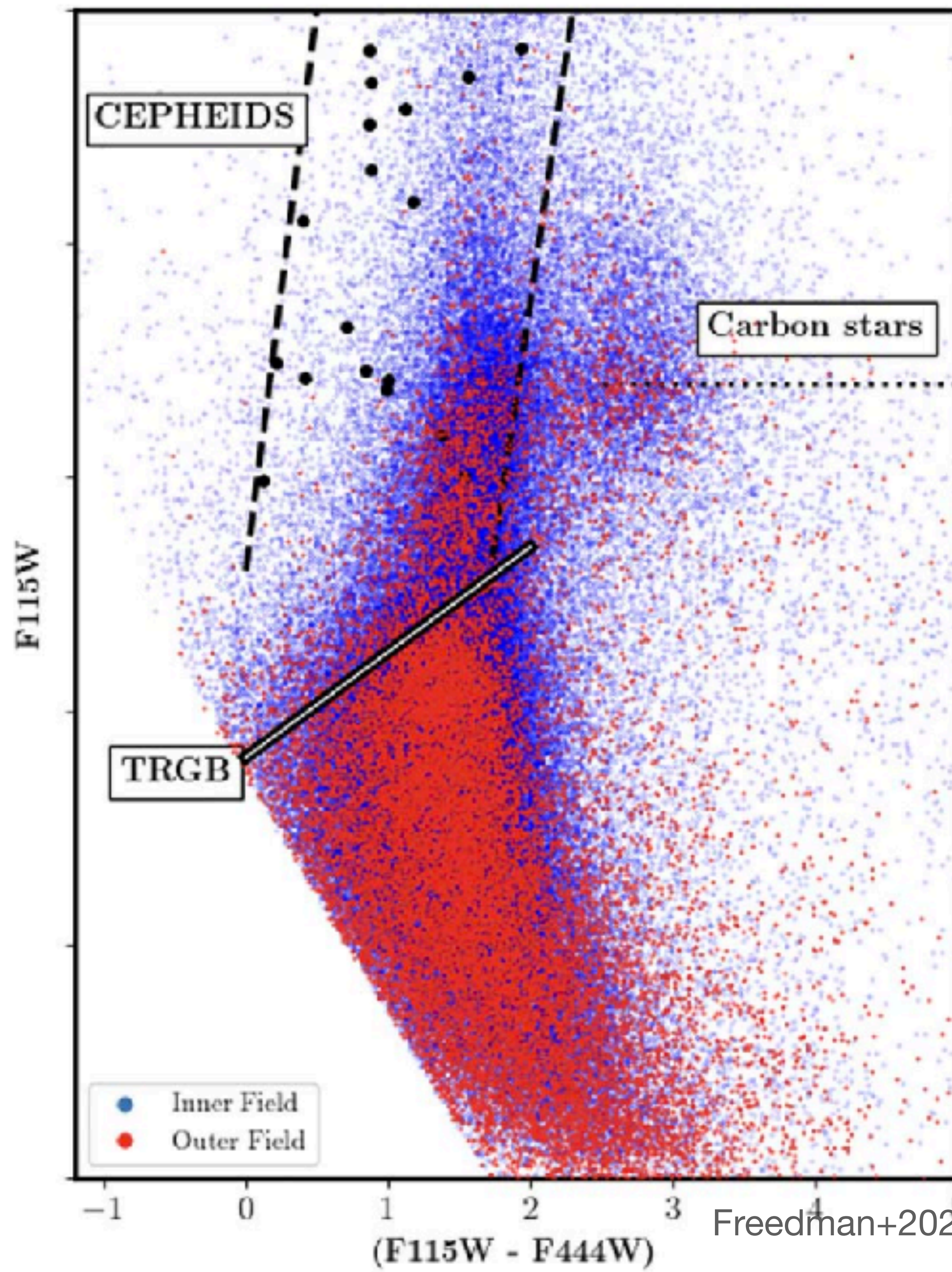




- Star gets more luminous and cool, and enters the asymptotic giant branch (AGB).
- Electron-degenerate Carbon core with no nuclear fusion (T not hot enough)
- H^- keeps the surface temperature almost constant, like in the Red Giant Branch (RGB)

AGB stars, RGB stars, and Cepheids on the HR Diagram





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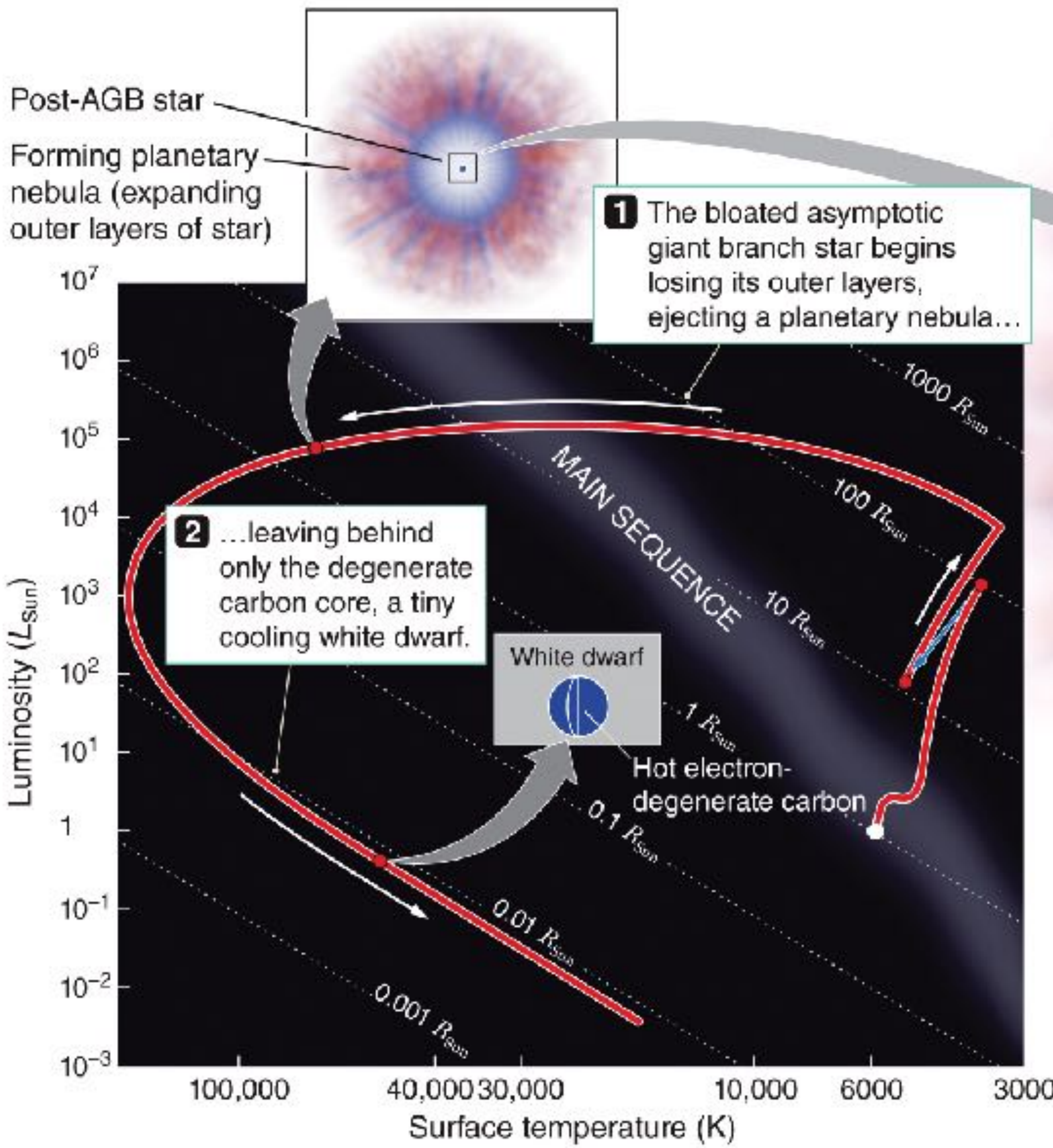
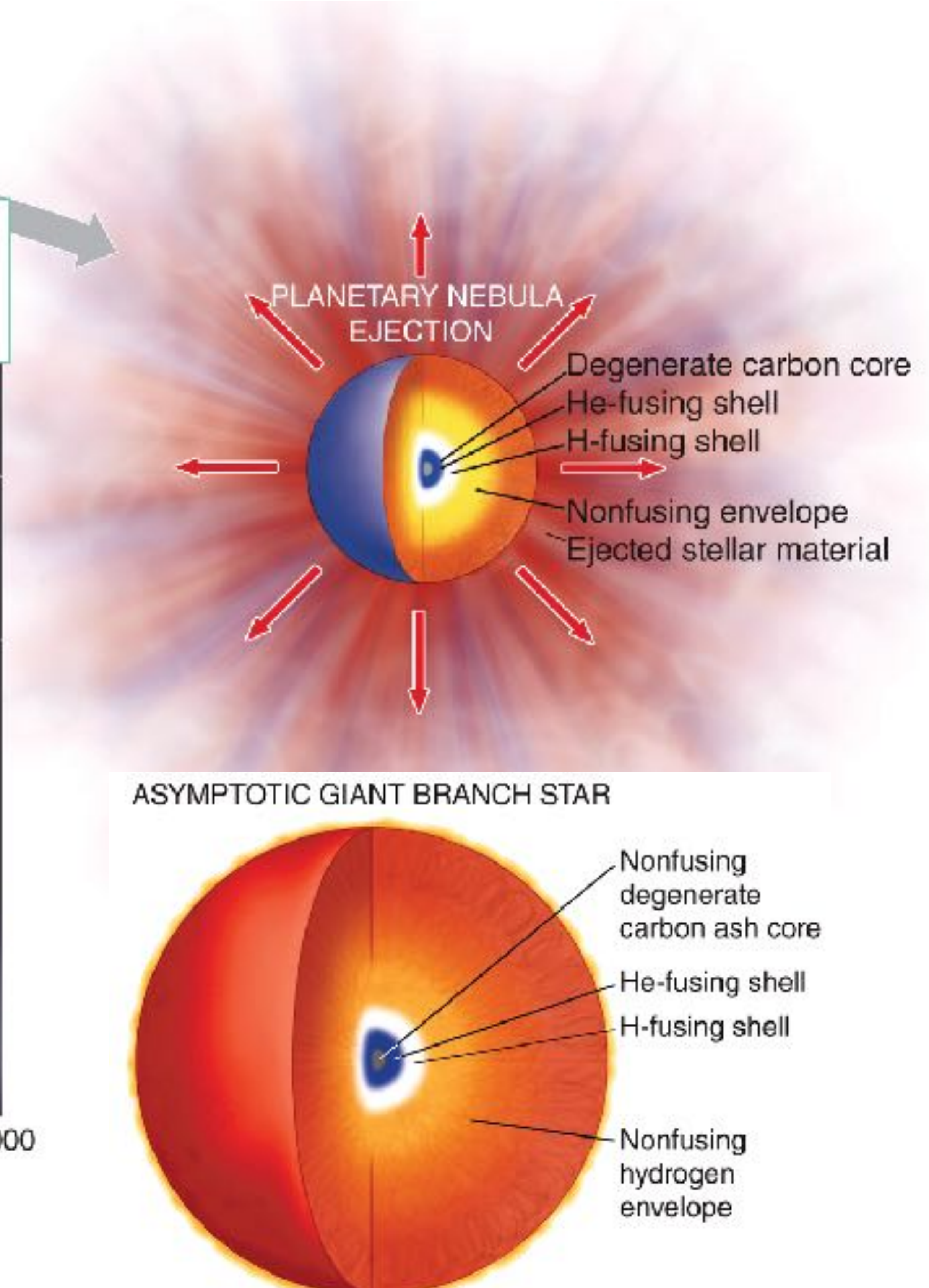
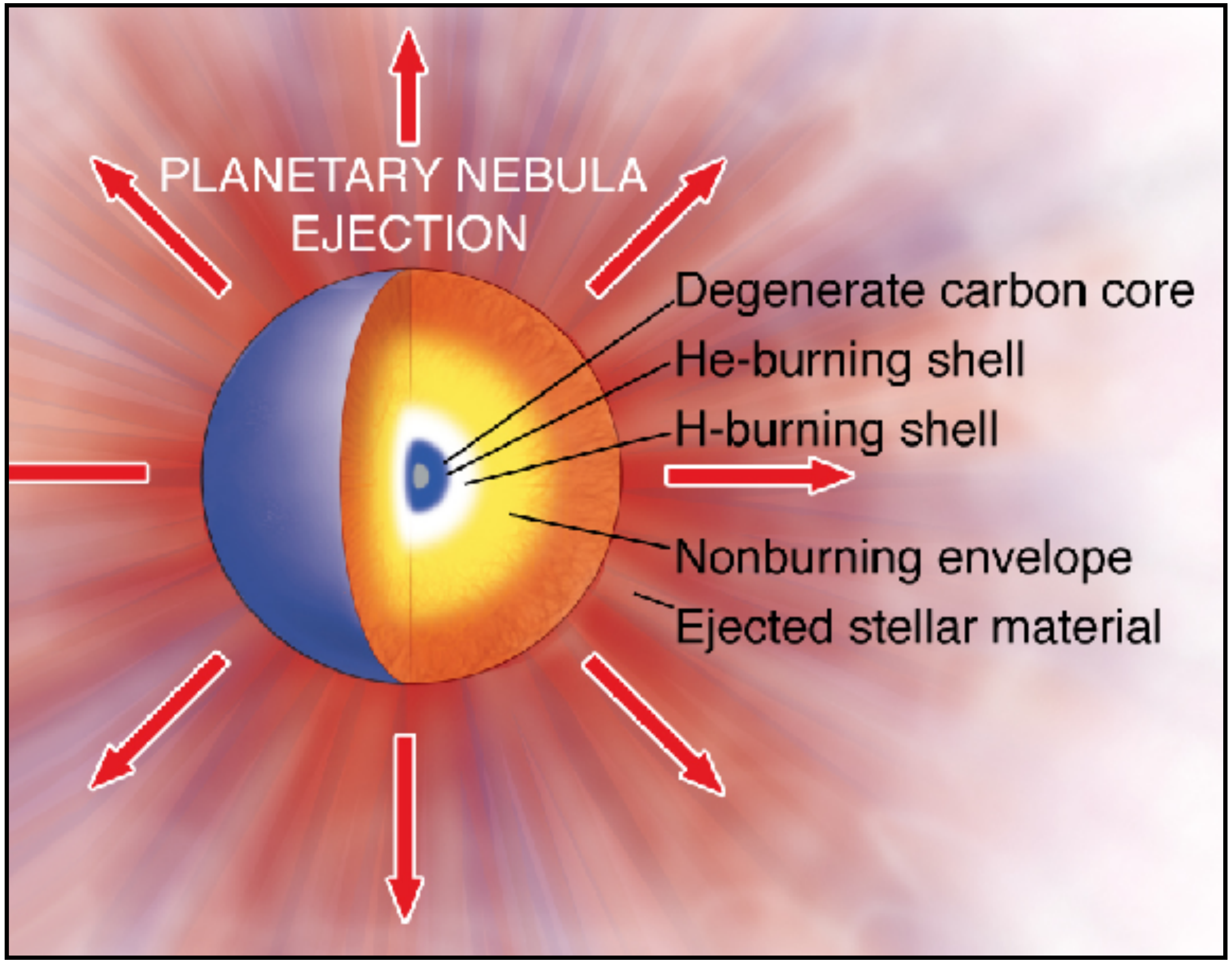
Post-**AGB** Mass Loss

- Asymptotic Giant Branch – 100 Kyr
- Post-AGB / Planetary Nebulae - 10 Kyr





Post-AGB - No "Carbon Flash" but a Planetary Nebula



- Temperature never gets high enough to initiate C burning (**500 million K** at core density)

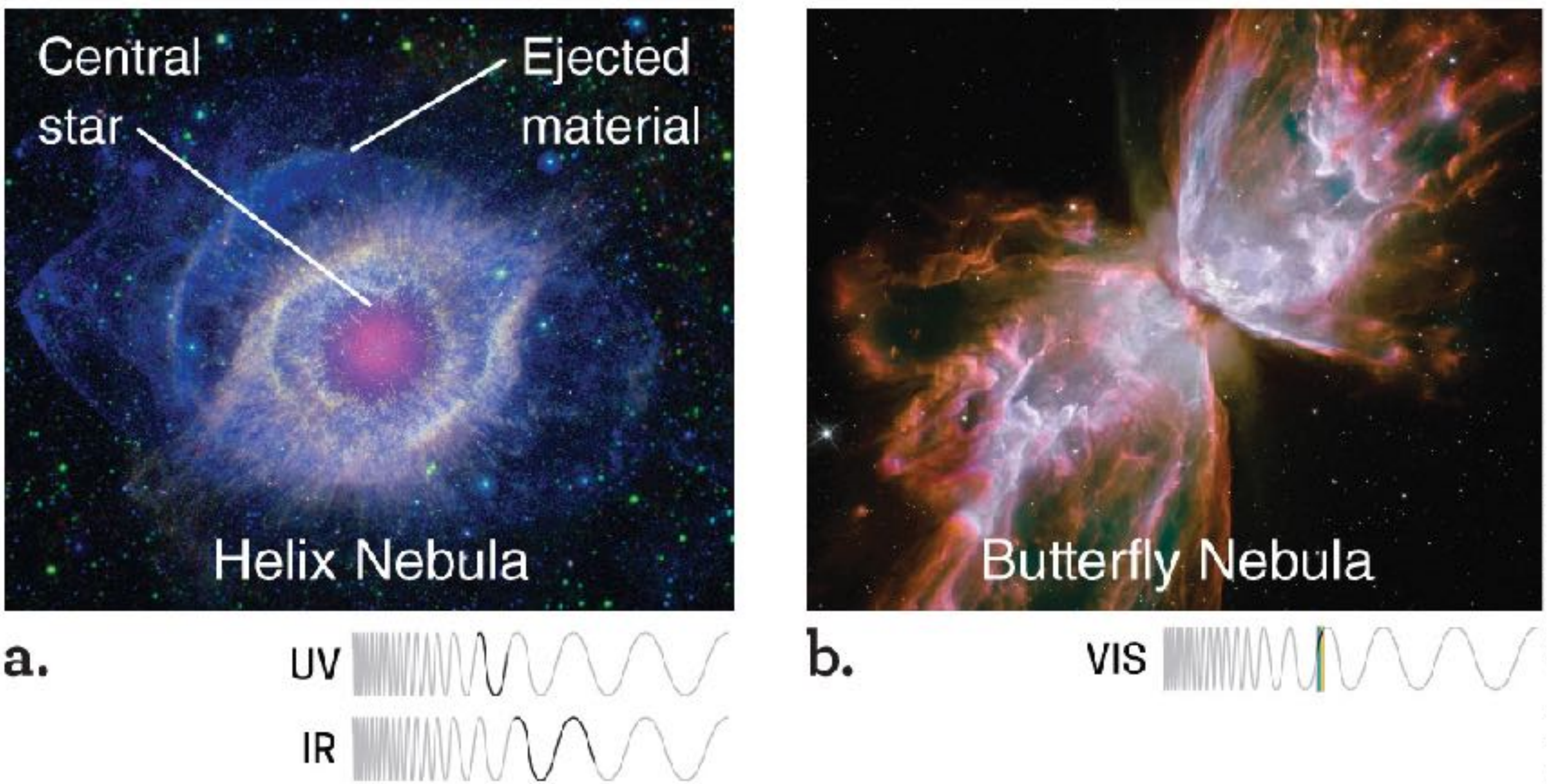
$$^{12}_{6}C + ^{12}_{6}C \rightarrow ^{24}_{12}Mg + \gamma$$

- **Mass loss becomes a runaway process** – forming **planetary nebulae**
- mass loss -> star puffs up -> less gravity -> more mass loss

M57 - the Ring Nebula in Lyra near Vega

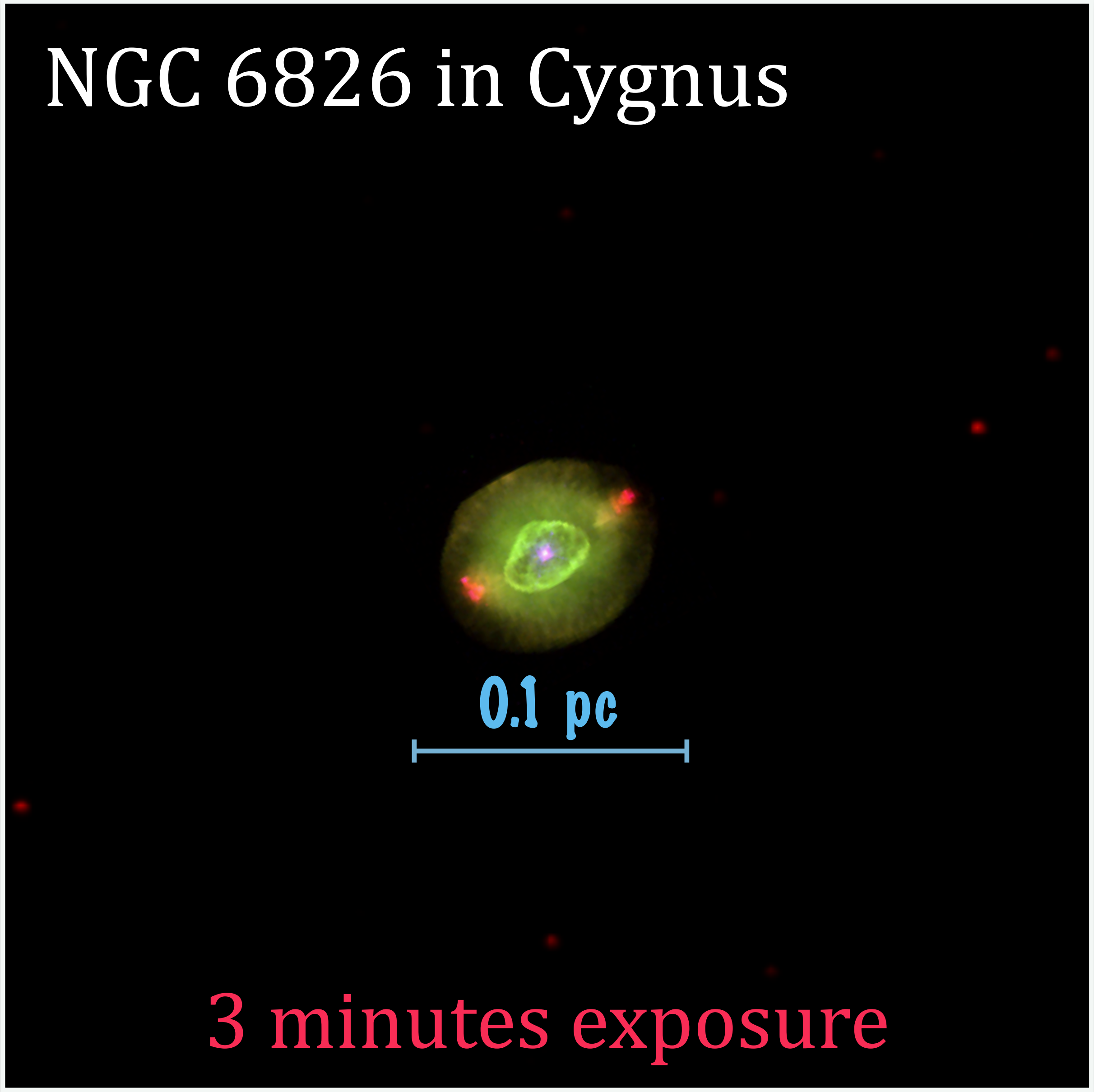
Planetary Nebulae

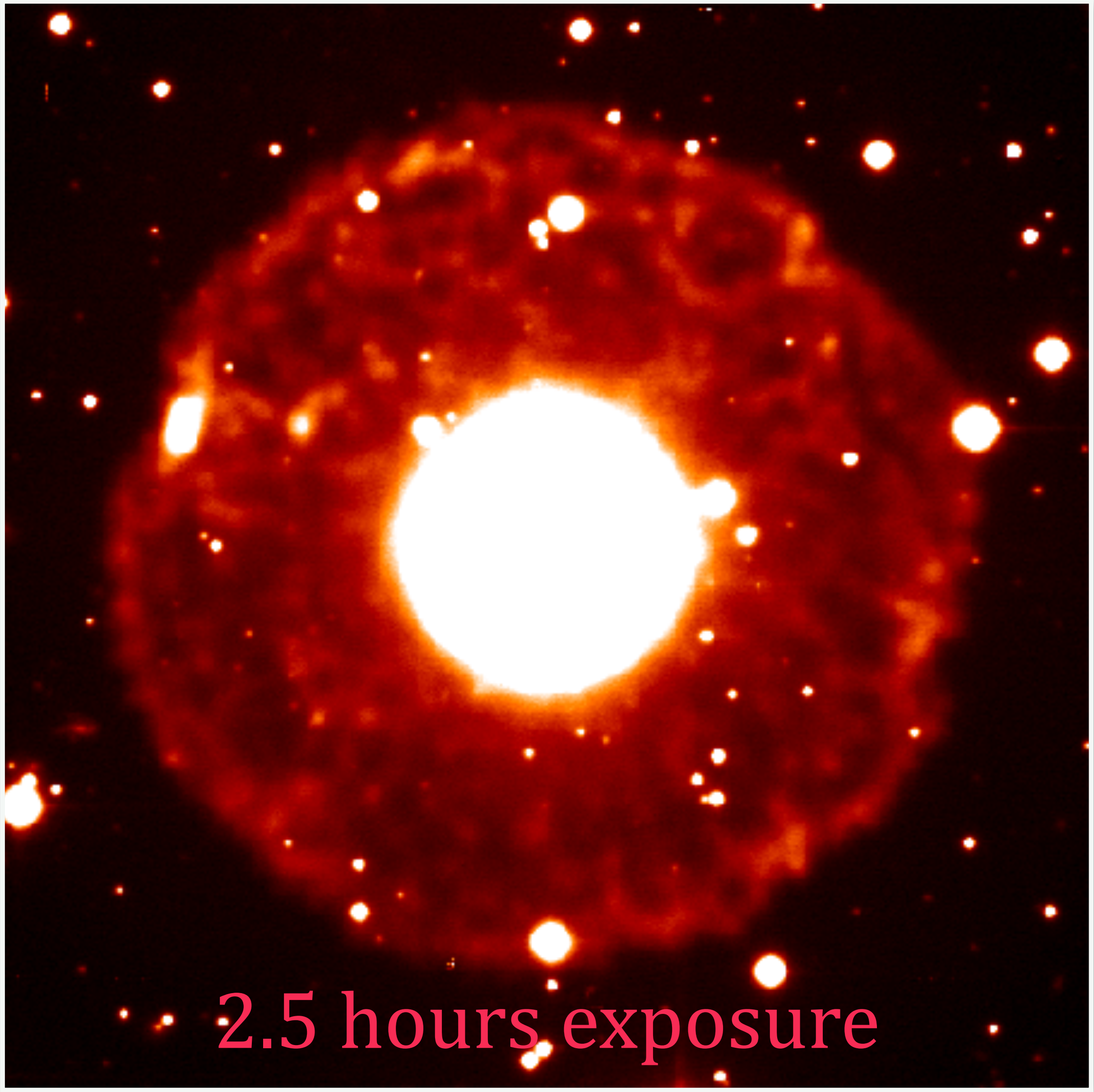
- Material farther from the star was ejected earlier.
- Radiation from the white dwarf ionizes the gas. The colors are due to specific atoms and bright spots indicate areas of denser gas.

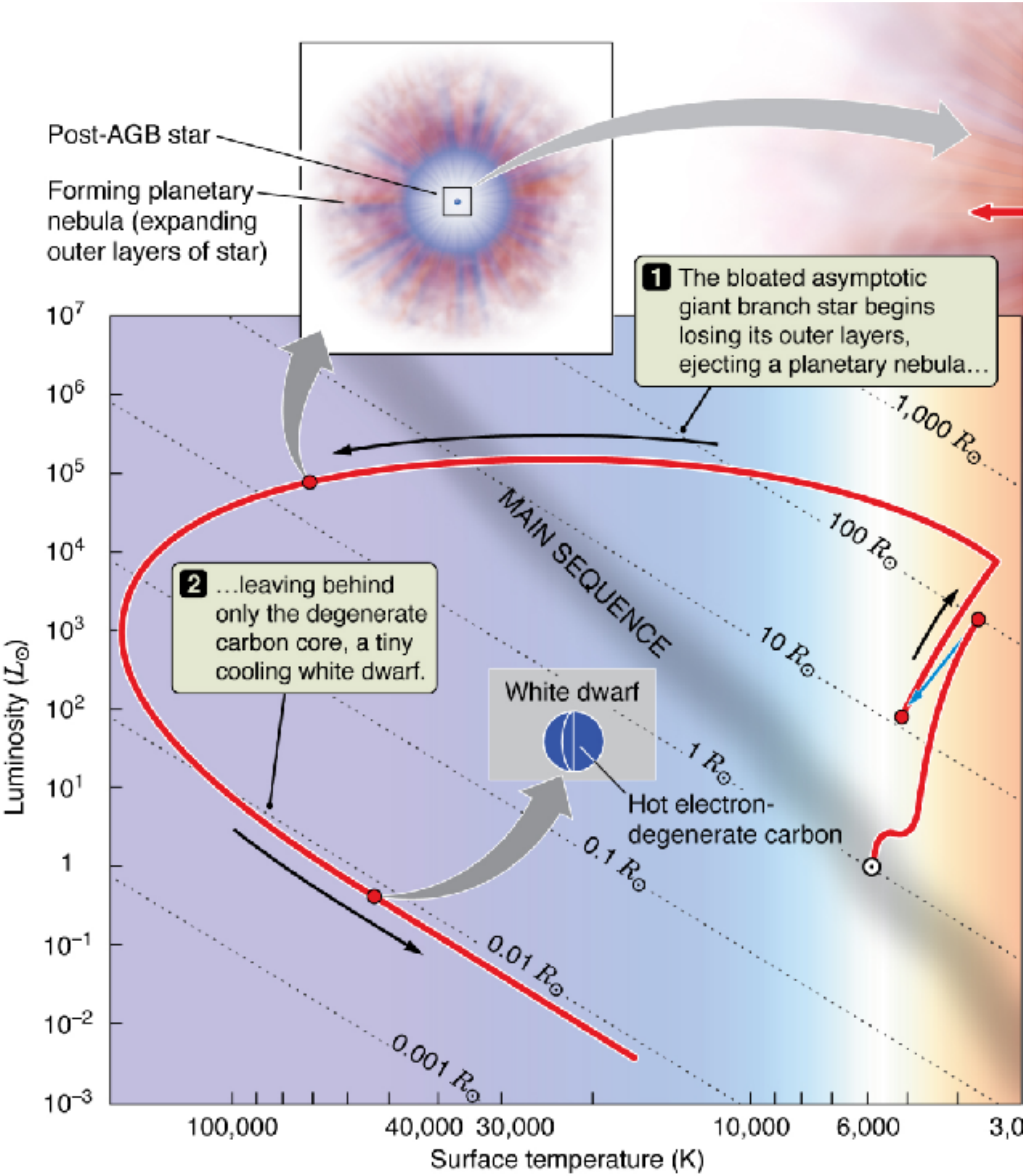


Severe Stellar Mass Loss illustrated in a Planetary Nebula

The planetary nebula extends much farther than the central bright area



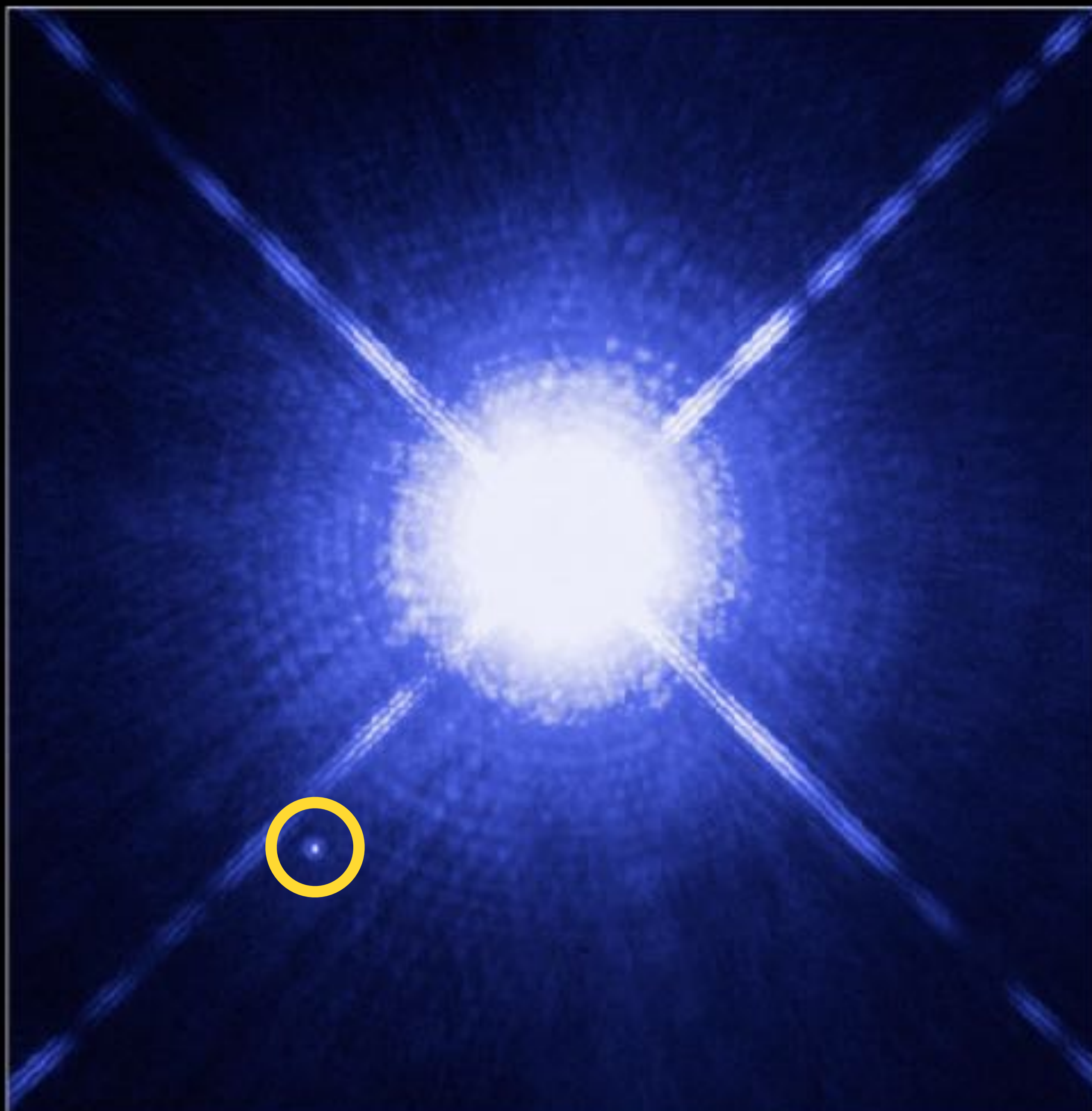




Post-AGB and White Dwarfs

- Post-AGB lasts **~10,000 yrs** before the gas expands too far and disperses into the ISM
- The hot **electron-degenerate Carbon core** gradually reveals itself as the star's outer envelope disperses.
- A **white dwarf** is born.

The Evolution End Point - White Dwarfs



Sirius A and Sirius B
Hubble Space Telescope • WFPC2

NASA, ESA, H. Bond (STScI), and M. Barstow (University of Leicester) STScI-PRC05-36a

- Inferred properties of Sirius B:
  - 1 Solar Mass
  - 0.03 Solar Luminosity
  - 27,000 K surface temperature
  - 5500 km radius (Earth-size)

- The physical conditions of WDs are extreme:
  - extreme density ($\rho \approx$ 3e9 kg/m^3)
    ($n_e \sim$ 1e36 /m^3)
  - extreme surface gravity
  - extreme pressure at the center:

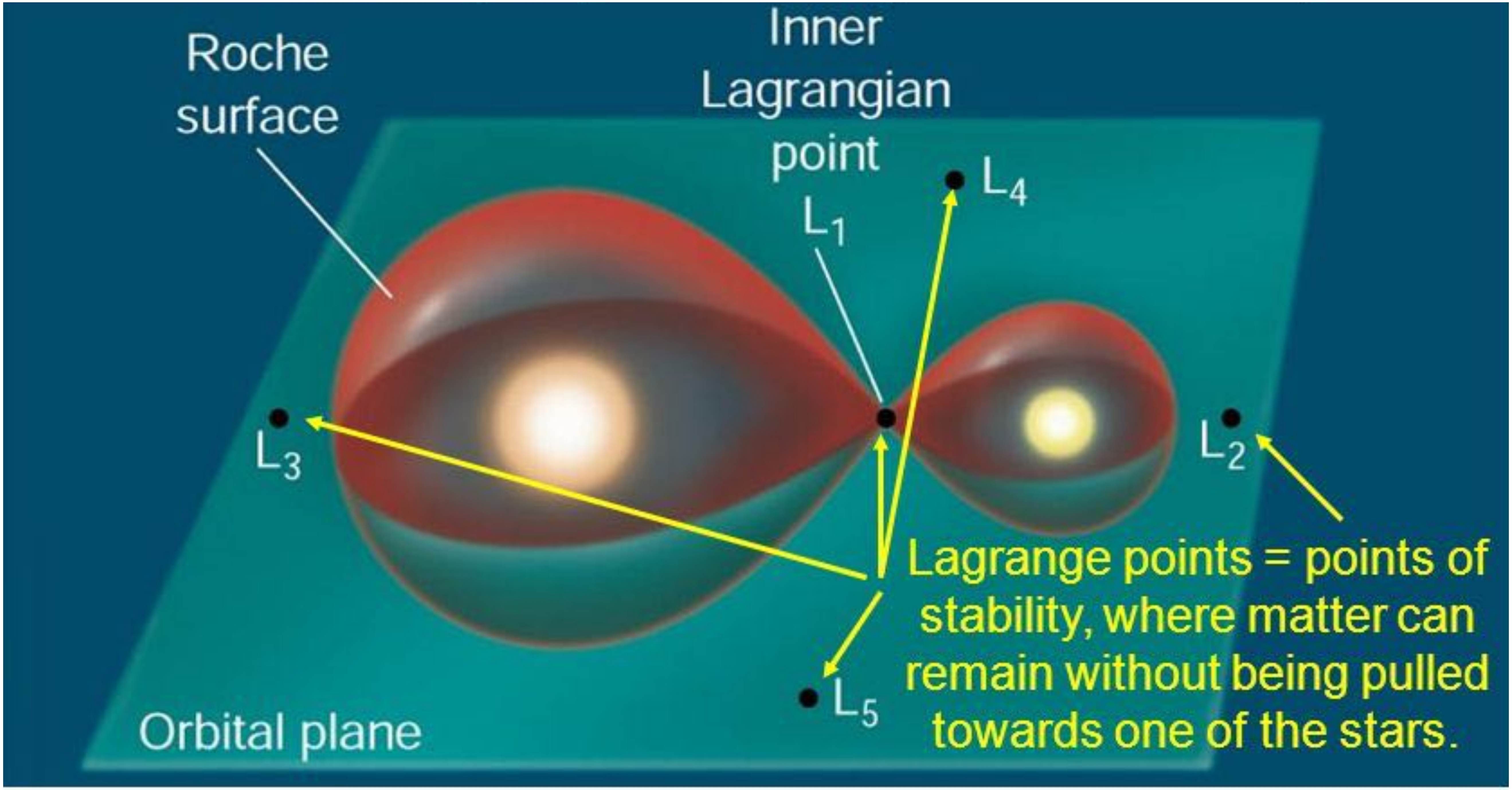
* a **white dwarf** will cool for eternity, without changing its size.
How would you calculate the time it takes for the temperature to half?

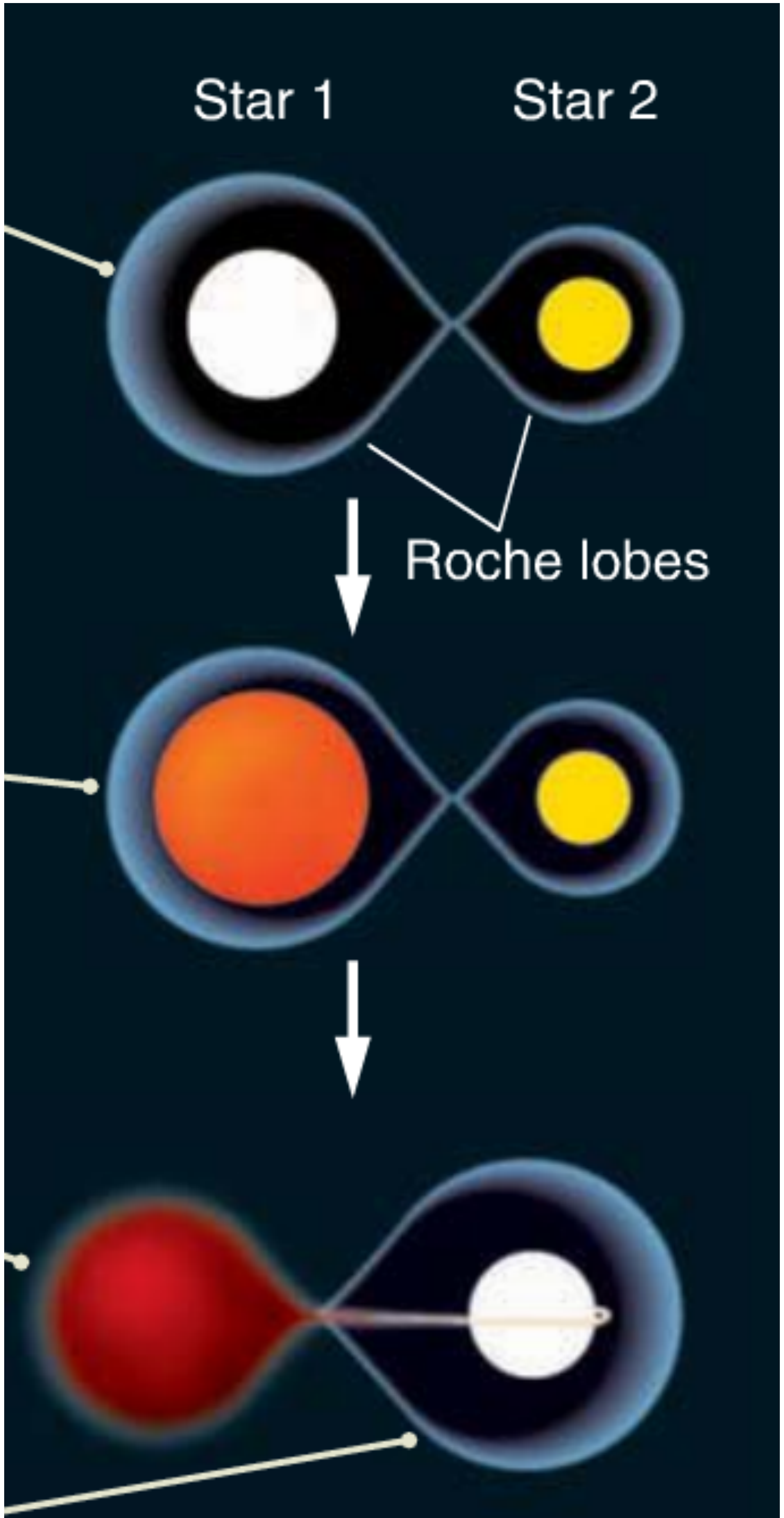
Mass-Transfer Binaries:

- Blue Stragglers
- Classic Novae
- Type Ia Supernovae

Roche Lobe (or Roche Surface)

- In 3D, the critical equipotential surface delineates two lobes in a binary system. In each lobe, small-mass objects are gravitationally bounded to the massive object at the center.

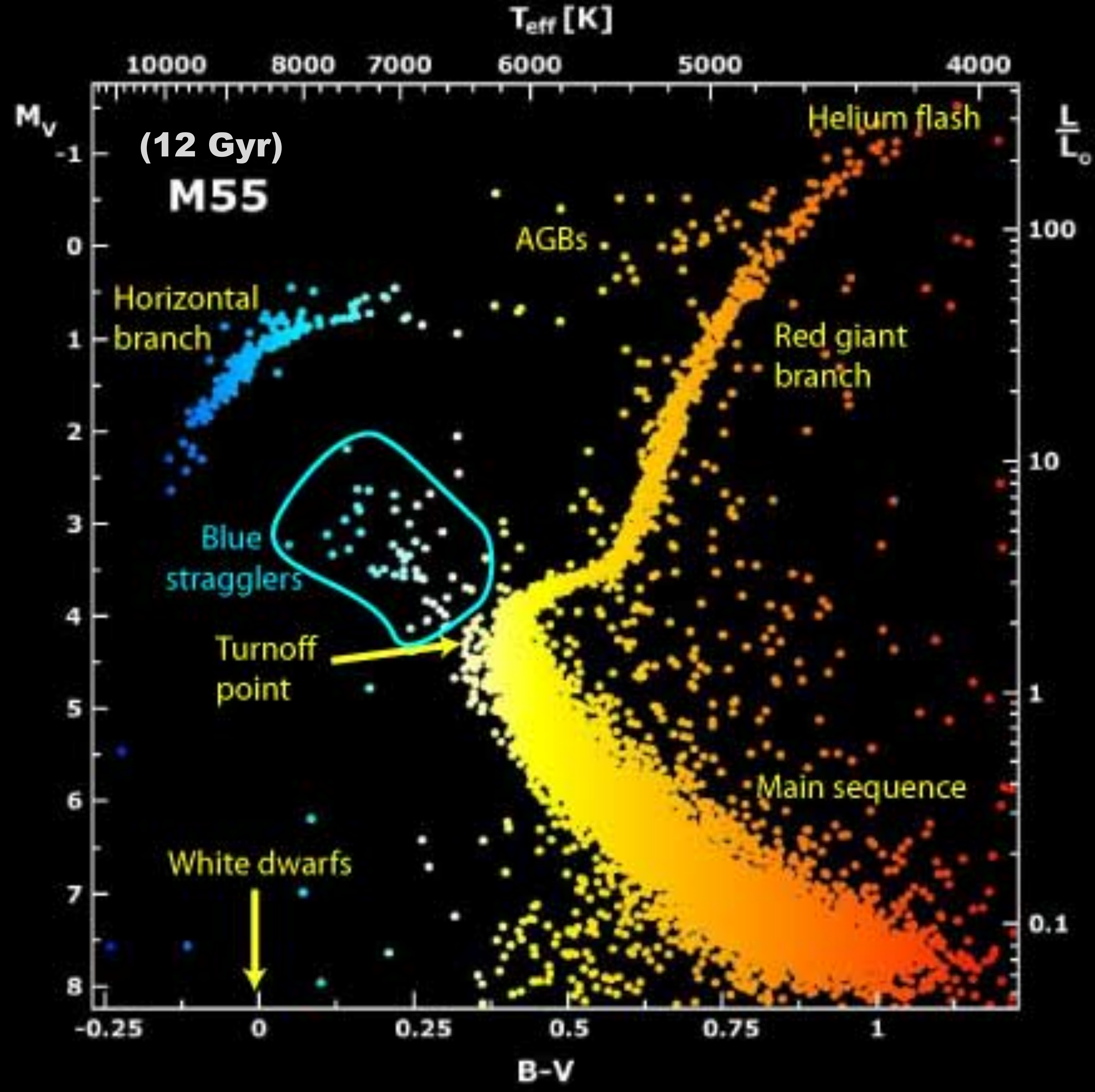


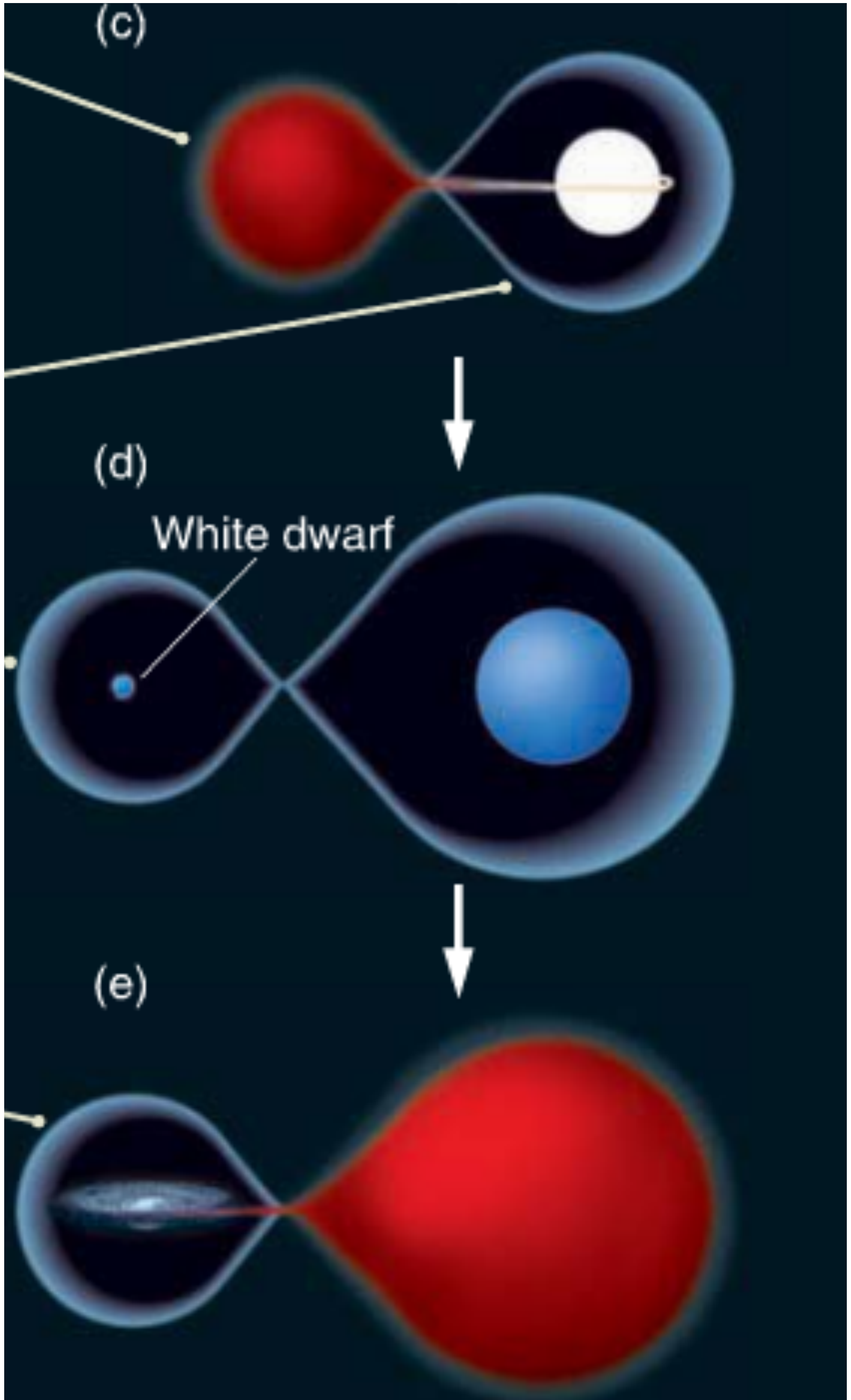


Mass-Transfer Binary Stars

- ~60% of stars are in binaries, a small fraction of which are very close binaries.
- The two stars in a binary have **different MS lifetimes** because of their different *initial* masses
- The more massive **primary** evolves into a **RGB** while the less massive **secondary** remains on the **MS** (middle figure)
- If the **Roche lobe** is smaller than the possible size of the RGB, the **red giant primary** can only expand so much before material is lost to the **MS secondary's** gravity (bottom fig)

Blue Stragglers:
MS stars in a cluster beyond the turnoff point

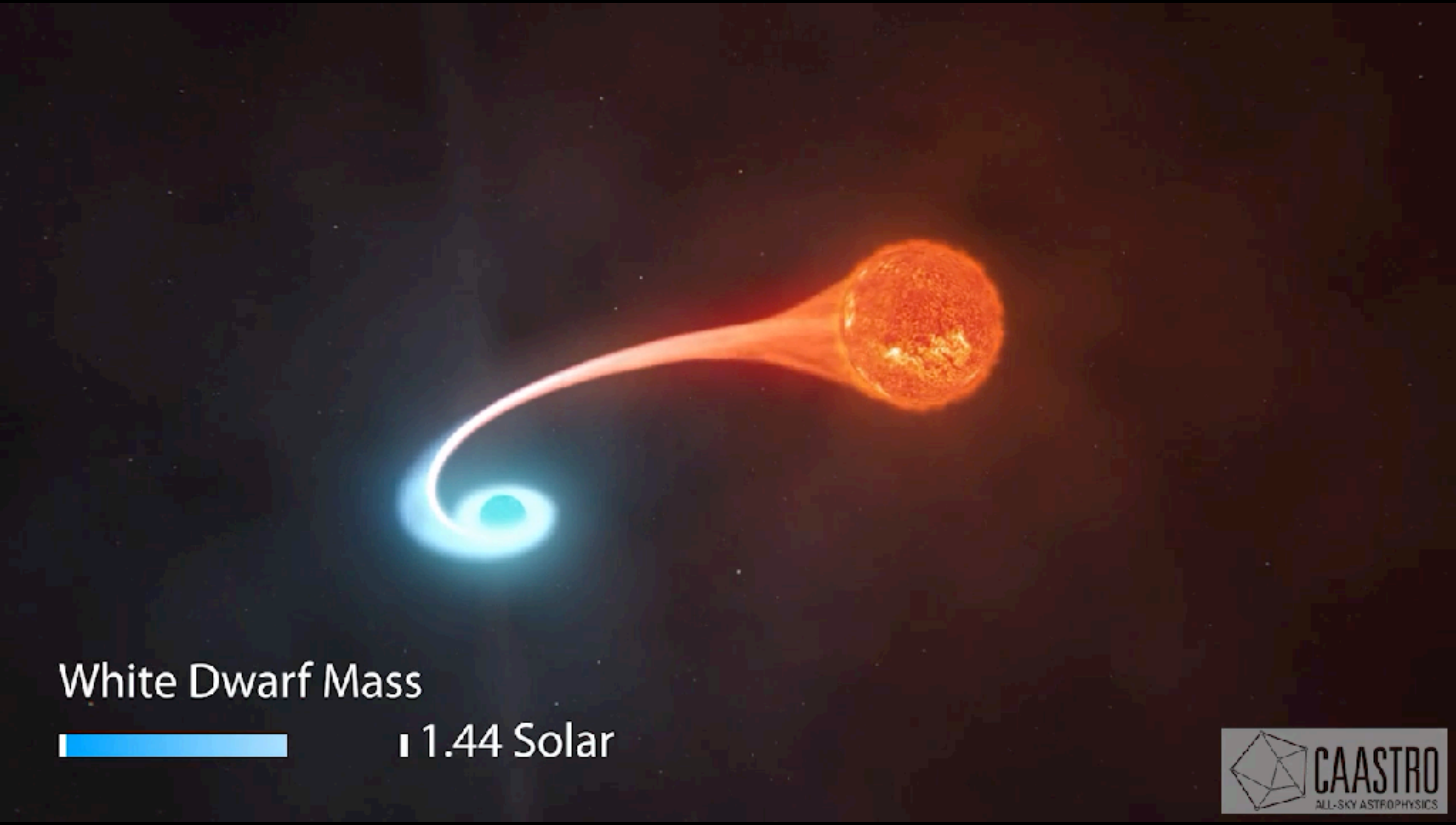


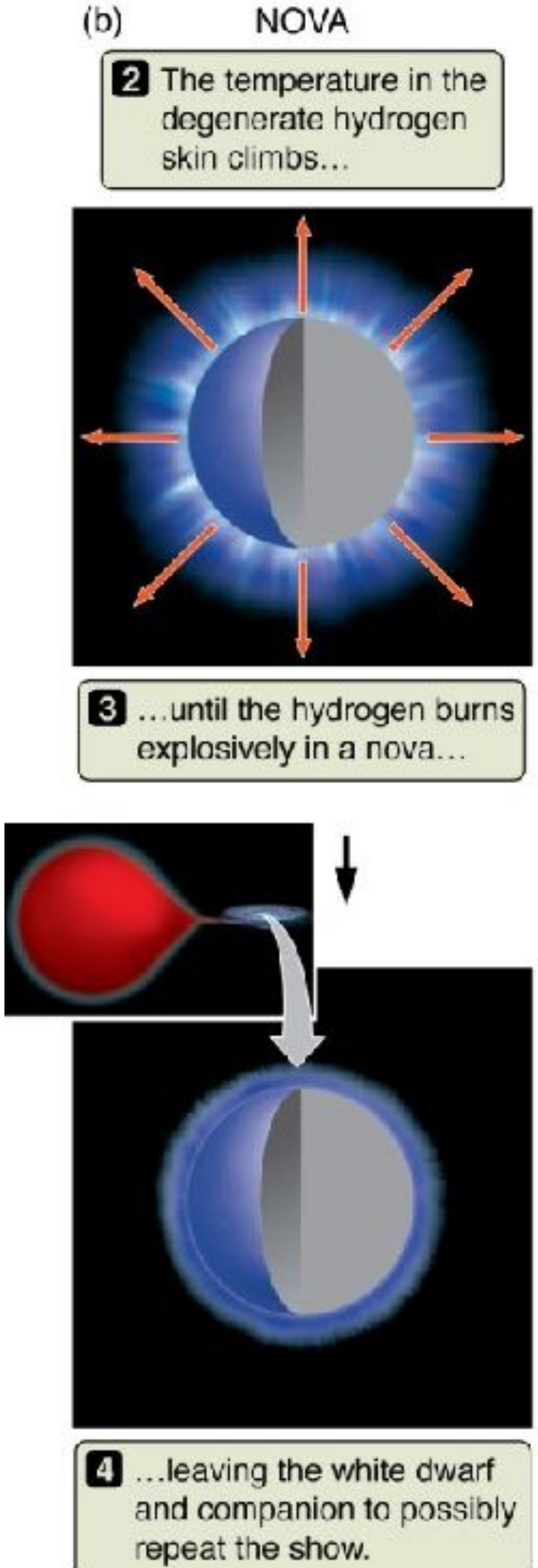


Mass-Transfer Binary Stars

- It's likely that by the time the less massive star evolves to a red giant, the originally more massive star already evolved into a WD (Fig d)
- So mass transfer reverses: the secondary star begins to lose its envelope to the WD's gravity, forming an accretion disk (Fig e)
- As the WD grows in mass because of accretion, there are two possible consequences.

Mass Transfer Binary Stars and Type Ia Supernovae



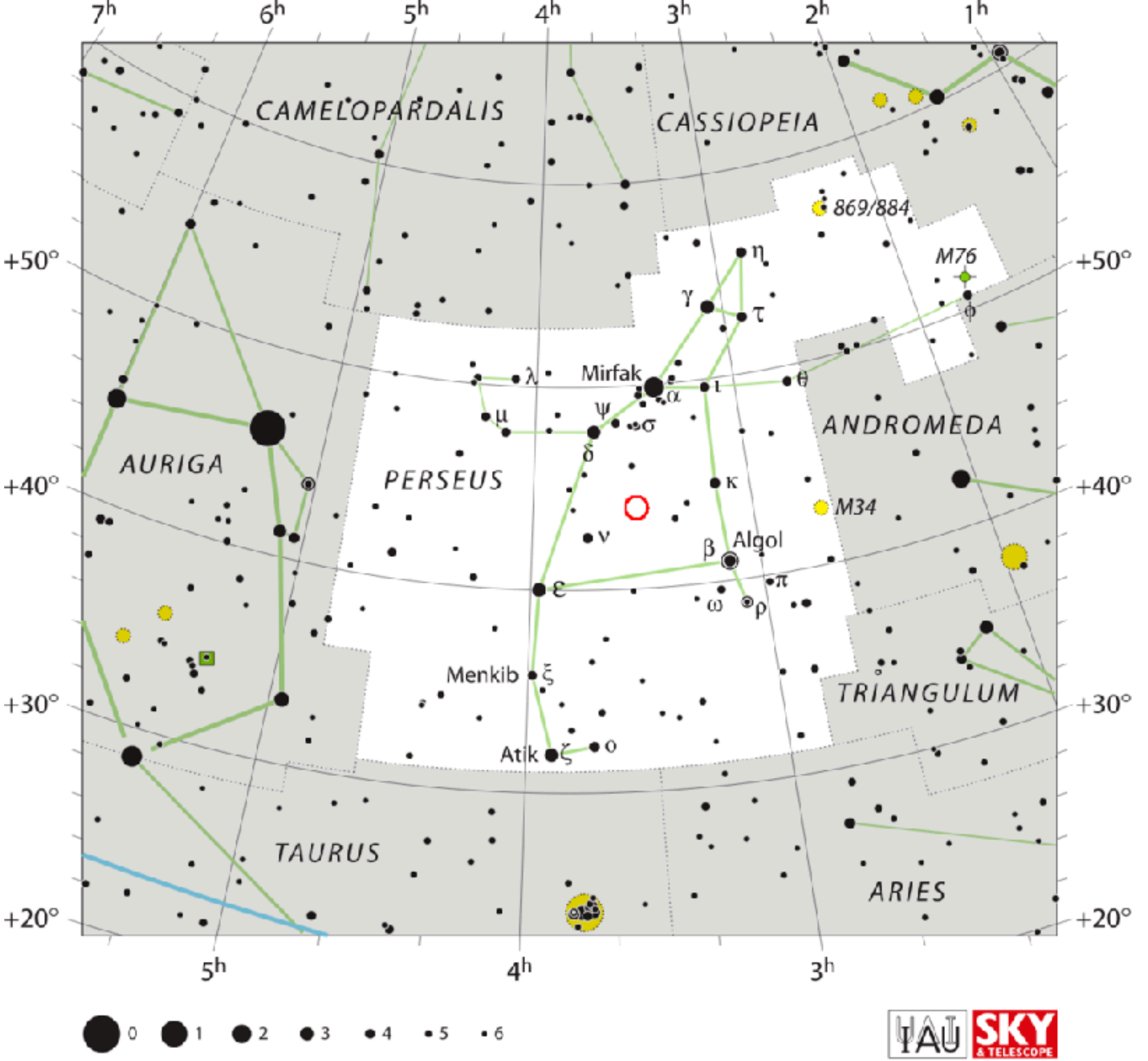


I: Classical Novae

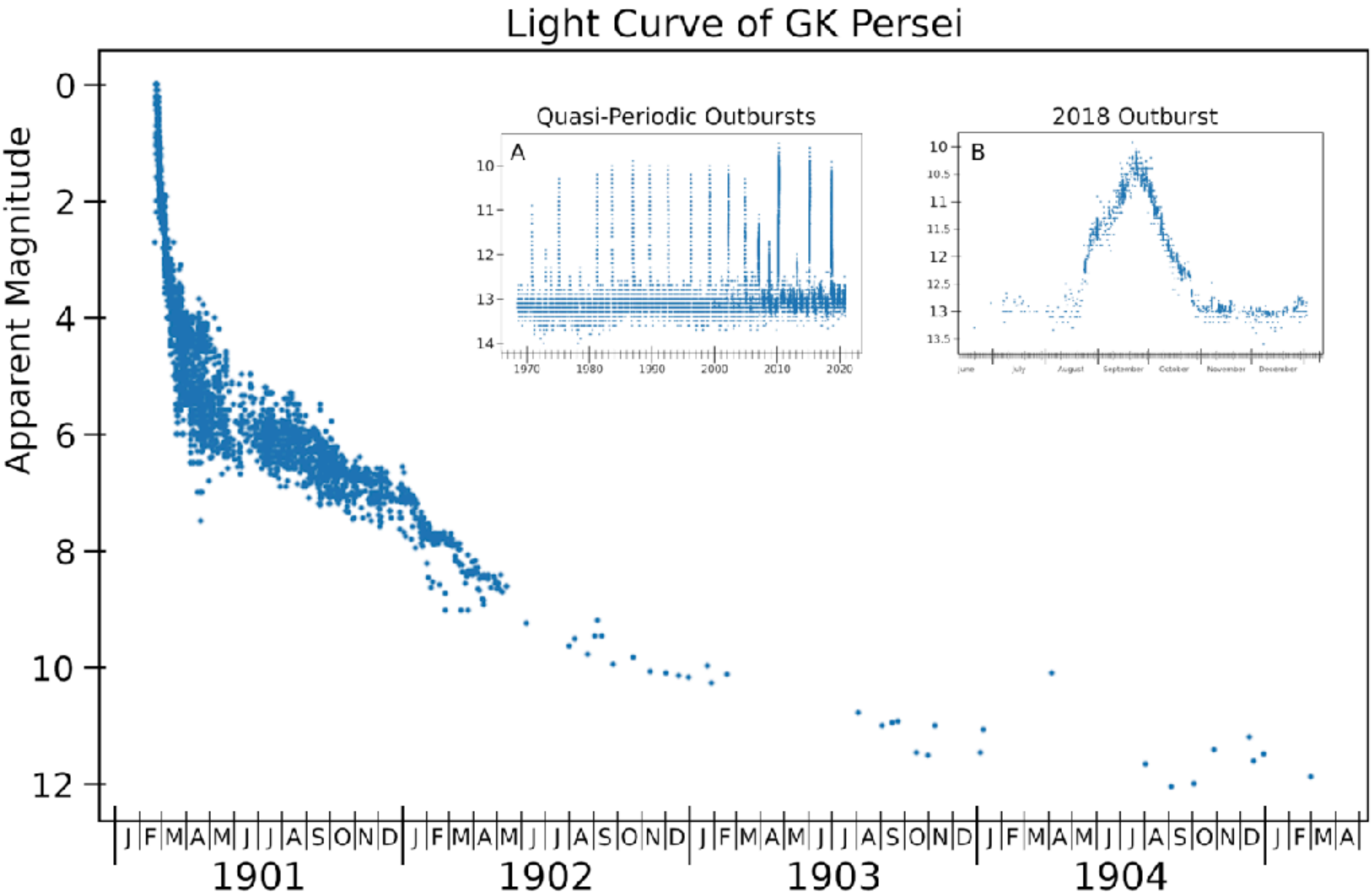
- H deposition on the surface of the white dwarf from the red giant star
- Condenses onto degenerate core and explosively burns episodically: **Nova**
- For a few hours, a Nova can be 10^5 times more luminous than the Sun.

GK Persei: Nova Persei 1901 red circle on the chart

m ~ 14
d ~ 442 pc

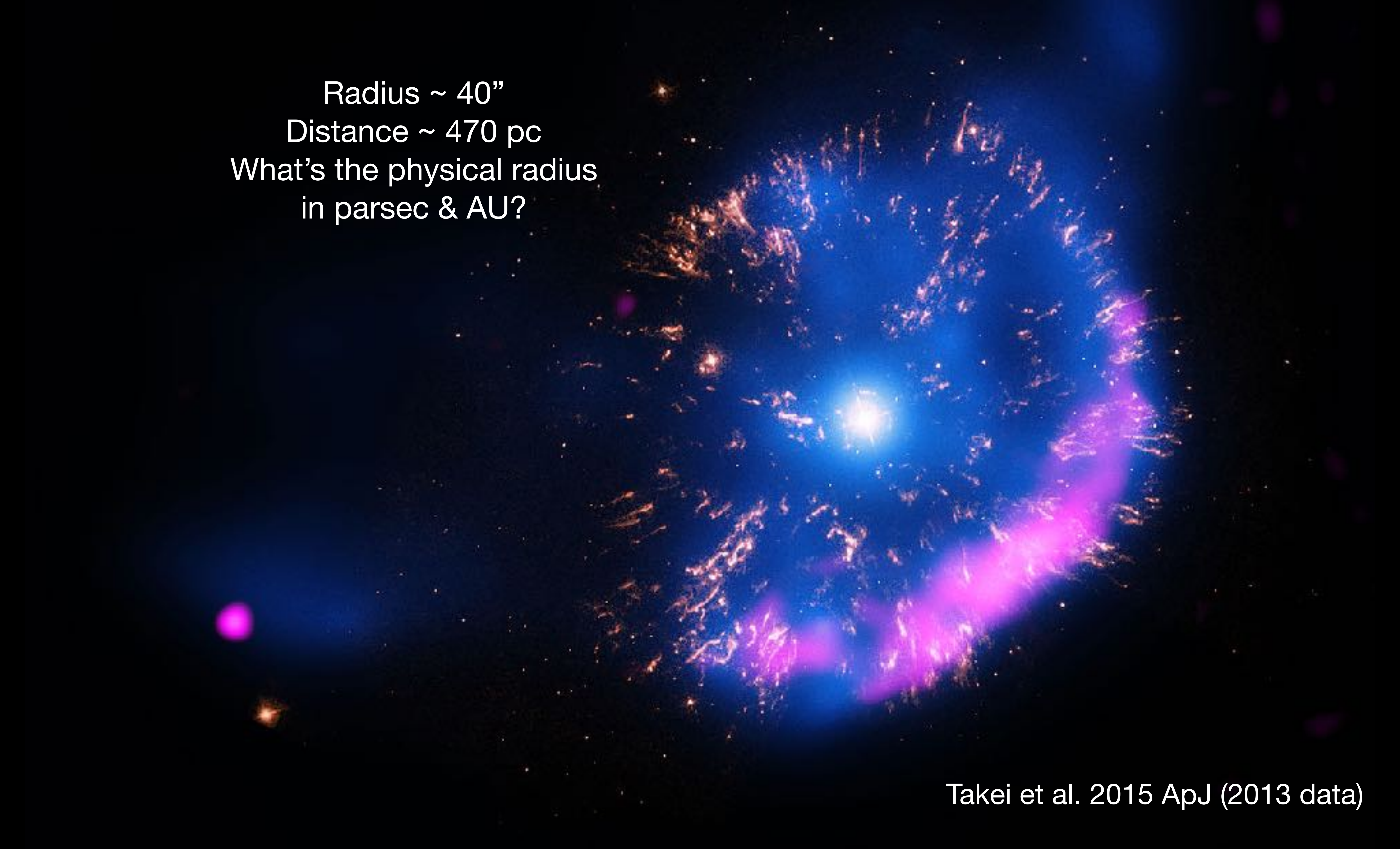


In 1901, GK Persei was one of the brightest star on the sky (for a few days)



GK Persei: Nova of 1901
X-ray (blue), optical (yellow), radio (pink)

Radius ~ 40”
Distance ~ 470 pc
What’s the physical radius
in parsec & AU?



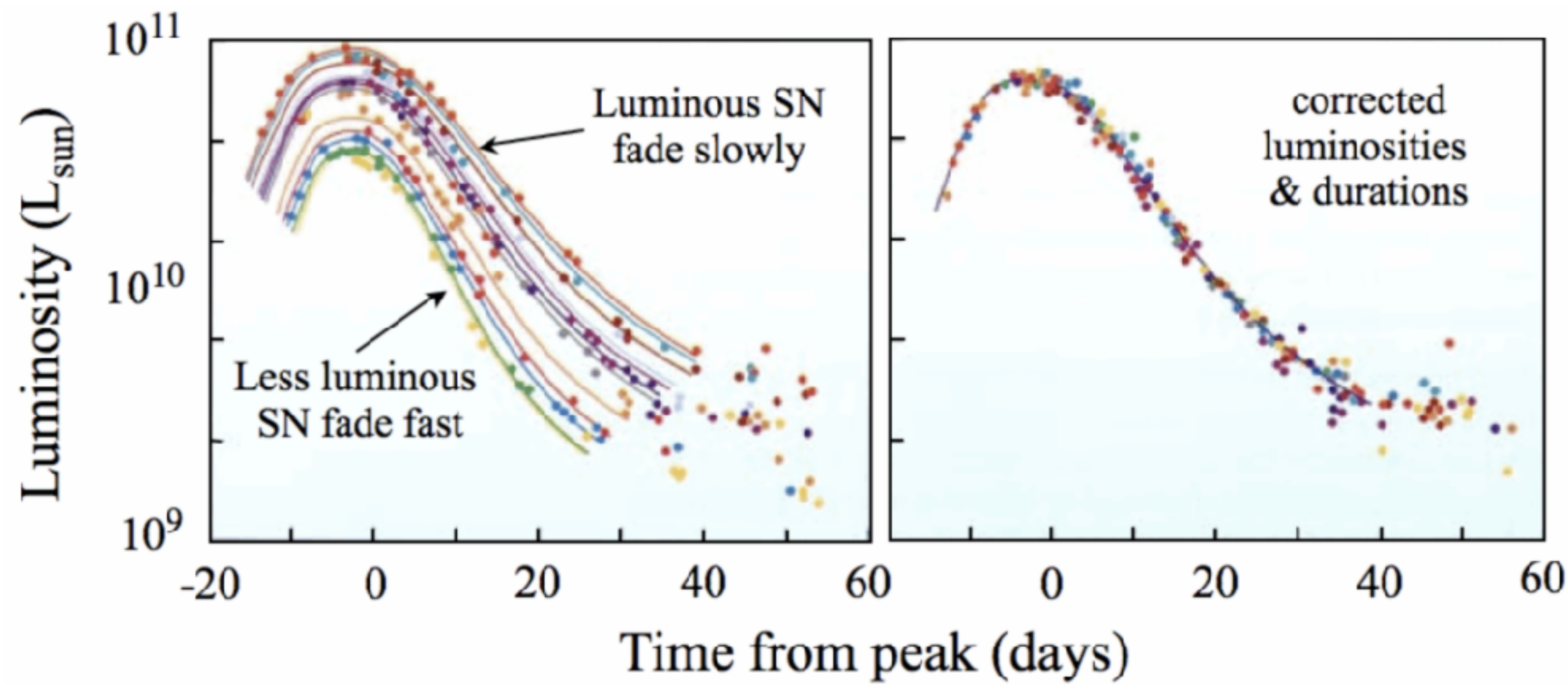
Takei et al. 2015 ApJ (2013 data)

II: Type Ia Supernovae

- WD *mass increases over time* because of accretion from the RGB
- When its mass reaches 1.4 M_{sun}, the **Chandrasekhar limit**, gravity overcomes the ***relativistic* electron degeneracy pressure**
- The WD collapses, heats up and triggers a **thermonuclear runaway**:
  - **"C Flash"**: C core burns out in <1 sec! Forms Mg, Ne, Na, Ni, Fe.
  - This is a Type Ia supernova.
- 10^{10} times brighter than the Sun – comparable to the luminosity of a galaxy!
- Note: Type Ia may also be WD mergers

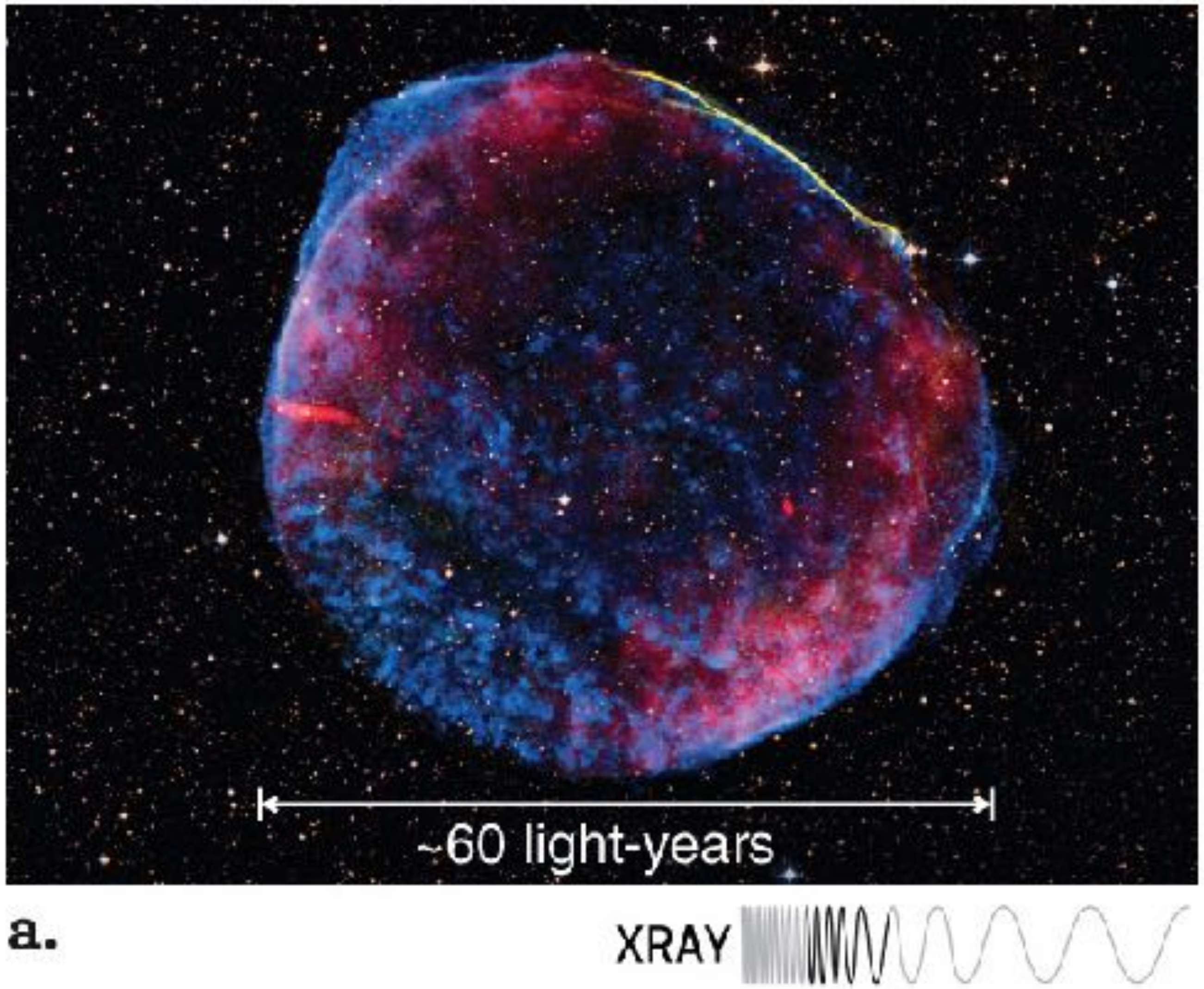
Type Ia Supernovae

- Over just a few days, the explosion releases about the same amount of energy as the Sun does over its entire main-sequence lifetime (10^{44} Joules).
- Type Ia supernovae are excellent **distance indicators** because they are **standardizable candles** (peak apparent magnitude can be standardized by the shape of its light curve): $m_B^0 = -2.5\log(x_0) + \alpha x_1 - \beta c$, where x_0, x_1, c are amplitude, stretch and color parameters.

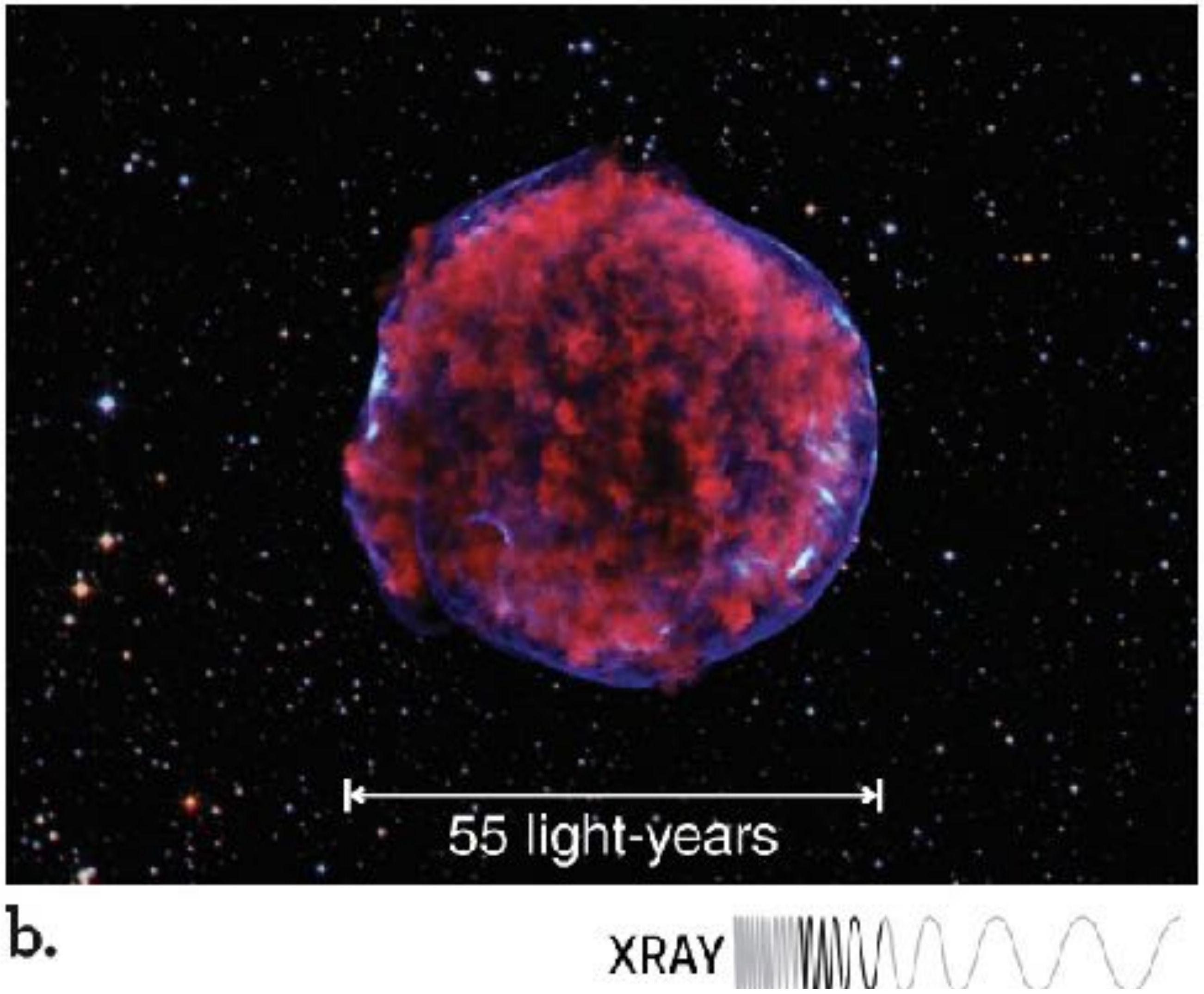


Type Ia Supernova Remnants

- The entire star explodes in the thermonuclear runaway — there is no central star left (unlike planetary nebulae and type II SNe)
- **Supernova remnants** are leftover shells of dust and gas from the explosion

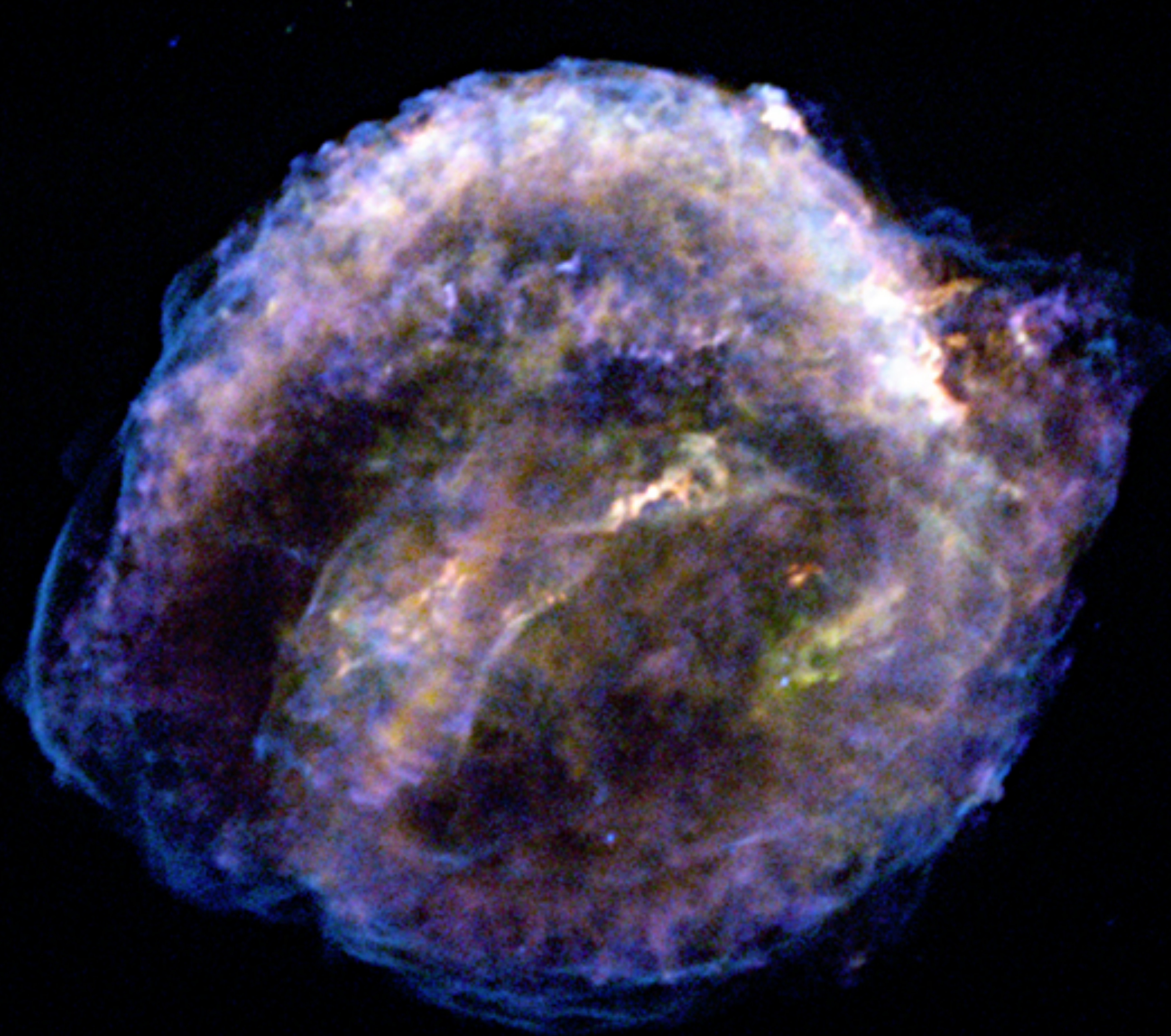


a.

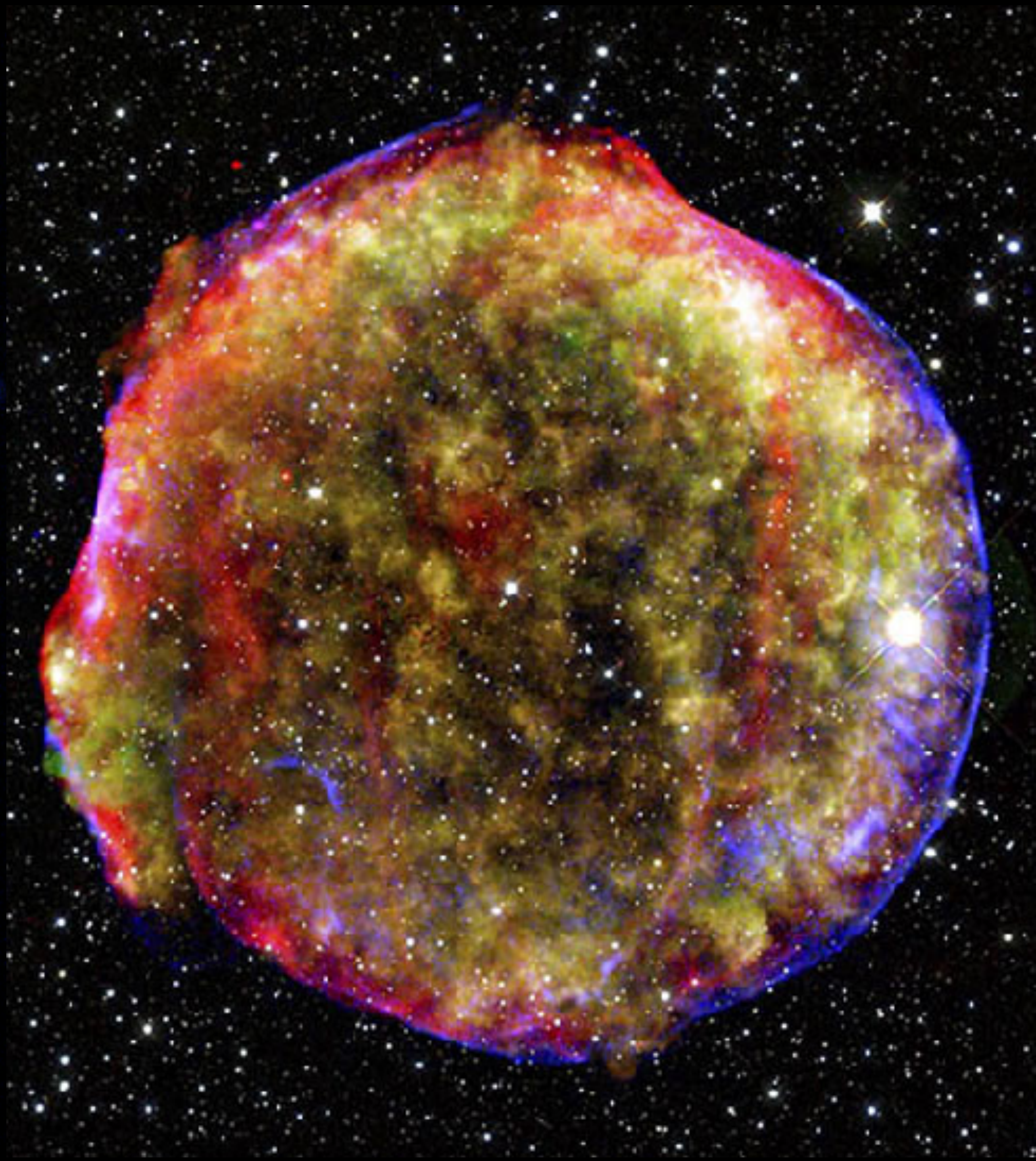


b.

More X-ray Images of Type Ia Supernova Remnants



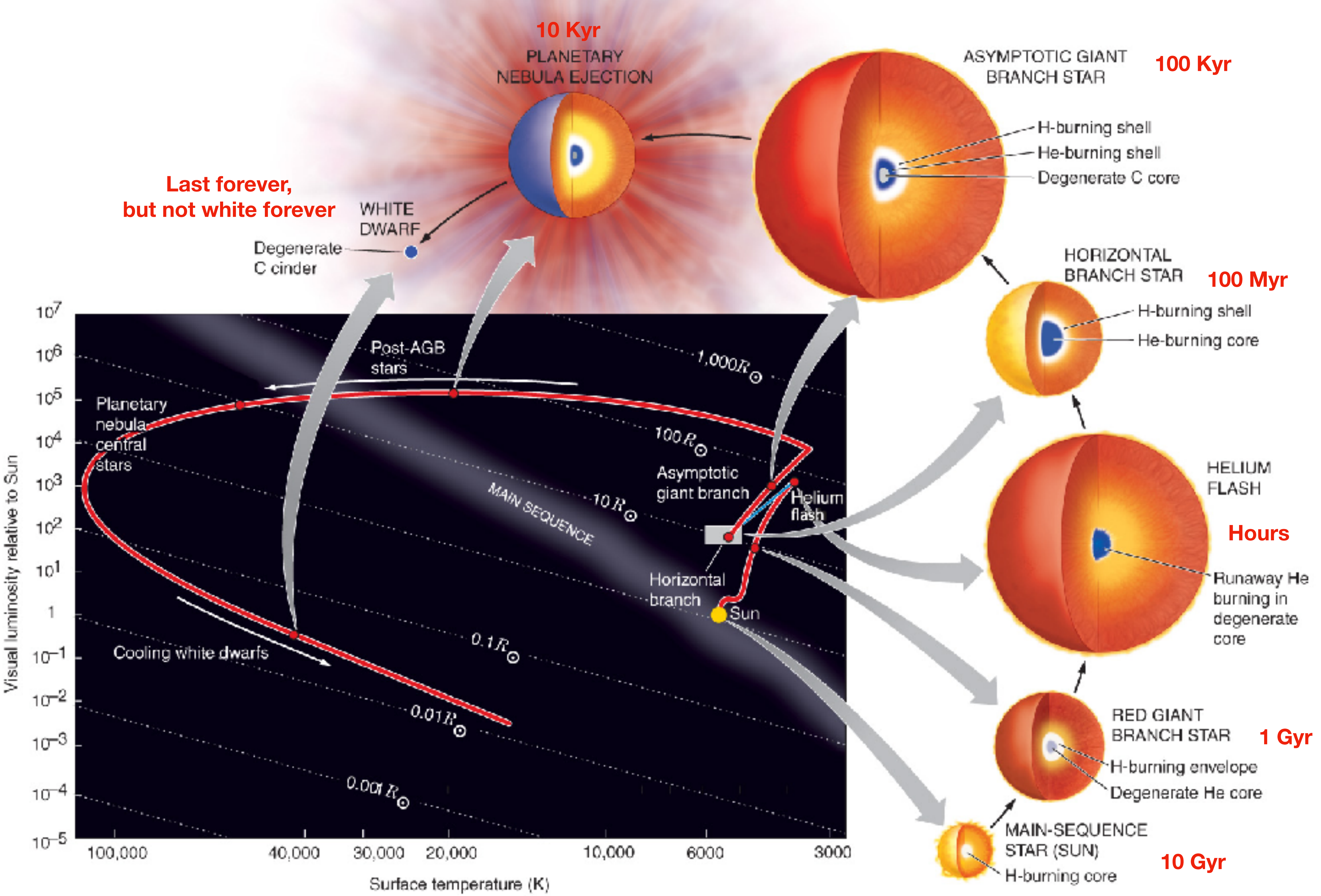
Kepler supernova (1604)



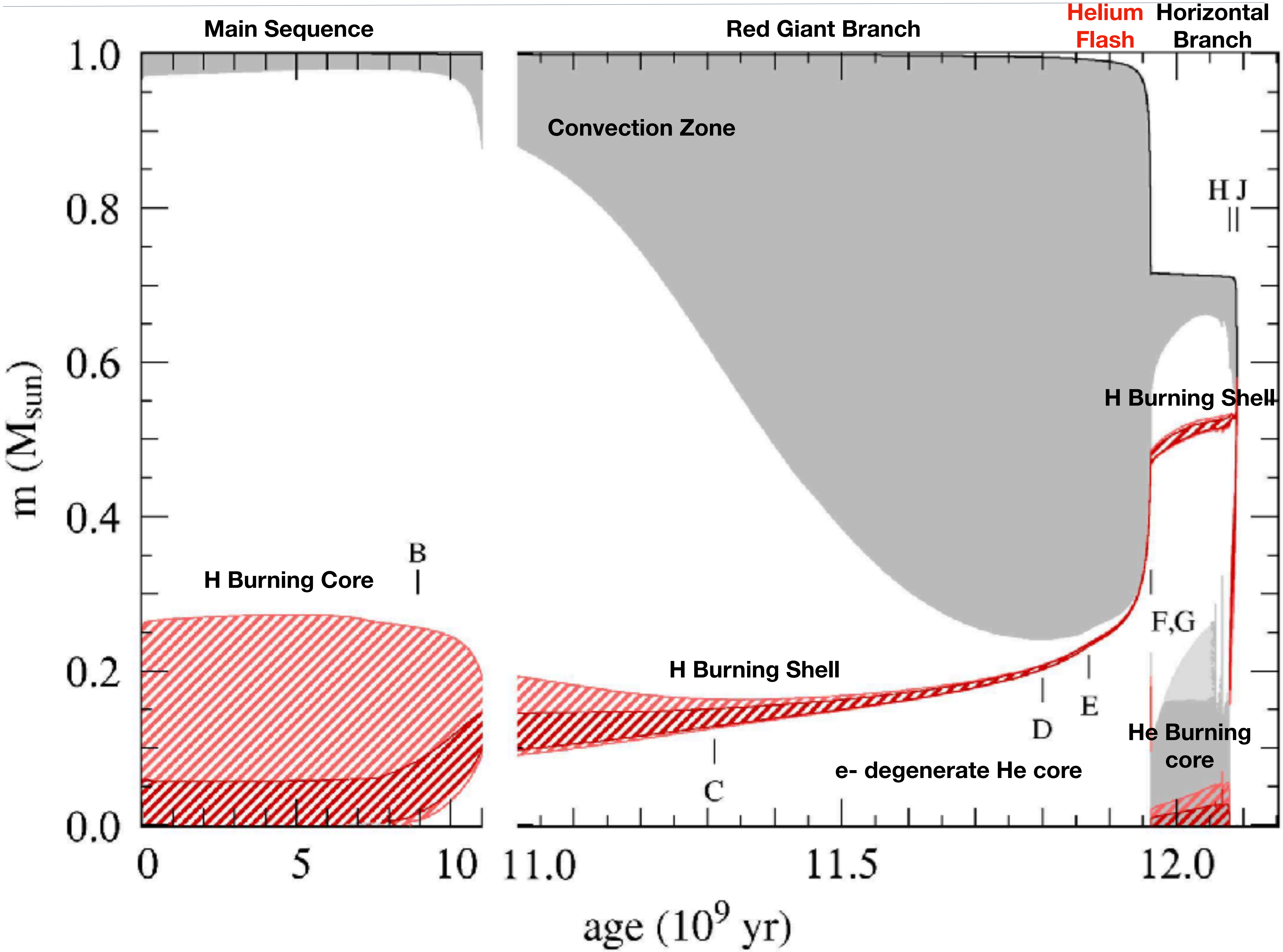
Tycho supernova (1572)

A Summary of Low-Mass Stellar Evolution

The evolutionary track of a 1 Solar mass star



Kippenhahn Diagram of a Star with an Initial Mass of 1.0 Solar Mass



Post-**MS** evolution:

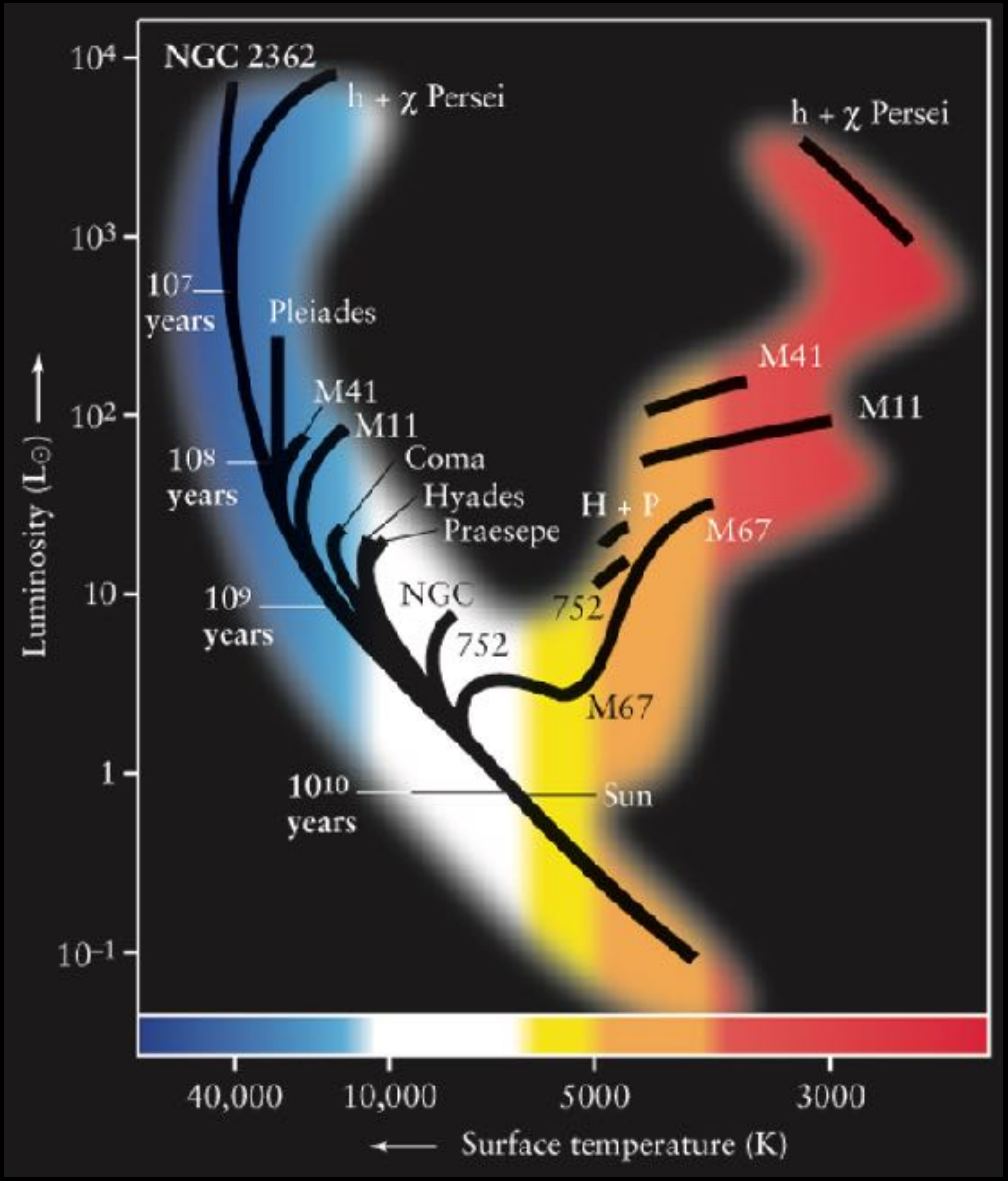
- non-fusing Helium core, H-burning shell
- He-core contracts and become **e- degenerate**
- **Red giant** phase (H^- controls surface T as L increases)
- uncontrolled Helium-burning in the e- degenerate core (thermonuclear runaway, **Helium flash**)
- core expands and become non-degenerate, allowing steady Helium burning (**Horizontal branch**)

Post-**HB** evolution:

- non-fusing C core, He-burning shell, H-burning shell
- C-core contracts and become **e- degenerate**
- **Asymptotic Giant Branch** Phase (H^- controls surface T as L increases, the radius of the star increases even more)
- core temperature never reaches 500 million K needed for Carbon-burning (**no Carbon flash**)
- But eventually the AGB star becomes so bloated that it loses its envelope (**post-AGB** phase) and reveals the **WD**

Chap 3: Key Concepts

- Observations
  - Nothing last forever, even stars
  - H-R diagram of star clusters
- Numerical Models
  - Equations of stellar structure and evolution
  - Stellar evolutionary tracks
- Fine-Tune Models
  - Isochrones (equal-age lines)
  - Fitting cluster H-R diagrams
  - Cluster age estimates
- Model Inferences
  - Main stages and rough lifetimes
  - Changes in the interiors of the stars: e- degenerate core + fusion shells



Chap 3: Key Equations

- Hydrostatic Equilibrium:

$$\frac{dP}{dr} = -\rho g(r) = -\rho \frac{GM_r}{r^2}$$

- The pressure from non-relativistic degenerate gas is:

$$P_{\text{degen}} = \frac{2}{3} n \frac{p^2}{2m} \approx \frac{h^2}{4\pi^2} \frac{n^{5/3}}{m}$$

- The pressure from ideal gas is:

$$P_{\text{ideal}} = \frac{2}{3} n \left(\frac{3}{2} kT \right) = nkT$$

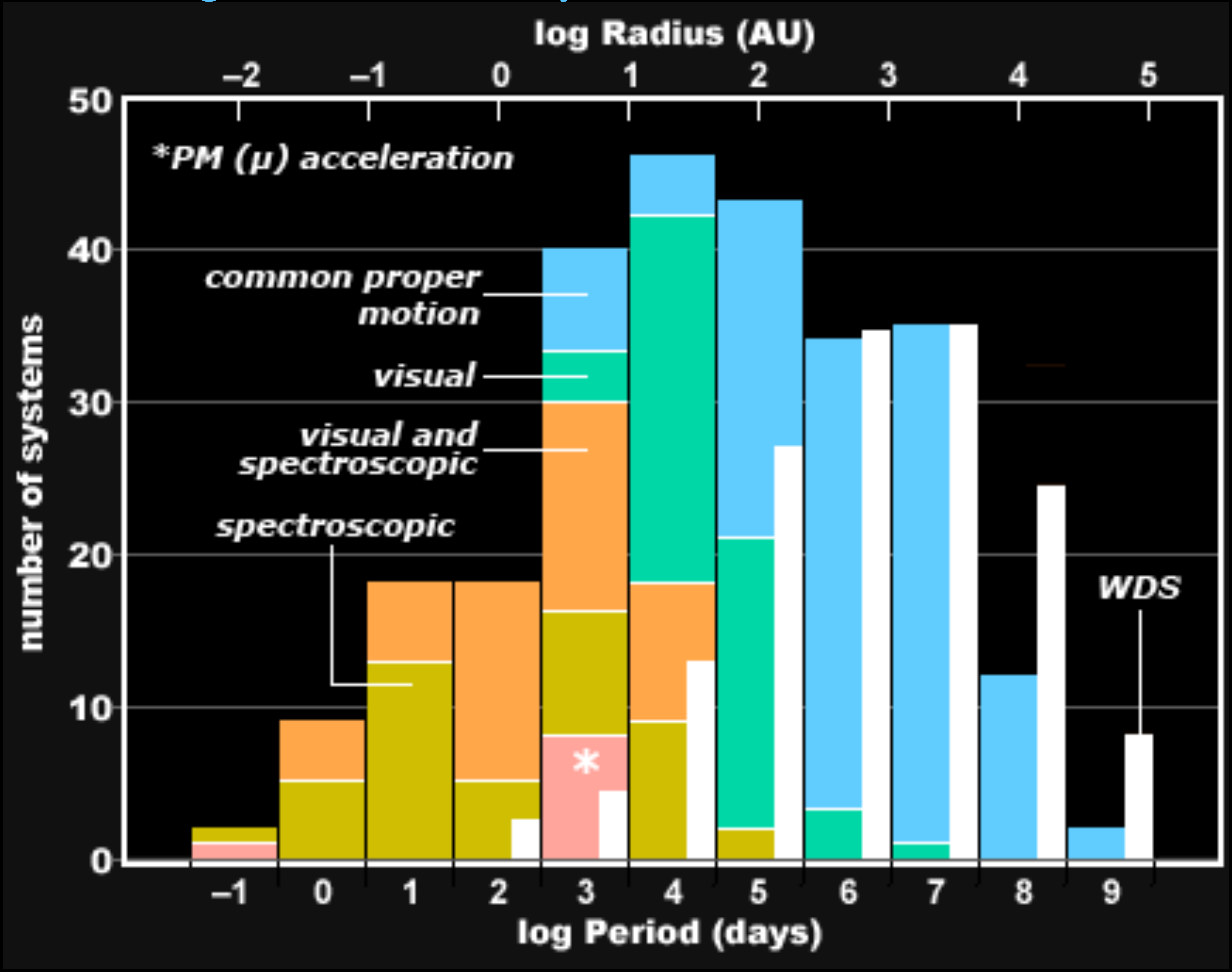
- The condition for degeneracy is

$$\frac{h^2}{4\pi^2} \frac{n^{2/3}}{m} > kT$$

- Mass-Radius relation:
  - $R \propto M^{-1/3}$ for white dwarfs (Chandrasekhar limit: 1.4 M_{sun})
  - $R \propto M^{0.7}$ for main-sequence stars

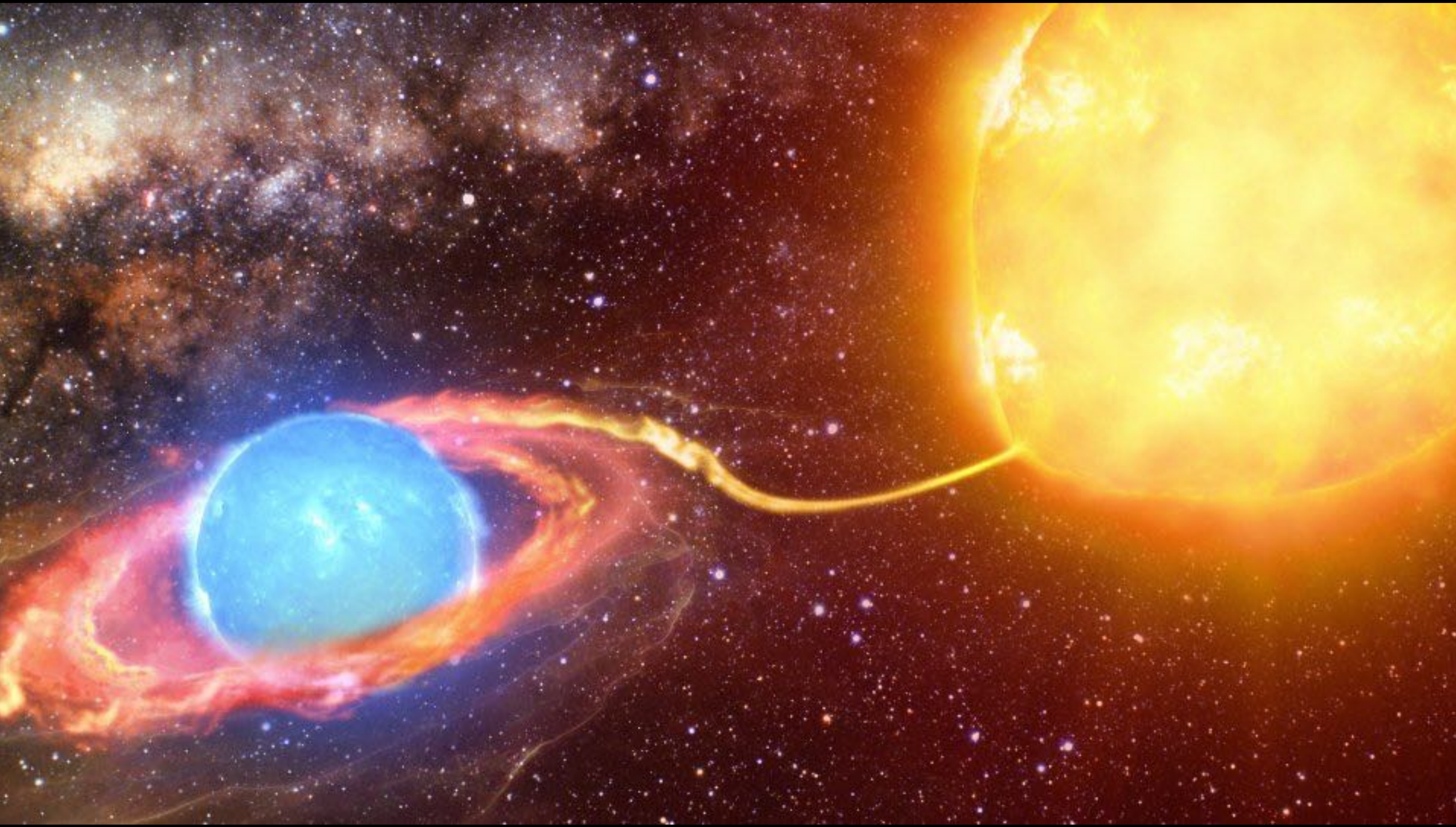
Effective Potential of a Binary & Lagrange Points

The logarithmic of Binary Periods follow a “Bell” curve



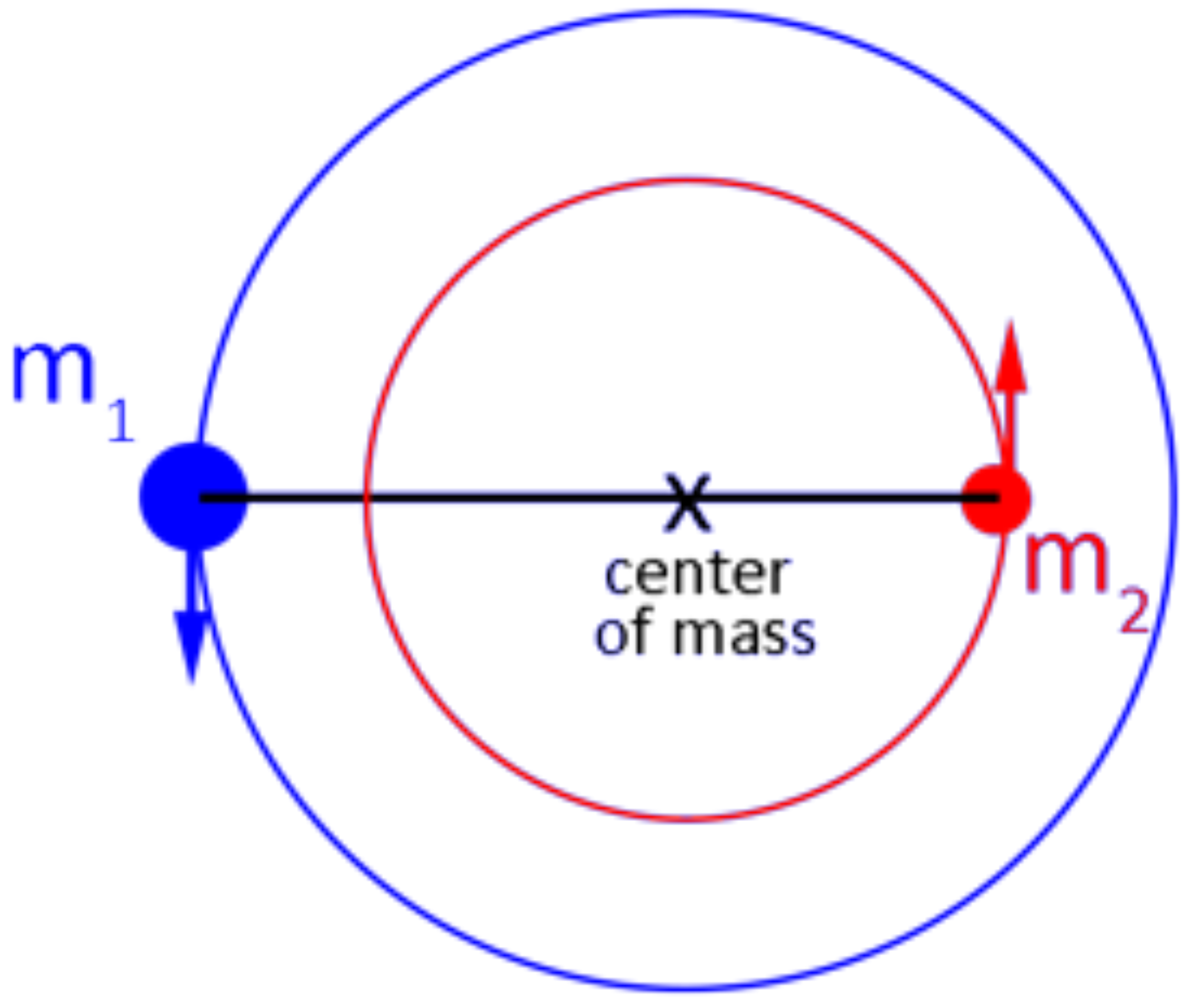
What about stars in close binary systems?

"Breaking the isolation and starting to share": co-evolution

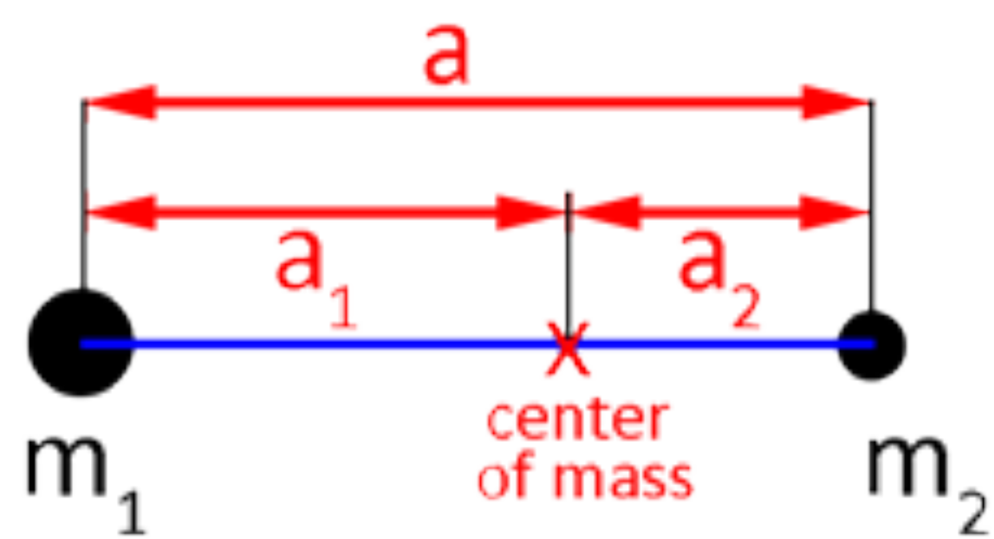


Introducing the non-inertial co-rotating reference frame

- Inertial reference frame for binaries on circular orbits



- Non-inertial co-rotating reference frame



A **non-inertial reference frame** (also known as an **accelerated reference frame**[1]) is a frame of reference that undergoes acceleration with respect to an inertial frame. In classical mechanics it is often possible to explain the motion of bodies in non-inertial reference frames by introducing additional fictitious forces (also called inertial forces, pseudo-forces[5] and d'Alembert forces) to Newton's second law. Common examples of this include the Coriolis force and the centrifugal force.

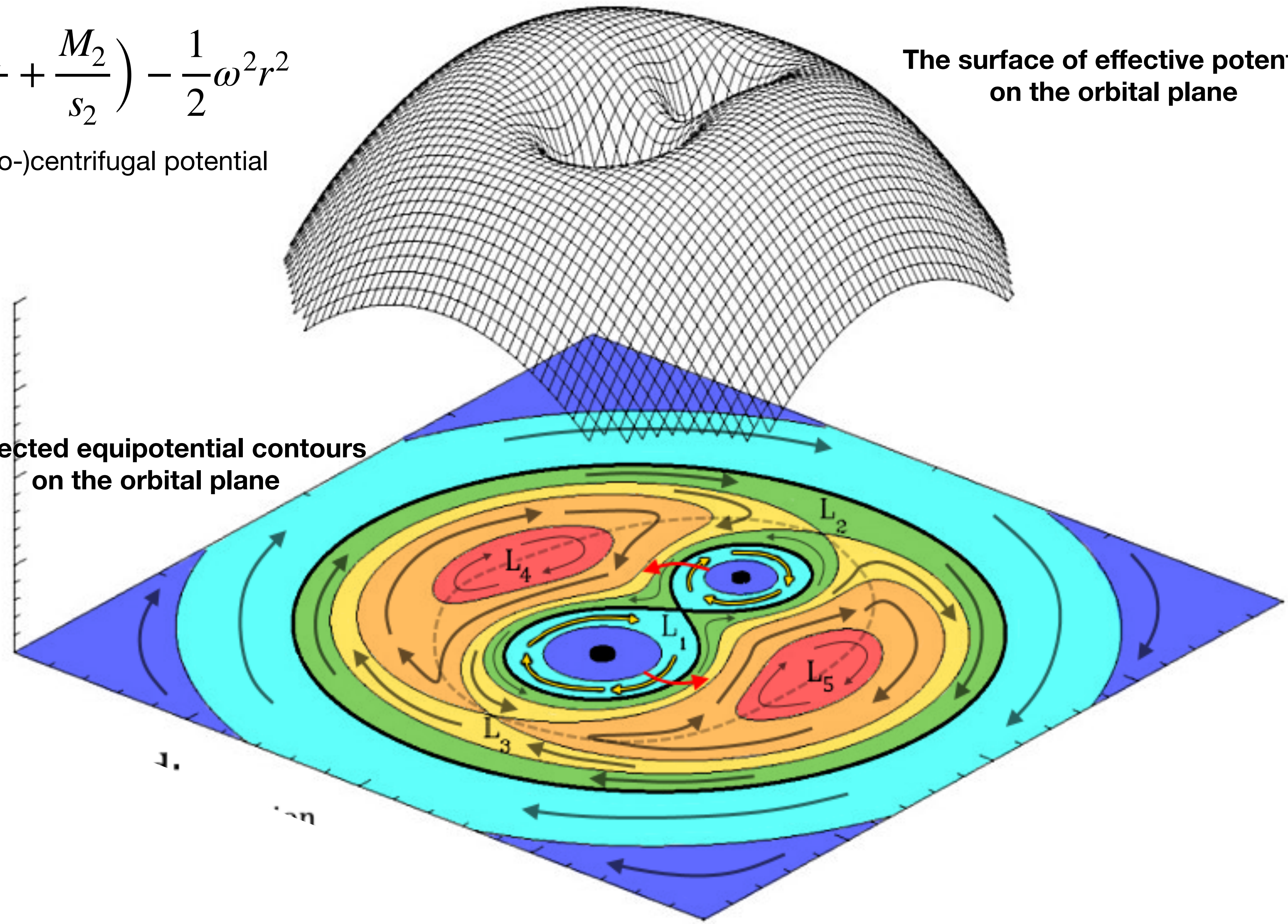
Effective potential in a co-rotating reference frame

The contours of **equipotential** show the ***effective*** **gravitational potential** on the orbital plane, the **Lagrange points** are the **local maxima** where the gradient of effective potential is zero (**no acceleration** in the **co-rotating non-inertial reference frame**).

$$\Phi_{\text{eff}} = -G\left(\frac{M_1}{s_1} + \frac{M_2}{s_2}\right) - \frac{1}{2}\omega^2 r^2$$

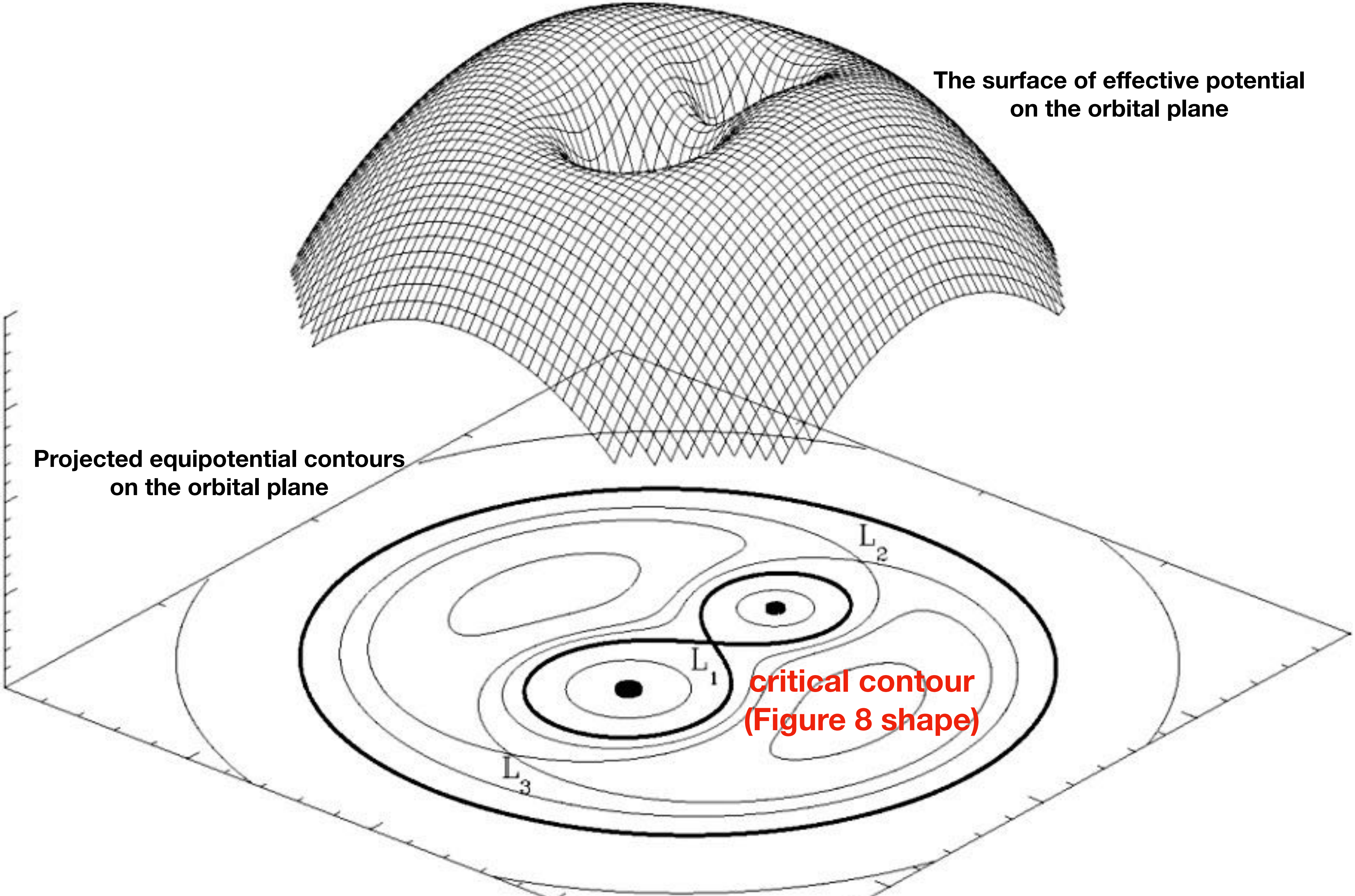
2nd term is (pseudo-)centrifugal potential

$$\vec{a}_{\text{eff}} = -\nabla\Phi_{\text{eff}}$$



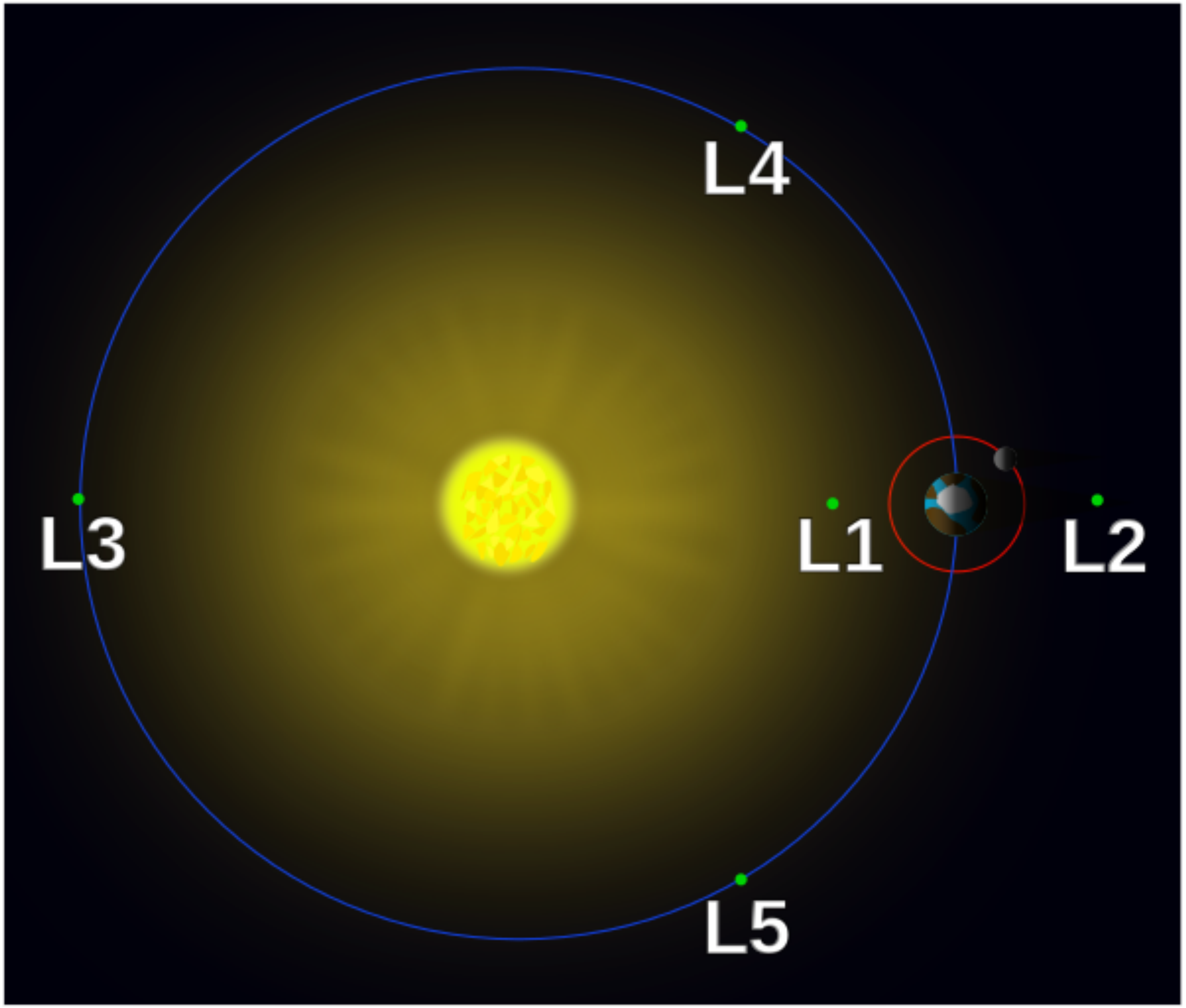
Roche Lobe (or Roche Surface)

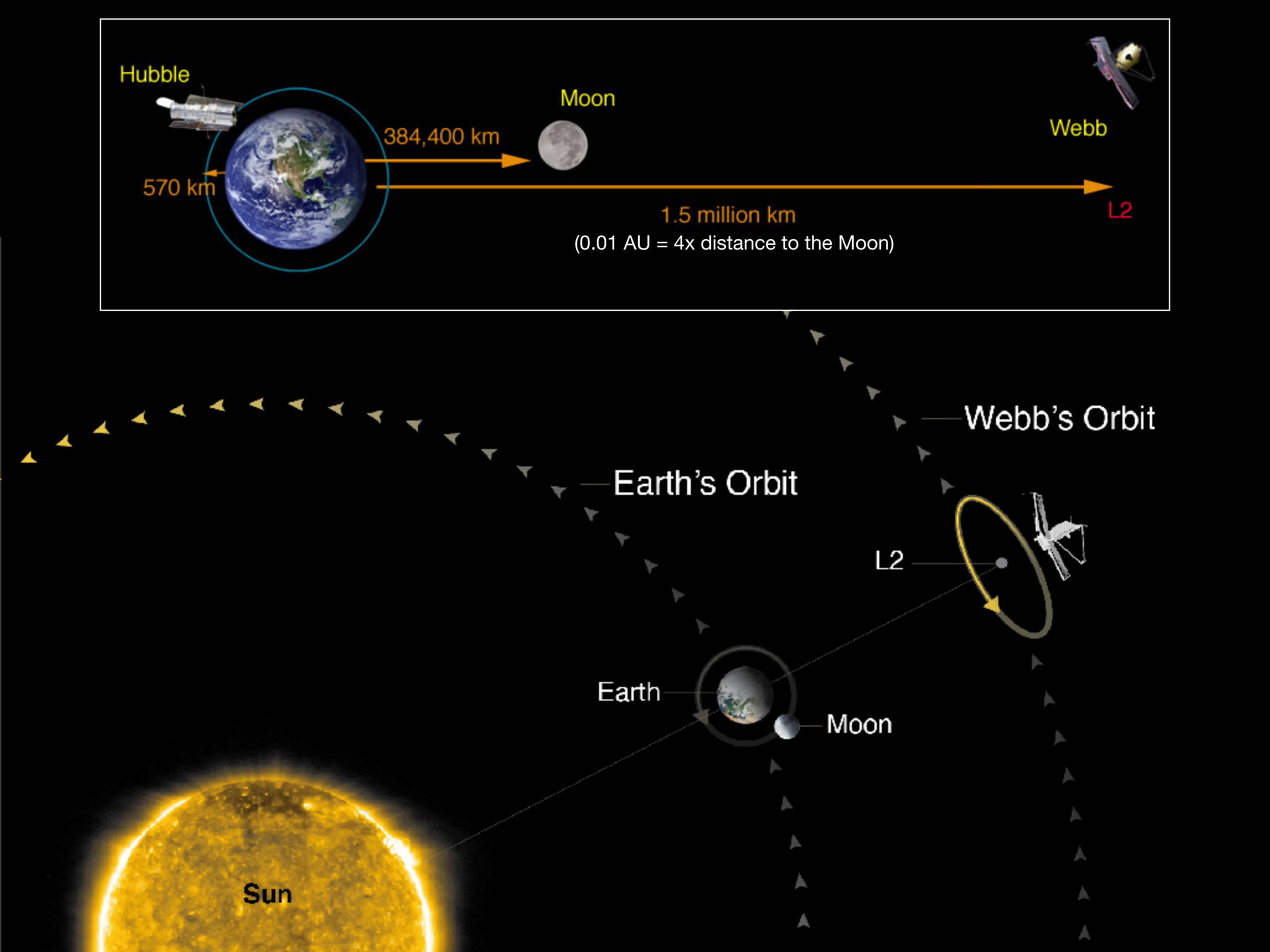
- On the orbital plane, there is a **critical equipotential contour** that intersects itself at the **L1 point**, forming a figure-of-eight.



Lagrange Points of a Binary System

- **Equilibrium points** for small mass objects under the influence of two massive orbiting bodies in a **co-rotating frame of reference (non-inertial)**



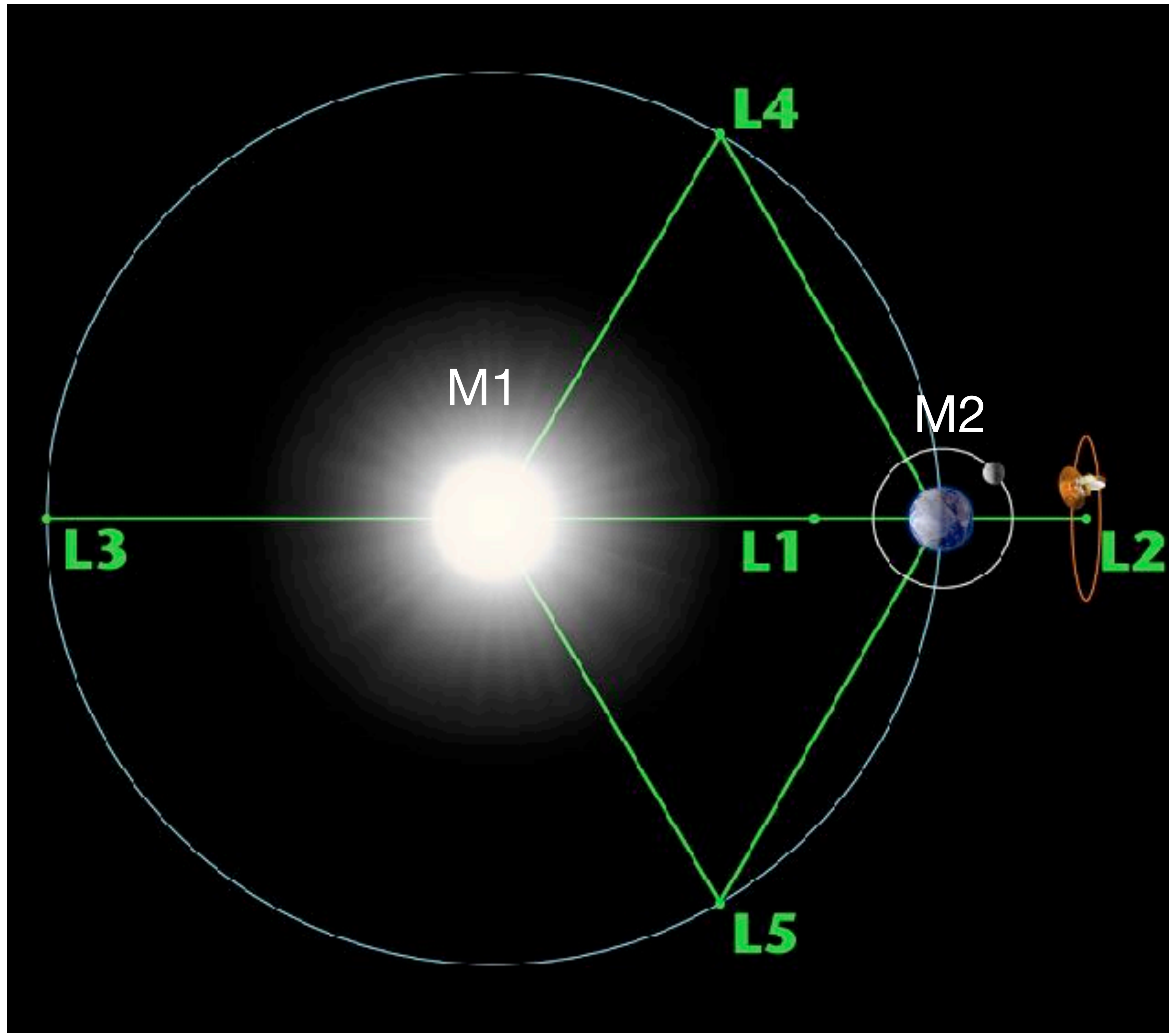


Hubble
570 km
384,400 km
Moon
Webb
1.5 million km
L2
(0.01 AU = 4x distance to the Moon)
Webb's Orbit
Earth's Orbit
L2
Earth
Moon
Sun

Location of the L1 Point: approximate formulae for large mass ratios (M2/M1 > 0.01), this provide a rough estimate of the sizes of Roche Lobes

$$d(L1 \rightarrow M_1) = a[0.5 - 0.227\log(M_2/M_1)]$$
$$d(L1 \rightarrow M_2) = a[0.5 + 0.227\log(M_2/M_1)]$$

Note that as the mass ratio changes, the **L1 point** moves in the ***opposite direction*** as the **center of mass**.



Earth-Moon free-return trajectory: 1959 Soviet Luna 3